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WELL-DEFINABLE TYPES OVER SUBSETS

by Anand PILLAY (*)

In this short note I give a "direct" proof of a beautiful result of elementary stability theory. The result is that for T stable, if d_1 and d_2 are "good" defining schemae over a set A , and $d_1(A) = d_2(A)$, then for all $B \supset A$, $d_1(B) = d_2(B)$, that is, d_1 and d_2 are equivalent. This result does not mention forking, although the "usual" proof of it uses forking. Our proof will be forking-free. In fact, we show directly that if the result fails then T has the order property.

T is complete, and we work, as usual, in a very saturated model of T . I recall the following definitions.

Definition 1. - Let A be a set of parameters (i. e. a subset of the big models), and $n < \omega$. Let $\bar{x}$ denote an n -tuple of variables. An n -schema over A , is a map d which associates to each L -formula $\varphi(\bar{x}, \bar{y})$ an $L(A)$ -formula $\psi(\bar{y})$. $\psi(\bar{y})$ is denoted $d\varphi(\bar{y})$. A schema over A is just an n -schema over A for some $n < \omega$.

Definition 2. - Let d be a schema over A . Let B be a set. Then

$$d(B) = \{\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in L \text{ and } \models d\varphi(\bar{b})\}.$$

Note. - B is usually taken to include A . $d(B)$ need neither be consistent nor complete.

Definition 3. - d is said to be a good defining schema over A , if d is a schema over A and moreover for all B , $d(B) \in S(B)$, i. e. $d(B)$ is consistent and complete.

Fact 4. - Let d be a schema over A . Then the following are equivalent,

- (i) d is a good defining schema over A ,
- (ii) for some model $M \supset A$, $d(M) \in S(M)$,
- (iii) for each L -formula $\varphi(\bar{x}, \bar{y})$ and finite collection $\{\varphi_i(\bar{x}, \bar{y}_i) : i < m\}$ of L -formulae, we have

$$\models (\forall \bar{y}) ((d \neg \varphi)(\bar{y}) \leftrightarrow \neg d\varphi(\bar{y}))$$

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and

$$\models (\forall \bar{y}_0 \dots \bar{y}_{m-1}) (\bigwedge_{i < m} d\varphi_i(\bar{y}_i) \rightarrow (\exists \bar{x}) (\bigwedge_{i < m} \varphi_i(\bar{x}, \bar{y}_i))) .$$

Remember that, even for T stable there may be a schema d over a set A such that $d(A) \in S(A)$ and d is not a good defining schema. It is also easy to manufacture examples of d_1, d_2 schemae over A such that d_1 is a good defining schema over A , d_2 is not a good defining schema over A and $d_1(A) = d_2(A)$.

If d is a schema over a model M , and $d(M) \in S(M)$, then, by Fact 4, d is a good defining schema. Moreover, it is easy to see that if d' is another schema over M such that $d(M) = d'(M)$, then d and d' are equivalent. (d and d' are said to be equivalent if, for all B , $d(B) = d'(B)$.) This holds whether T is stable or not.

The following is an example of a theory T (unstable of course) for which there are good defining schemae d_1 and d_2 over a set A such that $d_1(A) = d_2(A)$, but d_1 and d_2 are not equivalent. Let T be $\text{Th}(\underline{Q}, <)$. Let $M = (\underline{Q}, <)$, and let a, b be elements of the big model such that

$$\models a > q \text{ for all } q \in M \text{ and } \models b < q \text{ for all } q \in M .$$

So $\text{tp}(a/M) \neq \text{tp}(b/M)$. It is easy to see that both $\text{tp}(a/M)$ and $\text{tp}(b/M)$ are definable over $\emptyset$. (For example, for each $y \in M$, $\models a > y$ if, and only if, $\models y = y$.) Let d_1 and d_2 be defining schemae over $\emptyset$ for $\text{tp}(a/M)$ and $\text{tp}(b/M)$ respectively. So (by Fact 4) both d_1 and d_2 are good defining schemae over $\emptyset$. Also $d_1(\emptyset) = d_2(\emptyset) =$ the unique 1-type of T over $\emptyset$. But of course d_1 and d_2 are not equivalent (as $d_1(M) \neq d_2(M)$).

The main property of good defining schemae that we use, is the following (which is trivial) :

Fact 5. - Let d be a good defining schema over A . Let $\varphi(\bar{x}, \bar{y}) \in L$. Let B be a set and $\bar{b}, \bar{b}' \in B$ be such that $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A)$. Then $\varphi(\bar{x}, \bar{b}) \in d(B)$ if, and only if, $\varphi(\bar{x}, \bar{b}') \in d(B)$.

As we are proving things "from scratch" here, we give the following standard lemma :

LEMMA 6. - Let T be stable. Suppose that d_1, d_2 are good defining schemae over A and that $\text{tp}(\bar{a}/A) = d_1(A)$ and $\text{tp}(\bar{b}/A) = d_2(A)$. Then

$$\text{tp}(\bar{a}/A \cup \bar{b}) = d_1(A \cup \bar{b}) \text{ if, and only if, } \text{tp}(\bar{b}/A \cup \bar{a}) = d_2(A \cup \bar{a}) .$$

Proof. - Without loss of generality let us assume that $\text{tp}(\bar{b}/A \cup \bar{a}) = d_2(A \cup \bar{a})$, but, for some $L(A)$ -formula, $\varphi(\bar{x}, \bar{y})$, $\models \varphi(\bar{a}, \bar{b})$ and $\neg \varphi(\bar{x}, \bar{b}) \in d_1(A \cup \bar{b})$. Now we define $\bar{a}_i, \bar{b}_i$ for $i < \omega$ as follows, $\bar{a}_0 = \bar{a}$, $\bar{b}_0 = \bar{b}$, $\bar{a}_{n+1}$ is a

realisation of $d_1(\Lambda \cup \{\bar{a}_i \wedge \bar{b}_i : i \leq n\})$ and $\bar{b}_{n+1}$ is a realisation of $d_2(\Lambda \cup \{\bar{a}_i \wedge \bar{b}_i : i \leq n\} \cup \{\bar{a}_n\})$. It is then easy to see, using Fact 5, that $\models \varphi(\bar{a}_i, \bar{b}_j)$ if, and only if, $i \leq j$. Thus T has the order property, which contradicts stability.

PROPOSITION 7. - Let T be stable. Let d_1, d_2 be good defining schemae over A such that $d_1(\Lambda) = d_2(\Lambda)$. Then for all B , $d_1(B) = d_2(B)$.

Proof. - Without loss of generality, let us assume that $\Lambda \neq \emptyset$. If the proposition fails then we have, for some formula $\varphi(\bar{x}, \bar{y})$ and tuple $\bar{b}$,

$$\varphi(\bar{x}, \bar{b}) \in d_1(b) \quad \text{and} \quad \neg \varphi(\bar{x}, \bar{b}) \in d_2(b).$$

We now define inductively $\bar{a}_i$ and $\bar{b}_i$ for $i < \omega$ such that

- (i) $\text{tp}(\bar{b}_0/\emptyset) = \text{tp}(\bar{b}/\emptyset)$,
- (ii) $\models \varphi(\bar{a}_i, \bar{b}_j)$ if, and only if, $i \leq j$, for all $i, j < \omega$,
- (iii) $\text{tp}(\bar{a}_0 \wedge \bar{b}_0 \dots \bar{a}_{n-1} \wedge \bar{b}_{n-1}) = \text{tp}(\bar{a}_1 \wedge \bar{b}_1 \dots \bar{a}_n \wedge \bar{b}_n)$ for $1 \leq n < \omega$,
- (iv) $\text{tp}(\bar{a}_0/\bar{b}_0 \wedge \bar{a}_1 \wedge \bar{b}_1 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n) = d_1(\bar{b}_0 \wedge \bar{a}_1 \wedge \bar{b}_1 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n)$, for $n < \omega$
- (v) $\text{tp}(\bar{a}_n/\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1}) = d_2(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1})$, for $n < \omega$.

First let $\bar{a}_0$ be any realisation of $d_1(\emptyset)$, and let $\bar{b}_0$ be a tuple such that $\text{tp}(\bar{b}_0) = \text{tp}(\bar{b})$ and $\text{tp}(\bar{a}_0/\bar{b}_0) = d_1(\bar{b}_0)$. Clearly (i) is satisfied, as is (iv). The satisfaction of (ii) is given by Fact 5 and the fact that $\varphi(\bar{x}, \bar{b}) \in d_1(\bar{b})$. (v) follows from the fact that $d_1(\emptyset) = d_2(\emptyset)$.

Now suppose that $\bar{a}_i$ and $\bar{b}_i$ have been defined for $i \leq n$ satisfying the requirements. We proceed to define $\bar{a}_{n+1}$ and $\bar{b}_{n+1}$.

First let $\bar{a}_{n+1}$ be a realisation of $d_2(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n)$. Now by induction hypothesis

$$\text{tp}(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1}) = \text{tp}(\bar{a}_1 \wedge \bar{b}_1 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n)$$

and

$$\text{tp}(\bar{a}_n/\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1}) = d_2(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1}).$$

It follows from Fact 5 that

$$(*) \quad \text{tp}(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1} \wedge \bar{a}_n) = \text{tp}(\bar{a}_1 \wedge \bar{b}_1 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n \wedge \bar{a}_{n+1}).$$

As $\neg \varphi(\bar{x}, \bar{b}) \in d_2(\bar{b})$ and $\text{tp}(\bar{b}_i) = \text{tp}(\bar{b})$ for all $i \leq n$, we have

$$(**) \quad \models \neg \varphi(\bar{a}_{n+1}, \bar{b}_i) \quad \text{for all } i \leq n.$$

By (*), we can find $\bar{b}'$ such that

$$(*') \quad \text{tp}(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n) = \text{tp}(\bar{a}_1 \wedge \bar{b}_1 \wedge \dots \wedge \bar{a}_{n+1} \wedge \bar{b}').$$

Thus we have, using the induction hypothesis,

$$(***) \quad \models \varphi(\bar{a}_i, \bar{b}') \quad \text{for } 1 \leq i \leq n+1.$$

The trouble is that we might not have $\models \varphi(\bar{a}_0, \bar{b}')$. To overcome this, we let $\bar{a}'$ be a realisation of $d_1(\bar{b}_0 \wedge \bar{a}_1 \wedge \bar{b}_1 \wedge \dots \wedge \bar{a}_{n+1} \wedge \bar{b}')$. As $\text{tp}(\bar{b}_i) = \text{tp}(\bar{b})$ for all $i \leq n$ and also $\text{tp}(\bar{b}') = \text{tp}(\bar{b})$, it follows that

$$(***) \quad \models \varphi(\bar{a}', \bar{b}') \quad \text{and} \quad \models \varphi(\bar{a}', \bar{b}_i) \quad \text{for all } i \leq n.$$

Now by (iv) of the induction hypothesis and the definition of $\bar{a}'$, it follows that

$$(I) \quad \text{tp}(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n) = \text{tp}(\bar{a}' \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n).$$

Now lemma 6 (and the definition of $\bar{a}'$) imply that

$$(II) \quad \text{tp}(\bar{a}_{n+1}/\bar{a}' \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n) = d_2(\bar{a}' \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n).$$

Then (I) and (II) imply that

$$\text{tp}(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n \wedge \bar{a}_{n+1}) = \text{tp}(\bar{a}' \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n \wedge \bar{a}_{n+1}).$$

Thus we can find $\bar{b}_{n+1}$ such that

$$(III) \quad \text{tp}(\bar{a}_0 \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n \wedge \bar{a}_{n+1} \wedge \bar{b}_{n+1}) \\ = \text{tp}(\bar{a}' \wedge \bar{b}_0 \wedge \dots \wedge \bar{a}_n \wedge \bar{b}_n \wedge \bar{a}_{n+1} \wedge \bar{b}').$$

Now we check the satisfaction of conditions (ii)-(v), for

$$\{\bar{a}_0, \bar{b}_0, \dots, \bar{a}_{n+1}, \bar{b}_{n+1}\}.$$

(ii) follows from the induction hypothesis, (**), (***), (****) and (III). (iii) is a consequence of (*) and (III). (iv) (with $n+1$ in place of n) is by the definition of $\bar{a}'$ and (III). (v) (again with $n+1$ in place of n) is by the definition of $\bar{a}_{n+1}$ and (III).

Thus the induction can be carried out, whereby condition (ii) says that T has the order property, contradicting stability. So Proposition 7 is proved.

Let me briefly remark on how Proposition 7 follows easily given forking theory. One just needs to observe that (for T stable), if d is a good defining schema over A then

(a) $d(A)$ is stationary, and

(b) for any $B \supset A$, $d(B)$ is the nonforking extension to B of $d(A)$.