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Asymptotic estimates for the first hitting time of fluctuating additive functionals of Brownian motion

Y. Isozaki S. Kotani

1 Introduction

In [3], we obtained the following estimates for the first hitting time of the integrated Brownian motion: Let $B(t)$ be the linear Brownian motion started at 0. It holds with some explicit constant $k > 0$

$$P\left[\int_0^u B(s)ds < r \text{ for all } 0 \le u \le t\right] \sim kr^{1/6}t^{-1/4} \text{ as } r^{1/6}t^{-1/4} \to 0, \tag{1.1}$$

which is a refinement of Sinai's estimates[12].

The above formula as well as the other ones follow systematically from the theorem in [3]: Let $(X(t), Y(t))$ be the Kolmogorov diffusion ([5]).

$$Y(t) = y + B(t), \qquad X(t) = x + \int_0^t Y(s)ds. \tag{1.2}$$

Let T be the first hitting time to the positive y-axis:

$$T = \inf\{t \ge 0; X(t) = 0, Y(t) \ge 0\}. \tag{1.3}$$

Hence $Y(T)$ is the hitting place on the positive y-axis. We denote by $E_{(x,y)}$ and $P_{(x,y)}$ the expectation and the probability measure for this diffusion respectively.

Theorem *([3]) For $\mu, \kappa \ge 0$ and $x \le 0$, $y \in \mathbb{R}$ it holds*

$$1 - E_{(\bar{\sigma}x, \bar{\sigma}^{1/3}y)}\left[\exp\left\{-\sigma\mu T - \sqrt{\sigma}\sqrt{2\kappa}Y(T)\right\}\right] \sim \tilde{K}(x,y)\bar{\sigma}^{1/6}K(\kappa,\mu)\sigma^{1/4} \tag{1.4}$$

as $\bar{\sigma}^{1/6}\sigma^{1/4}$ tends to 0, where

$$K(\kappa,\mu) = \frac{3(\sqrt{2\kappa} + \sqrt{2\mu})\Gamma\left(\frac{1}{3}\right)3^{1/3}}{\sqrt{\pi}\sqrt{\sqrt{2\kappa} + 2\sqrt{2\mu}}\Gamma\left(\frac{1}{6}\right)2^{1/6}}$$

and

$$\tilde{K}(x,y) = \frac{|x|^{5/6}e^{-2(y^+)^3/9|x|}}{\Gamma\left(\frac{1}{3}\right)}\int_0^\infty dt e^{-t}\left(|x|t + 2|y^-|^3/9\right)^{1/6}\left(|x|t + 2(y^+)^3/9\right)^{-5/6}.$$

The proof depends heavily on a formula obtained by McKean[8].

We considered in [4] a generalization for this problem. We redefine $(X(t), Y(t))$, the odd additive functional, as

$$Y(t) = y + B(t), \qquad X(t) = x + \int_0^t |Y(s)|^\alpha \operatorname{sgn}(Y(s))ds. \tag{1.5}$$

and we retain the notations T, $E_{(x,y)}$ and $P_{(x,y)}$. In [4], we were able to prove some weaker estimates:

Theorem *([4]) For $\alpha \geq 0$, $\nu := 1/(\alpha+2)$, $x \leq 0$ and $y = 0$, there exist positive constants $k'(\alpha)$, $k''(\alpha)$ such that*

$$k'(\alpha)|x|^{\nu/2}t^{-1/4} < P_{(x,0)}[T > t] < k''(\alpha)|x|^{\nu/2}t^{-1/4} \tag{1.6}$$

for all small $|x|^{\nu/2}t^{-1/4}$.

The present paper proves the existence of the limit value for $|x|^{-\nu/2}t^{1/4}P_{(x,0)}[T > t]$, and more generally, we obtain similar results for some additive fuctionals that are not odd, or symmetric. We shall observe that the exponent $-1/4$ of time parameter in the above theorems varies between 0 and $-1/2$ in accordance with the skewness of additive functionals.

There are at least two approaches for our problem: the analytical one using Krein's spectral theory of strings(cf. Kotani–Watanabe[6]) and the probabilistic one based on the excursion theory, among which we mainly take the latter course.

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2 The main theorem

In the remainder of this paper, almost all quantities depend on the parameter $\alpha > -1$ and $\bar{c} > 0$ without any mentioning. Let V be a function on the real line which is positive on $(0, \infty)$ and negative on $(-\infty, 0)$.

$$V(x) = x^\alpha \text{ for } x > 0; V(0) = 0; V(x) = -|x|^\alpha/\bar{c} \text{ for } x < 0. \tag{2.7}$$

We define a diffusion $(X(t), Y(t))$ on $\mathbb{R}^2$ in a similar way and denote it by the same symbol:

$$Y(t) = y + B(t), \qquad X(t) = x + \int_0^t V(Y(s))ds. \tag{2.8}$$

We denote by $E_{(x,y)}$ and $P_{(x,y)}$ the expectation and the probability measure for the diffusion started at $(x, y) \in \mathbb{R}^2$. Let T be the first hitting time to the positive y-axis as usual. Let T_0^Y be the first hitting time to x-axis:

$$T_0^Y = \inf\{t \geq 0; Y(t) = 0\} \tag{2.9}$$

and for $\kappa, \lambda, \mu \geq 0$, $x \leq 0$, $y \in \mathbb{R}$ define $u_0(x, y) \equiv u_0(x, y; \mu)$ by

$$u_0(x, y) = E_{(x,y)}[\exp(-\mu T)] \tag{2.10}$$

and more generally $u(x,y) \equiv u(x,y;\kappa,\lambda,\mu)$ by

$$u(x,y) = E_{(x,y)}\left[\exp\left\{-\mu T - \lambda X(T_0^Y \circ \theta_T) - \kappa(T_0^Y \circ \theta_T - T)\right\}\right] \tag{2.11}$$

$$\equiv E_{(x,y)}[\exp\{-\mu T\}\, F(\lambda V + \kappa; Y(T))] \tag{2.12}$$

here θ_t is the usual shift operator on the path space and the function $F(\lambda V + \kappa; z)$ is the unique bounded solution of $\frac{1}{2}F''(z) = (\lambda V(z) + \kappa)F(z)$ on $(0,\infty)$ with $F(0) = 1$. It is clear that $0 \le u(x,y) \le 1$, $u(0,0) = 1$ and $u_0(0,y) = 1$ for $y > 0$.

Theorem 1 *Define positive numbers $0 < \nu < 1$, $0 < \rho < 1$ by $\nu = 1/(\alpha+2)$ and $\bar{c}^\nu \sin \pi\nu(1-\rho) = \sin \pi\nu\rho$. Then for $\kappa,\lambda,\mu \ge 0$ there exists a positive constant $C(\kappa,\lambda,\mu)$ such that it holds*

$$1 - u(x,0;\sigma\kappa,\sigma^{1/2\nu}\lambda,\sigma\mu) \sim |x|^{\nu\rho} C(\kappa,\lambda,\mu)\sigma^{\rho/2} \tag{2.13}$$

as $|x|^{\nu\rho}\sigma^{\rho/2}$ tends to 0.

Corollary 1 *It holds that*

$$1 - u_0(x,0;\mu) \sim C(0,0,1)|x|^{\nu\rho}\mu^{\rho/2} \tag{2.14}$$

as $|x|^{\nu\rho}\mu^{\rho/2}$ tends to 0, in other words,

$$P\left[\int_0^s V(B(u))du < |x| \text{for all } 0 \le s \le t\right] \sim \frac{C(0,0,1)}{\Gamma(1-\rho/2)}|x|^{\nu\rho}t^{-\rho/2} \tag{2.15}$$

as $|x|^{\nu\rho}t^{-\rho/2}$ tends to 0.

We have, more generally, the following theorem.

Theorem 2 *There exist a positive constant $\tilde{C}(x,y)$ such that, for $\kappa,\lambda,\mu \ge 0$, $x \le 0$ and $y \in \mathbb{R}$, it holds that*

$$1 - u(\bar{\sigma}x, \bar{\sigma}^{1/\nu}y; \sigma\kappa, \sigma^{1/2\nu}\lambda, \sigma\mu) \sim \tilde{C}(x,y)\bar{\sigma}^{\nu\rho}C(\kappa,\lambda,\mu)\sigma^{\rho/2} \tag{2.16}$$

for positive $\sigma,\bar{\sigma}$ such that $\bar{\sigma}^{\nu\rho}\sigma^{\rho/2}$ tends to 0, where $C(\kappa,\lambda,\mu)$ is the same as in Theorem 1 and $\tilde{C}(x,y)$ is given by

$$\tilde{C}(x,y) = \frac{|x|^{1-\nu+\nu\rho}\exp\left\{-2\nu^2(y^+)^{1/\nu}/|x|\right\}}{\Gamma(\nu)} \int_0^\infty dt e^{-t}\left(|x|t + \frac{2\nu^2}{\bar{c}}|y^-|^{1/\nu}\right)^{\nu\rho}\left(|x|t + 2\nu^2(y^+)^{1/\nu}\right)^{-1+\nu-\nu\rho}.$$

Remark 1. The function u has the following scaling property: for any $c > 0$

$$u(x,y;\kappa,\lambda,\mu) \equiv u(c^{1/\nu}x, cy; c^{-2}\kappa, c^{-1/\nu}\lambda, c^{-2}\mu)$$

$$\equiv E_{(c^{1/\nu}x,cy)}\left[\exp\left\{-c^{-2}\mu T\right\} F(\lambda V + \kappa; c^{-1}Y(T))\right]$$

and the theorems are stated accordingly.

Remark 2. The distribution of $Y(T)$ under $P_{(0,y)}$ is known explicitly by Rogers-Williams[10], see also McGill[7]: For $y < 0$,

$$P_{(0,y)}[Y(T) \in d\eta] = \frac{\sin \pi\nu\rho}{\pi\nu\bar{c}^{\nu\rho}} |y|^{\rho} \eta^{1/\nu - 1 - \rho} \frac{d\eta}{\bar{c}^{-1}|y|^{1/\nu} + \eta^{1/\nu}}, \text{on } \{\eta > 0\}. \tag{2.17}$$

Their methods do not seem to cover, however, the cases involving the stopping time T.

Remark 3. We denote by $\tau(t)$ the inverse of the local time of Y at 0. It is well known that $\int_0^{\tau(t)} V(B_u)du$ is a stable process with index ν and it holds

$$P\left[\int_0^{\tau(s)} V(B_u)du < |x| \text{ for all } s \leq t\right] \sim \text{const}\, |x|^{\nu\rho} t^{-\rho} \tag{2.18}$$

as $|x|^{\nu\rho}t^{-\rho}$ tends to 0. See e.g. Bertoin[2]. This result has the same order as our Corollary 1 in the space variable $|x|$, but differs in the time variable t.

Remark 4. Note also that ρ is equal to the probability $P[\int_0^{\tau(t)} V(B_u)du > 0]$ independent of t, which can be proved using the result by Zolotarev[13].

3 Proof of Theorem 1

We denote by $L(t)$ the local time at 0 of $Y(T)$: $L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y(u))du$ and by τ_t or $\tau(t)$ the right continuous inverse of L_t: $\tau_t \equiv \tau(t) = \inf\{u > 0;\ L_u > t\}$. Let n^+ and n^- be the Itô measure for positive and negative excursions respectively, and set $n = n^+ + n^-$.

We denote a general excursion by $\varepsilon = (\varepsilon_t; t \geq 0)$, its lifetime by $\zeta = \zeta(\varepsilon)$ and define a random time for $x \leq 0$,

$$T(\varepsilon, x) = \inf\left\{0 \leq t \leq \zeta; x + \int_0^t V(\varepsilon_s)ds \geq 0\right\}. \tag{3.19}$$

We set $T(\varepsilon, x) = \zeta$ if there is no such t. It follows, through calculations of the Lévy measure of $X(\tau_t)$, that

$$\begin{aligned} n^+\left[1 - \exp\left\{-\lambda \int_0^{\zeta} V(\varepsilon_s)ds\right\}\right] &= \frac{\nu^{2\nu-1}2^{\nu}\Gamma(1-\nu)}{\Gamma(\nu)}\lambda^{\nu}, \\ n^-\left[1 - \exp\left\{\lambda \int_0^{\zeta} V(\varepsilon_s)ds\right\}\right] &= \frac{\nu^{2\nu-1}2^{\nu}\Gamma(1-\nu)}{\Gamma(\nu)}(\lambda/\bar{c})^{\nu} \end{aligned}$$

for positive λ and that

$$n^+\left[\int_0^{\zeta} V(\varepsilon_s)ds > \xi\right] = \frac{\nu^{2\nu-1}2^{\nu}}{\Gamma(\nu)}\xi^{-\nu}, \tag{3.20}$$

$$n^-\left[\int_0^{\zeta} V(\varepsilon_s)ds < -\xi\right] = \frac{\nu^{2\nu-1}2^{\nu}}{\Gamma(\nu)}(\bar{c}\xi)^{-\nu} \tag{3.21}$$

for positive ξ.

We have an integral equation for $u(x, 0)$.

Lemma 1 *We extend u for positive x by $u(x,0)=1$. Then it holds for $x<0$*

$$
\begin{aligned}
& n\left[u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)-u(x,0)\right] \\
(3.22) \quad &= n\left[u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)\left\{1-e^{-\mu T(\varepsilon,x)}F(\lambda V+\kappa;\varepsilon(T(\varepsilon,x)))\right\}\right].
\end{aligned}
$$

Proof. Let $F(z)=F(\lambda V+\kappa;z)$. Define $a\vee b=\max(a,b)$, $a\wedge b=\min(a,b)$ and

$$M(t)=u(X(\tau_t\wedge T),Y(\tau_t\wedge T))e^{-\mu(\tau(t)\wedge T)}$$

then $u(x,0)=E_{(x,0)}[M(t)]$ holds for any $t\geq 0$ and $x\leq 0$.

If $T\leq\tau_{t-}$ then $M(t)-M(t-)=0$.

If $\tau_{t-}<T\leq\tau_t$ then τ_t is the first hitting time of 0 by Y after T, i.e., $\tau_t=T_0^Y\circ\theta_T$. In this case,

$$M(t)-M(t-)=e^{-\mu T}F(Y(T))-e^{-\mu\tau(t-)}u(X(\tau_{t-}),0)$$

and $T-\tau_{t-}=T(\varepsilon,X(\tau_{t-}))$, here ε denotes the excursion started at τ_{t-} and ended at τ_t: $\varepsilon_s=Y(s+\tau_{t-})$, $s<\tau_t-\tau_{t-}$.

Finally if $\tau_t<T$ then

$$M(t)-M(t-)=e^{-\mu\tau(t)}u((X(\tau_t),0)-e^{-\mu\tau(t-)}u((X(\tau_{t-}),0)$$

and $\tau_t-\tau_{t-}=\zeta(\varepsilon)$. The master formula of excursion theory(cf. Revuz-Yor[11] page 439) tells us

$$
\begin{aligned}
& E_{(x,0)}\left[M(s)-M(0)\right]=\int_0^s dtE_{(x,0)}\Bigg[\\
& e^{-\mu\tau(t-)}n^+\left[e^{-\mu T(\varepsilon,X(\tau(t-)))}F(\varepsilon(T(\varepsilon,X(\tau_{t-}))))-u((X(\tau_{t-}),0);T(\varepsilon,X(\tau_{t-}))<\zeta\right] \\
& +e^{-\mu\tau(t-)}n\left[e^{-\mu\zeta}u\left(X(\tau_{t-})+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)-u(X(\tau_{t-}),0);T(\varepsilon,X(\tau_{t-}))=\zeta\right].
\end{aligned}
$$

Recalling $u(x,0)=E_{(x,0)}\left[M(s)\right]$, we know that the integrand of the right hand side is identically null.

Since $X(\tau_t)$ is a ν-stable Lévy process, the paths are right continuous and the transition density decays as t goes to 0 uniformly outside any neighbohood of $X(0)$. The proof is hence complete if we show

$$
\begin{aligned}
n^+ \quad & \left[e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))-u(x,0);T(\varepsilon,x)<\zeta\right] \\
& +n\left[e^{-\mu\zeta}u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)-u(x,0);T(\varepsilon,x)=\zeta\right],
\end{aligned}
$$

which coincides with

$$n\left[u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))-u(x,0)\right],$$

is continuous on $\{x<0\}$ and its absolute value is dominated by an integrable function plus a constant. We need the following lemma.

Lemma 2 *The function $u(x,0)$ is infinitely differentiable on $\{x < 0\}$ and $\dfrac{\partial u}{\partial x}$ is positive.*

Moreover, if $\alpha \geq 0$, $\nu \leq 1/2$ then $\dfrac{\partial^2 u}{\partial x^2}$ is positive, in particular $\dfrac{\partial u}{\partial x} = o(1/|x|)$ as $x \to -\infty$. If $-1 < \alpha < 0$, $1/2 < \nu < 1$ then $\dfrac{\partial u}{\partial x} = O(1/|x|)$ as $x \to -\infty$.

Remark. It can be proved for any $m > 0$ and $n > 0$, $\dfrac{\partial^n u}{\partial x^n} = O(|x|^{-m})$. However the statemant above is sufficient for our purpose.

Proof. Let $F(z) = F(\lambda V + \kappa; z)$. By the scaling property it holds that

$$u(x,0;\kappa,\lambda,\mu) = E_{(-1,0)}[e^{-|x|^{2\nu}\mu T} F(|x|^{\nu} Y(T))].$$

Since $F(z)$ decays exponentially as $z \to \infty$, the differentiation inside the expectation can be justified. Hence

$$\begin{aligned}\frac{\partial u}{\partial x}(x,0;\kappa,\lambda,\mu) &= E_{(-1,0)}\left[2\nu|x|^{2\nu-1}\mu T e^{-|x|^{2\nu}\mu T} F(|x|^{\nu}Y(T)) \right. \\ &\quad \left. + e^{-|x|^{2\nu}\mu T}\nu|x|^{\nu-1} Y(T)(-F'(|x|^{\nu}Y(T)))\right].\end{aligned}$$

Here $-F'(z)$ is a positive decreasing function. The integrand is obviously positive and if $2\nu - 1 \leq 0$ it is strictly decreasing in $|x|$. If $\nu > 1/2$, we use again the scaling property:

$$\frac{\partial u}{\partial x}(x,0;\kappa,\lambda,\mu) = \frac{1}{|x|} E_{(x,0)}\left[2\nu\mu T e^{-\mu T} F(Y(T)) + e^{-\mu T}\nu Y(T)(-F'(Y(T)))\right].$$

The integrand is a bounded function of two variables T and $Y(T)$. □

End of the proof of Lemma 1. The difference between

$$n\left[u\left\{ x + \int_0^{\zeta} V(\varepsilon_s)ds, 0\right\} e^{-\mu T(\varepsilon,x)} F(\varepsilon(T(\varepsilon,x))) - u(x,0)\right]$$

and $n\left[u(x + \int_0^{\zeta} V(\varepsilon_s)ds, 0) - u(x,0)\right]$ is bounded since it is dominated by

$$\begin{aligned}& n\left[1 - e^{-\mu T(\varepsilon,x)} F(\varepsilon(T(\varepsilon,x)))\right] \\ \equiv\ & n\left[1 - \exp\left\{ -\mu T(\varepsilon,x) - \kappa(\zeta - T(\varepsilon,x)) - \lambda \int_{T(\varepsilon,x)}^{\zeta} V(\varepsilon_s)ds\right\}\right],\end{aligned}$$

which is also bounded by

$$\begin{aligned}& n\left[1 - \exp\left\{ -(\mu \vee \kappa)\zeta - \lambda \int_0^{\zeta} V(\varepsilon_s) \vee 0 ds\right\}\right] \\ <\ & n\left[1 - \exp\left\{ -(\mu \vee \kappa)\zeta\right\}\right] + n\left[1 - \exp\left\{ -\lambda \int_0^{\zeta} V(\varepsilon_s) \vee 0 ds\right\}\right] < \infty.\end{aligned}$$

We divide $n\left[u(x + \int_0^{\zeta} V(\varepsilon_s)ds, 0) - u(x,0)\right]$ into two parts.

$n\left[|u(x+\int_0^\zeta V(\varepsilon_s)ds,0)-u(x,0)|;|\int_0^\zeta V(\varepsilon_s)ds|>1\right]$ is bounded because $0\le u\le 1$ and $n\left[|\int_0^\zeta V(\varepsilon_s)ds|>1\right]<\infty$ by (3.20) and (3.21). Integrating by parts,

$$\begin{aligned} & n\left[\left|u\left\{x+\int_0^\zeta V(\varepsilon_s)ds,0\right\}-u(x,0)\right|;\left|\int_0^\zeta V(\varepsilon_s)ds\right|<1\right] \\ = & \int_0^1 d\xi\frac{\partial u}{\partial x}(x+\xi,0)n^+\left[\xi<\int_0^\zeta V(\varepsilon_s)ds<1\right] \\ & -\int_{-1}^0 d\xi\frac{\partial u}{\partial x}(x+\xi,0)n^-\left[-1<\int_0^\zeta V(\varepsilon_s)ds<\xi\right], \end{aligned}$$

which is integrable, since it is a convolution of two integrable functions $\frac{\partial u}{\partial x}$ and $n^\pm[\xi<|\int_0^\zeta V(\varepsilon_s)ds|]$.

The continuity also follows using the above arguments since $T(\varepsilon,x)$ and $\varepsilon(T(\varepsilon,x))$ are continuous in x. □

Putting the explicit value of $n^\pm[\xi<|\int_0^\zeta V(\varepsilon_s)ds|]$ into the left side of Lemma 1, we have

$$\begin{aligned} & n\left[u(x+\int_0^\zeta V(\varepsilon_s)ds,0)-u(x,0)\right] \\ = & \frac{\nu^{2\nu-1}2^\nu}{\Gamma(\nu)}|x|^{-\nu}\left(\{1-u(x,0)\}-\nu\int_0^1|1-t|^{-\nu-1}(\{1-u(xt,0)\}-\{1-u(x,0)\})dt\right. \\ & \left.-\nu\int_1^\infty\frac{|1-t|^{-\nu-1}}{\bar{c}^\nu}(\{1-u(xt,0)\}-\{1-u(x,0)\})dt\right) \end{aligned}$$

The integral transform on this right side can be inverted.

Lemma 3 *For $v\in C^1((-\infty,0))$ such that $\frac{dv}{dx}$ is integrable, define $Lv(x)\in C((-\infty,0))$ by*

$$Lv(x)=\nu\int_0^1|1-t|^{-\nu-1}(v(xt)-v(x))dt+\nu\int_1^\infty\frac{|1-t|^{-\nu-1}}{\bar{c}^\nu}(v(xt)-v(x))dt.$$

If $v(x)-Lv(x)=f(x)$ then it holds

$$v(x)=\int_{-\infty}^0\frac{dt}{|t|}f(t)G(-\frac{|t|}{|x|}) \tag{3.23}$$

with a function $G(b)$ defined by

$$\begin{aligned} G(b) &= \tilde{G}(-\log(-b)),\quad b<0, \\ \tilde{G}(\xi) &= \lim_{A\to+\infty}\int_{-A}^A\frac{e^{-i\xi x}}{2\pi r(ix)}dx,\quad \xi\in\mathbb{R} \\ r(z) &= \frac{1}{\Gamma(\nu)\sin\pi\nu\rho}\Gamma(1-z)\Gamma(\nu+z)\sin\pi(\nu\rho+z),\quad z\in\mathbb{C} \end{aligned}$$

and with $\rho\in(0,1)$ defined by $\bar{c}^\nu=\frac{\sin\pi\nu\rho}{\sin\pi\nu(1-\rho)}$.

Moreover, if $\int_{-\infty}^{0} |x|^{-1-\nu\rho} |f(x)| dx < \infty$ *then*

$$\lim_{x \to -0} \frac{v(x)}{|x|^{\nu\rho}} = \frac{\Gamma(\nu) \sin \pi\nu\rho}{\pi\nu\rho\Gamma(\nu\rho)\Gamma(\nu - \nu\rho)} \int_{-\infty}^{0} |x|^{-1-\nu\rho} f(x) dx. \tag{3.24}$$

Remark. The Markov process associated to L turns into a Lévy process by taking the logarithm. This property enables us to calculate $\tilde{G}(\xi)$ and $r(z)$ explicitly. We prove this lemma at the end of this section.

Proof of Theorem 1. We set

$$f(x) = \frac{\Gamma(\nu)|x|^{\nu}}{\nu^{2\nu-1}2^{\nu}} n \left[u\left(x + \int_0^{\zeta} V(\varepsilon_s) ds, 0 \right) \left(1 - e^{-\mu T(\varepsilon,x)} F(\varepsilon(T(\varepsilon, x))) \right) \right]$$

for $x < 0$. It is obvious that $f(x)$ is positive everywhere and continuous. As we saw in the proof of Lemma 2, $n[1 - e^{-\mu T(\varepsilon,x)} F(\varepsilon(T(\varepsilon, x)))]$ is bounded, hence $f(x) = O(|x|^{\nu})$ as x tends to 0.

We have also

$$f(x) = \frac{\Gamma(\nu)|x|^{\nu}}{\nu^{2\nu-1}2^{\nu}} n \left[u\left(x + \int_0^{\zeta} V(\varepsilon_s) ds, 0 \right) - u(x, 0) \right].$$

By integration by parts, $n\left[|u(x + \int_0^{\zeta} V(\varepsilon_s) ds, 0) - u(x, 0)|; \frac{|x|}{2} > |\int_0^{\zeta} V(\varepsilon_s) ds| \right]$ is dominated by const $\int_{-|x|/2}^{|x|/2} d\xi |\xi|^{-\nu} \frac{\partial u}{\partial x}(x+\xi, 0)$. It is shown in Lemma 2 that $\frac{\partial u}{\partial x} = O(1/|x|)$ as $x \to -\infty$, which implies

$$n\left[\left| u\left(x + \int_0^{\zeta} V(\varepsilon_s) ds, 0 \right) - u(x, 0) \right|; \frac{|x|}{2} > \left| \int_0^{\zeta} V(\varepsilon_s) ds \right| \right] = O(|x|^{-\nu}).$$

Finally, $n\left[|u(x + \int_0^{\zeta} V(\varepsilon_s) ds, 0) - u(x, 0)|; \frac{|x|}{2} < |\int_0^{\zeta} V(\varepsilon_s) ds| \right]$ is easily dominated by $n\left[\frac{|x|}{2} < |\int_0^{\zeta} V(\varepsilon_s) ds| \right] = O(|x|^{-\nu})$ since $0 \le u(x, 0) \le 1$ for every x.

Therefore we have shown that $f(x) = O(1)$ as $x \to -\infty$, hence the integrability of $f(x)$ with respect to $|x|^{-1-\nu\rho} dx$ and the existence of the limit value for $(1 - u(x, 0))/|x|^{\nu\rho}$ as $x \to -0$.

The statement of the theorem follows from the scaling property of u: for any $c > 0$,

$$u(x, y; c^2\kappa, c^{1/\nu}\lambda, c^2\mu) = u(c^{1/\nu}x, cy; \kappa, \lambda, \mu).$$

Setting $y = 0$, $c = \sqrt{\sigma}$, $u(x, 0; \sigma\kappa, \sigma^{1/2\nu}\lambda, \sigma\mu)$ is equal to $u(\sigma^{1/2\nu}x, 0; \kappa, \lambda, \mu)$, which satisfies $1 - u(\sigma^{1/2\nu}x, 0; \kappa, \lambda, \mu) \sim$ const $|\sigma^{1/2\nu}x|^{\nu\rho}$ as $\sigma^{1/2\nu}|x|$ tends to 0. □

Proof of Lemma 3. Define the functions $\tilde{v}$ and $\tilde{f}$ on $\mathbb{R}$ by $\tilde{v}(x) = v(-\exp(-x))$ and $\tilde{f}(x) = f(-\exp(-x))$. Define the integral operators $\tilde{L}$ and $\tilde{L}'$ by

$$\begin{aligned} \tilde{L}g(x) &:= \nu \int_0^{\infty} |1 - e^{-X}|^{-\nu-1} e^{-X} \{g(x + X) - g(x)\} dX \\ &\quad + \frac{\nu}{\tilde{c}^{\nu}} \int_{-\infty}^{0} |1 - e^{-X}|^{-\nu-1} e^{-X} \{g(x + X) - g(x)\} dX \end{aligned} \tag{3.25}$$

$$\begin{aligned} \tilde{L}'g(x) &:= \frac{\nu}{\tilde{c}^{\nu}} \int_0^{\infty} |1 - e^{X}|^{-\nu-1} e^{X} \{g(x + X) - g(x)\} dX \\ &\quad + \nu \int_{-\infty}^{0} |1 - e^{X}|^{-\nu-1} e^{X} \{g(x + X) - g(x)\} dX \end{aligned} \tag{3.26}$$

for $g \in C^1(\mathbb{R} \to \mathbb{C})$ with integrable $\frac{dg}{dx}$. It is obvious that $\tilde{L}\tilde{v}(x) \equiv Lv(-e^{-x})$.

To prove the lemma, it is sufficient to show that

$$\tilde{v}(x) = \int_{-\infty}^{\infty} \tilde{f}(y)\tilde{G}(y-x)dy. \tag{3.27}$$

We can show, by an standard argument, that $\tilde{f}$, $\tilde{v}$ and $\tilde{L}\tilde{v}$ belong to $\mathcal{S}'(\mathbb{R})$ and for any $\phi \in \mathcal{S}(\mathbb{R})$ it holds

$$(\tilde{v} - \tilde{L}\tilde{v})(\mathcal{F}\phi) = \tilde{v}((1 - \tilde{L}')\mathcal{F}\phi) \tag{3.28}$$

It is elementary but tedious to verify that

$$\begin{aligned} 1 - r(i\xi) &= \frac{\nu}{\tilde{c}^\nu} \int_0^\infty |1 - e^X|^{-\nu-1} e^X \left\{ e^{-i\xi X} - 1 \right\} dX \\ &\quad + \nu \int_{-\infty}^0 |1 - e^X|^{-\nu-1} e^X \left\{ e^{-i\xi X} - 1 \right\} dX, \end{aligned} \tag{3.29}$$

$$(1 - \tilde{L}')\mathcal{F}\phi(x) = \mathcal{F}[\phi(\xi)r(i\xi)](x), \quad \phi \in \mathcal{S} \tag{3.30}$$

and that the function $\frac{1}{r(i\xi)}$ on $\mathbb{R}$ is infinitely diffenrentiable and

$$r(-i\xi) = \overline{r(i\xi)}, \qquad \frac{1}{r(i\xi)} = \frac{\text{const}}{x^\nu} + O(x^{-1-\nu}) \text{ as } x \to \infty. \tag{3.31}$$

We next show for any $\chi \in \mathcal{S}(\mathbb{R})$

$$\mathcal{F}\left[\frac{\chi(x)}{r(ix)} \right] = \mathcal{F}\chi * \tilde{G}. \tag{3.32}$$

We start with

$$\mathcal{F}\left[\chi(x) \frac{1_{[-A,A]}(x)}{r(ix)} \right] = \frac{1}{\sqrt{2\pi}} \mathcal{F}\chi * \mathcal{F}\left[\frac{1_{[-A,A]}(x)}{r(ix)} \right], \quad A > 0. \tag{3.33}$$

It is clear that the left side of (3.33) converges to the left side of (3.32) as A tends to ∞.

The difference between $\tilde{G}(y)$ and $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\frac{1_{[-A,A]}(x)}{r(ix)}](y)$ is dominated by $c_0 + c_1|y|^{\nu-1}$. To see this, it is sufficient to estimate $\int_A^\infty \frac{\exp(-iyx)}{r(ix)}dx$ for positive A.

By (3.31), $|\int_A^\infty \frac{\exp(-iyx)}{r(ix)}dx|$ is less than $c_2 \int_A^\infty \frac{|\exp(-iyx)|}{x^{1+\nu}}dx + c_3|\int_A^\infty \frac{\exp(-iyx)}{x^\nu}dx|$ with some positive constants c_2, c_3. $\int_A^\infty \frac{\exp(-iyx)}{x^\nu}dx = |y|^{\nu-1}\int_{|y|A}^\infty \frac{\exp(-i(\operatorname{sgn} y)x)}{x^\nu}dx$ is dominated by $M|y|^{\nu-1}$ with

$$M = \sup_{\xi > 0} \left| \int_\xi^\infty \frac{\exp(-ix)}{x^\nu} dx \right|.$$

Since $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\frac{1_{[-A,A]}(x)}{r(ix)}](y)$ converges to $\tilde{G}(y)$ for fixed $y > 0$, the right side of (3.33) also converges to the right side of (3.32) as A tends to ∞. Hence we have established the equation (3.32).

We now show (3.27). Set $\chi(x) = \phi(x)r(ix) \in \mathcal{S}(\mathbb{R})$. $\tilde{v}(\mathcal{F}\chi)$ is equal to $\tilde{v}((1 - \tilde{L}')\mathcal{F}\phi)$ by (3.30), which is further equal to the left side of (3.28). Since $\tilde{v} - \tilde{L}\tilde{v} = \tilde{f}$, we have

$$\tilde{v}(\mathcal{F}\chi) = \tilde{f}(\mathcal{F}\phi).$$

Here $\mathcal{F}\phi$ is equal to the left side of (3.32). Hence we have

$$\begin{aligned}\tilde{v}(\mathcal{F}\chi) &= \tilde{f}(\mathcal{F}\chi * \tilde{G}) \\ &= \int_{-\infty}^{\infty} dy \tilde{f}(y) \int_{-\infty}^{\infty} d\xi \mathcal{F}\chi(\xi)\tilde{G}(y-\xi) \\ &= \int_{-\infty}^{\infty} d\xi \mathcal{F}\chi(\xi) \int_{-\infty}^{\infty} dy \tilde{f}(y)\tilde{G}(y-\xi).\end{aligned}$$

The both sides of (3.27) are continuous and bounded, and coincide in $\mathcal{S}'(\mathbb{R})$, hence they also coincide in $C_b(\mathbb{R})$.

If, moreover, $f(x)$ is integrable with respect to $|x|^{-1-\nu\rho}dx$ on the negative half line, then $\int_{-\infty}^{0} \frac{dt}{|t|} f(t) \frac{G(-|t|/|x|)}{|x|^{\nu\rho}}$ converges to the right side of (3.24) as x tends to -0 because of the following asymptotics:

$$\begin{aligned}G(b) &\sim \frac{\Gamma(\nu)\sin\pi\nu\rho}{\pi\nu\rho\Gamma(\nu\rho)\Gamma(\nu-\nu\rho)}|b|^{-\nu\rho} \text{ as } b \to -\infty, \\ G(b) &\sim \frac{\Gamma(\nu)\sin\pi\nu\rho}{\pi\nu(1-\rho)\Gamma(\nu\rho)\Gamma(\nu-\nu\rho)}|b|^{1-\nu\rho} \text{ as } b \to -0, \\ G(b) &\sim O(|b+1|^{\nu-1}) \text{ as } b \to -1.\end{aligned}$$

□

4 Proof of Theorem 2.

By the scaling property of Brownian motion, we have for positive c

$$u^c(x,y) := u(c^{1/\nu}x, cy; \kappa, \lambda, \mu) = u(x, y; c^2\kappa, c^{1/\nu}\lambda, c^2\mu).$$

In the previous section it is established with some constant $C > 0$

$$1 - u^c(x,0) \sim Cc^{\rho}|x|^{\nu\rho} \text{ as } c \to +0 \tag{4.34}$$

while in this section we prove

$$1 - u^c(x,y) \sim Cc^{\rho}\tilde{C}(x,y) \text{ as } c \to +0 \tag{4.35}$$

for fixed $x \leq 0$, $y \in \mathbb{R}$.

4.1 The case of the starting point (x,y) in the third quadrant.

Let $Y_0 = y < 0$. In this case Y_t is negative until the hitting time T_0^Y. Applying the optional sampling theorem to the martingale $F(\lambda V_-; |Y_t|)\exp\left\{\lambda \int_0^t V(Y(s))ds\right\}$, $\lambda > 0$, we obtain

$$E_{(0,y)}[\exp\{\lambda X(T_0^Y)\}] = F(\lambda V_-; |y|),$$

where $F(\lambda V_-; z)$ is the unique bounded solution of $\frac{1}{2}F''(z) = \frac{\lambda}{c}z^{\alpha}F(z)$ on $\{z > 0\}$ with $F(0) = 1$.

The function $F(\lambda V_-; z)$ is expressed in terms of modified Bessel functions:

$$F(\lambda V_-; z) = \frac{2\nu^\nu}{\Gamma(\nu)}(2\lambda/\bar{c})^{\nu/2}\sqrt{z}K_\nu(2\nu z^{1/2\nu}(2\lambda/\bar{c})^{1/2}).$$

Here $\nu = 1/(2+\alpha)$ as usual. Using the formula (2.13.42) in Oberhettinger-Badii [9], we can invert the Laplace transform to obtain

$$(4.36)\quad E_{(0,y)}[X(T_0^Y) \in d\xi] = \frac{\nu^{2\nu}2^\nu|y|}{\Gamma(\nu)\bar{c}^\nu|\xi|^{1+\nu}} \exp\left\{-\frac{2\nu^2|y|^{1/\nu}}{\bar{c}|\xi|}\right\} d\xi \text{ on } \{\xi < 0\}.$$

It is obvious that the law of $X(T_0^Y)$ under $P_{(x,y)}$ is identical to that of $x + X(T_0^Y)$ under $P_{(0,y)}$.

By the strong Markov property of $(X(t), Y(t))$,

$$\begin{aligned} 1 - u^c(x,y) &= 1 - E_{(x,y)}[u^c(X(T_0^Y), 0)\exp\{-c^2\mu T_0^Y\}] \\ &= E_{(x,y)}[1 - u^c(X(T_0^Y), 0)] + O(E[1 - \exp\{-c^2\mu T_0^Y\}]). \end{aligned}$$

We see from (4.35) that $\dfrac{1-u^c(x,0)}{c^\rho}$ is dominated by $C'|x|^{\nu\rho}$ with some constant C', and it is well known that $E[1-\exp\{-c^2\mu T_0^Y\}] = 1 - \exp\{-\sqrt{2\mu}c|y|\} = O(c)$.

Combining this with the integrability of $|x + X(T_0^Y)|^{\nu\rho}$ we know

$$\lim_{c\to+0} \frac{1-u^c(x,y)}{Cc^\rho} = E_{(0,y)}\left[\,|x + X(T_0^Y)|^{\nu\rho}\,\right].$$

Putting (4.36) into the right hand side,

$$\begin{aligned} \tilde{C}(x,y) &= E_{(0,y)}\left[\,|x + X(T_0^Y)|^{\nu\rho}\,\right] \\ &= \int_0^\infty d\xi\,(|x| + \xi)^{\nu\rho} \frac{\nu^{2\nu}2^\nu|y|}{\Gamma(\nu)\bar{c}^\nu|\xi|^{1+\nu}} \exp\left\{-\frac{2\nu^2|y|^{1/\nu}}{\bar{c}|\xi|}\right\}. \end{aligned}$$

Replacing $\frac{2\nu^2|y|^{1/\nu}}{\bar{c}|\xi|}$ by t, we obtain

$$\tilde{C}(x,y) = \Gamma(\nu)^{-1}\int_0^\infty dt e^{-t}\left(|x|t + \frac{2\nu^2|y|^{1/\nu}}{\bar{c}}\right)^{\nu\rho} t^{-1+\nu-\nu\rho}. \qquad \square$$

4.2 The case of the starting point (x, y) in the second quadrant.

The function u^c satisfies in the left half plain $\{x < 0\}$ the differential equation

$$\frac{1}{2}\frac{\partial^2 u^c}{\partial y^2} + V(y)\frac{\partial u^c}{\partial y} = c^2\mu u^c$$

with the boundary condition on the positive y-axis:

$$u^c(0,y) = F(c^{1/\nu}\lambda V + c^2\kappa; y) \equiv F(\lambda V + \kappa; cy), \qquad y > 0,\ c > 0.$$

Let $U_c(y) = \int_{-\infty}^{0} dx e^{zx} u^c(x,y)$, $z \geq 0$. It follows from Theorem 1

$$1/z - U_c(0) = Cc^\rho \Gamma(1+\nu\rho) z^{-1-\nu\rho} \text{ as } c \to 0. \tag{4.37}$$

An integration by parts shows

$$\frac{1}{2} U_c''(y) = (zy^\alpha + c^2\mu) U^c(y) - y^\alpha F(\lambda V + \kappa; cy), \qquad y > 0.$$

Let $\phi_c(y)$, $\psi_c(y)$, $F_c(y)$ be the solutions of the equation $\frac{1}{2} f''(y) = (zy^\alpha + c^2\mu) f(y)$ on $(0,\infty)$ determined by the following conditions:

$$\begin{aligned} &\phi_c(0) = 1, \quad \phi'_c(0) = 0 \\ &\psi_c(0) = 0, \quad \psi'_c(0) = 1 \\ &F_c(0) = 1, \quad F_c(y) \text{ is bounded, i.e., } F_c(y) = F(zV + c^2\mu; y). \end{aligned}$$

Let $\phi_0(y)$, $\psi_0(y)$, $F_0(y)$ be the solutions of $\frac{1}{2} f''(y) = zy^\alpha f(y)$ normalized similarly.

We have by the method of variation of constants that

$$\begin{aligned} U_c(y) &= U_c(0) F_c(y) + 2F_c(y) \int_0^y \psi_c(\xi) \xi^\alpha F(\lambda V + \kappa; c\xi) d\xi \\ &\quad + 2\psi_c(y) \int_y^\infty F_c(\xi) \xi^\alpha F(\lambda V + \kappa; c\xi) d\xi. \end{aligned}$$

Since $F(\lambda V + \kappa; c\xi)$ is a convex decreasing function it holds the inequality $0 < 1 - F(\lambda V + \kappa; c\xi) < \left| \frac{dF}{d\xi}(\lambda V + \kappa; 0) \right| c\xi$. Hence we have, for each fixed $y > 0$,

$$\begin{aligned} \int_0^y \psi_c(\xi) \xi^\alpha F(\lambda V + \kappa; c\xi) d\xi &= \int_0^y \psi_c(\xi) \xi^\alpha d\xi + O(c) \\ \int_y^\infty F_c(\xi) \xi^\alpha F(\lambda V + \kappa; c\xi) d\xi &= \int_y^\infty F_c(\xi) \xi^\alpha d\xi + O(c) \end{aligned}$$

as c tends to 0. Noting the differential equation of ψ_c and F_c we obtain

$$\begin{aligned} &2F_c(y) \int_0^y \psi_c(\xi) \xi^\alpha F(\lambda V + \kappa; c\xi) d\xi + 2\psi_c(y) \int_y^\infty F_c(\xi) \xi^\alpha F(\lambda V + \kappa; c\xi) d\xi \\ = \; & \frac{F_c(y)(\psi'_c(y) - 1) - \psi_c(y) F'_c(y)}{z} + O(c) = \frac{1 - F_c(y)}{z} + O(c). \end{aligned}$$

We need to prove that, for each fixed $y > 0$, $F_c(y) - F_0(y) = O(c)$ as $c \to 0$. By the Feynmann-Kac formula, $F_c(y) \equiv F(zV + c^2\mu; y)$ is the same as $E_y[\exp(-\int_0^{T_0} (zV(B_s) + c^2\mu) ds)]$. Here T_0 is the first hitting time to 0 by a standard Brownian motion B_s. Now it is clear that $0 < F_0(y) - F_c(y) = E_y[\exp(-\int_0^{T_0} (zV(B_s) ds)\,(1 - \exp(-c^2\mu T_0))] < E_y[1 - \exp(-c^2\mu T_0)] = 1 - \exp(-cy\sqrt{2\mu}) = O(c)$.

Combining these with (4.37) we have

$$1/z - U_c(y) = Cc^\rho \Gamma(1+\nu\rho) F(zV; y) z^{-1-\nu\rho} + O(c) \text{ as } c \to +0. \tag{4.38}$$

We can conclude by a standard argument that

$$1 - u^c(x,y) \sim Cc^\rho \tilde{C}(x,y) \text{ as } c \to +0$$

with

$$\int_0^\infty e^{-zx} dx \tilde{C}(x,y) = \Gamma(1+\nu\rho) F(zV;y) z^{-1-\nu\rho}.$$

Since $F(zV;y) = \frac{2\nu^\nu}{\Gamma(\nu)}(2z)^{\nu/2}\sqrt{y}K_\nu(2\nu y^{1/2\nu}(2z)^{1/2})$, we can invert the Laplace transform (see Oberhettinger-Badii [9] (13.45)) to obtain

$$\tilde{C}(x,y) = \frac{\Gamma(1+\nu\rho)|x|^{1/2+\nu\rho-\nu/2}}{\Gamma(\nu)2^{(1-\nu)/2}\nu^{1-\nu}y^{(1-\nu)/2\nu}} \exp\left\{-\frac{\nu^2 y^{1/\nu}}{|x|}\right\} W_{\frac{\nu}{2}-\frac{1}{2}-\nu\rho,\frac{\nu}{2}}(2\nu^2 y^{1/\nu}/|x|)$$

where $W_{\kappa,\mu}(z)$ is a Whittaker function defined by(see Abramowitz-Stegun [1] (13.1.33) and (13.2.5))

$$W_{\kappa,\mu}(z) = \frac{z^{1/2+\mu}e^{-z/2}}{\Gamma(1/2+\mu-\kappa)} \int_0^\infty dt e^{-zt} t^{-1/2+\mu-\kappa}(1+t)^{\mu+\kappa-1/2}.$$

Replacing $2\nu^2 y^{1/\nu} t/|x|$ by t, we obtain

$$\tilde{C}(x,y) = \frac{|x|^{1-\nu+2\nu\rho}\exp\left\{-2\nu^2 y^{1/\nu}/|x|\right\}}{\Gamma(\nu)} \int_0^\infty dt e^{-t} t^{\nu\rho} \left(|x|t + 2\nu^2 y^{1/\nu}\right)^{-1+\nu-\nu\rho}. \qquad \square$$

References

[1] M. Abramowitz, I. A. Stegun, *A Handbook of mathematical functions,* Dover, New York, 1964.

[2] J. Bertoin, *Lévy processes*, Cambridge Univ. Press, Cambridge, 1997.

[3] Y. Isozaki, S. Watanabe, An asymptotic formula for the Kolmogorov diffusion and a refinement of Sinai's estimates for the integral of Brownian motion, Proc. Japan Acad., **70A**, (1994), pp. 271–276.

[4] Y. Isozaki, Asymptotic estimates for the distribution of additive functionals of Brownian motion by the Wiener-Hopf factorization method, J. Math. Kyoto Univ., **36**, (1996), pp. 211-227.

[5] A. N. Kolmogorov, Zuffälige Bewegungen, Ann. Math. II., **35** (1934), pp. 116–117.

[6] S. Kotani, S. Watanabe, Krein's spectral theory of strings and generalized diffusion processes, *Functional Analysis in Markov Porcesses, ed. M. Fukushima*, Lecture Notes in Mathematics **923**, pp. 235–259, Springer-Verlag, Berlin, 1982.

[7] P. McGill, Wiener-Hopf factorization of Brownian motion, Prob. Th. Rel. Fields, **83**, (1989), pp. 355–389.

[8] H. P. McKean,Jr., A winding problem for a resonator driven by a white noise, J. Math. Kyoto Univ., **2** (1963), pp. 227–235.

[9] F. Oberhettinger, L. Badii, *Tables of Laplace Transforms,* Springer-Verlag, Berlin, 1973.

[10] L. C. G. Rogers, D. Williams, A differential equation in Wiener-Hopf theory, *Stochastic analysis and applications, ed. A. Truman, D. Williams*, Lecture Notes in Mathematics **1095**, pp. 187–199, Springer-Verlag, Berlin, 1984.

[11] D. Revuz, M. Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, 1991.

[12] Ya. G. Sinai, Distribution of some functionals of the integral of a random walk, Theor. Math. Phys., **90** (1992), pp. 219–241.

[13] V. M. Zolotarev, Mellin-Stieltjes transforms in probability theory, Theor. Prob. Appl., **2** (1957), pp. 433–460.

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