Séminaire de probabilités (Strasbourg)

Samia Beghdadi-Sakrani
Michel Émery

On certain probabilities equivalent to coin-tossing, d'après Schachermayer

Séminaire de probabilités (Strasbourg), tome 33 (1999), p. 240-256
<http://www.numdam.org/item?id=SPS_1999__33__240_0>

ON CERTAIN PROBABILITIES EQUIVALENT TO COIN-TOSSING, D'APRÈS SCHACHERMAYER

S. Beghdadi-Sakrani and M. Émery

L. Dubins, J. Feldman, M. Smorodinsky and B. Tsirelson have constructed in [4] and [5] a probability $\mathbb{Q}$ on Wiener space, equivalent to Wiener measure, but such that the canonical filtration on Wiener space is not generated by any $\mathbb{Q}$-Brownian motion whatsoever! Dreadfully complicated, their construction is almost as incredible as the existence result itself, and these articles are far from simple. In this volume, W. Schachermayer [14] proposes a shorter, more accessible redaction of their construction. We have rewritten it once more, for two reasons: First, the language used by these five authors is closer to that of dynamical systems than to a probabilist's mother tongue: where they use a dependence of the form $X = f(Y)$, we work with X measurable with respect to a σ-field; having both versions enables the reader to choose her favorite setup. Second, a straightforward adaptation of Tsirelson's ideas in [16] gives a stronger result (non-cosiness instead of non-standardness).

We warmly thank Walter Schachermayer for many fruitful conversations and helpful remarks, and for spotting an error (of M. É.) in an earlier version. We also thank Boris Tsirelson for his comments (though we disagree with him on one point: he modestly insists that we have given too much credit to him and to his co-authors) and Marc Yor for his many observations and questions.

1. — Notations

Probability spaces will always be complete: if $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, $\mathcal{A}$ contains all negligible events. Similarly, we consider only sub-σ-fields of $\mathcal{A}$ containing all negligible events of $\mathcal{A}$. For instance, the product $(\Omega^1, \mathcal{A}^1, \mathbb{P}^1)\otimes(\Omega^2, \mathcal{A}^2, \mathbb{P}^2)$ of two probability spaces is endowed with its completed product σ-field $\mathcal{A}^1\otimes\mathcal{A}^2$, containing all null events for $\mathbb{P}^1\times\mathbb{P}^2$. $\mathrm{L}^0(\Omega, \mathcal{A}, \mathbb{P})$ (or shortly $\mathrm{L}^0(\Omega, \mathbb{P})$, or $\mathrm{L}^0(\mathcal{A})$ etc. if there is no ambiguity) denotes the space of equivalence classes of all a.s. finite r.v.'s; so $\mathrm{L}^0(\Omega, \mathcal{A}, \mathbb{P}) = \mathrm{L}^0(\Omega, \mathcal{A}, \mathbb{Q})$ if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probabilities.

When there is an ambiguity on the probability $\mathbb{P}$, expectations and conditional expectations will be written $\mathbb{P}[X]$ and $\mathbb{P}[X|\mathcal{B}]$ instead of the customary $\mathbb{E}[X]$ and $\mathbb{E}[X|\mathcal{B}]$.

An *embedding* of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into another one $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$ is a mapping Ψ from $\mathrm{L}^0(\Omega, \mathcal{A}, \mathbb{P})$ to $\mathrm{L}^0(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$ that commutes with Borel operations on finitely many r.v.'s:

$$\Psi\big(f(X_1, \dots, X_n)\big) = f\big(\Psi(X_1), \dots, \Psi(X_n)\big) \quad \text{for every Borel } f$$

and preserves the probability laws:

$$\bar{\mathbb{P}}\big[\Psi(X) \in E\big] = \mathbb{P}\big[X \in E\big] \quad \text{for every Borel } E.$$

If Ψ embeds $(\Omega, \mathcal{A}, \mathbf{P})$ into $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}})$ and if $\mathbf{Q}$ is a probability absolutely continuous with respect to $\mathbf{P}$, then Ψ also embeds $(\Omega, \mathcal{A}, \mathbf{Q})$ into $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{Q}})$, where $\bar{\mathbf{Q}}$ is defined by $d\bar{\mathbf{Q}}/d\bar{\mathbf{P}} = \Psi(d\mathbf{Q}/d\mathbf{P})$. If $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, so are also $\bar{\mathbf{P}}$ and $\bar{\mathbf{Q}}$.

An embedding is always injective and transfers not only random variables, but also sub-σ-fields, filtrations, processes, etc. It is called an *isomorphism* if it is surjective: it then has an inverse. An embedding Ψ of $(\Omega, \mathcal{A}, \mathbf{P})$ into $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}})$ is always an isomorphism between $(\Omega, \mathcal{A}, \mathbf{P})$ and $(\bar\Omega, \Psi(\mathcal{A}), \bar{\mathbf{P}})$.

Processes and filtrations will be parametrized by time, represented by a subset $\mathbf{T}$ of $\mathbf{R}$. We shall use three special cases only: $\mathbf{T} = \mathbf{R}_+$, the usual time-axis for continuous-time processes, $\mathbf{T} = -\mathbf{N} = \{\dots, -2, -1, 0\}$ and $\mathbf{T}$ finite (and non-empty). In the first case ($\mathbf{T} = [0, \infty)$), filtrations are always right-continuous; for instance, the product of two filtrations $\mathcal{F}^1$ and $\mathcal{F}^2$ is the smallest right-continuous filtration $\mathcal{G}$ on $(\Omega^1, \mathcal{A}^1, \mathbf{P}^1) \otimes (\Omega^2, \mathcal{A}^2, \mathbf{P}^2)$ such that $\mathcal{G}_t \supset \mathcal{F}_t^1 \otimes \mathcal{F}_t^2$. If $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is a filtration, we set

$$\mathcal{F}_{-\infty} = \bigcap_{t\in\mathbf{T}} \mathcal{F}_t \quad \text{and} \quad \mathcal{F}_{\infty} = \bigvee_{t\in\mathbf{T}} \mathcal{F}_t ;$$

so $\mathcal{F}_{-\infty} = \mathcal{F}_0$ when $\mathbf{T} = \mathbf{R}_+$ and $\mathcal{F}_\infty = \mathcal{F}_0$ when $\mathbf{T} = -\mathbf{N}$. If $(\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$ and $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}}, \bar{\mathcal{F}})$ are two filtered probability spaces, the filtrations $\mathcal{F}$ and $\bar{\mathcal{F}}$ are called isomorphic if there exists an isomorphism Ψ between $(\Omega, \mathcal{F}_\infty, \mathbf{P})$ and $(\bar\Omega, \bar{\mathcal{F}}_\infty, \bar{\mathbf{P}})$ such that $\Psi(\mathcal{F}_t) = \bar{\mathcal{F}}_t$ for each t. For instance, if two processes have the same law, their natural filtrations are isomorphic.

A filtration indexed by $\mathbf{R}_+$ is *Brownian* if it is generated by one, or finitely many, or countably many independent real Brownian motions, started at the origin. It is well known that two Brownian filtrations are isomorphic if and only if they are generated by the same number ($\leqslant \infty$) of independent Brownian motions. (For if X is an m-dimensional Brownian motion in the filtration generated by an n-dimensional Brownian motion Y, there exist predictable processes H^{ij} such that $dX^i = \sum_j H^{ij}\, dY^j$; they form an $m\times n$ matrix H verifying $H\,{}^tH = \mathrm{Id}_m$ almost everywhere on $(\Omega\times\mathbf{R}_+, d\mathbf{P}\otimes dt)$, whence $m \leqslant n$.)

A filtration indexed by $-\mathbf{N}$ is *standard* if it is generated by a process $(Y_n)_{n\leqslant 0}$ where the Y_n's are independent r.v.'s with diffuse laws. It is always possible to choose each Y_n with law $\mathcal{N}(0,1)$, so all standard filtrations are isomorphic.

2. — Immersions

DEFINITION. — *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. The filtration $\mathcal{F}$ is* immersed *in $\mathcal{G}$ if every $\mathcal{F}$-martingale is a $\mathcal{G}$-martingale.*

This is inspired by the definition of an *extension* of a filtration, by Dubins, Feldman, Smorodinsky and Tsirelson in [4], and by that of a *morphism* from a filtered probability space to another, by Tsirelson in [16]; see also the *liftings* in Section 7 of Getoor and Sharpe [6]. (In these definitions, both filtrations are not necessarily on the same Ω.) Our definition is but a rephrasing of Brémaud, Jeulin and Yor's *Hypothèse (H)* in [2] and [9].

Immersion implies in particular that $\mathcal{F}_t$ is included in $\mathcal{G}_t$ for each t, but it is a much stronger property. As shown in [2], it amounts to requiring that the $\mathcal{G}$-optional projection of any $\mathcal{F}_\infty$-measurable process is $\mathcal{F}$-optional (and hence equal to the $\mathcal{F}$-optional projection), and it is also equivalent to demanding, for each t, that $\mathcal{F}_\infty \cap \mathcal{G}_t = \mathcal{F}_t$ and that $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$. We shall only need the weaker sufficient condition given by Lemma 1 below, concerning independent enlargements.

LEMMA 1. — *Let $\mathcal{F}$ and $\mathcal{G}$ be two independent filtrations (that is, $\mathcal{F}_t$ and $\mathcal{G}_t$ are independent for each t). If $\mathcal{H}$ is the smallest filtration containing both $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F}$ is immersed in $\mathcal{H}$.*

PROOF. — By a density argument, it suffices to show that every square-integrable $\mathcal{F}$-martingale M is an $\mathcal{H}$-martingale. For $s<t$, $F_s \in L^\infty(\mathcal{F}_s)$ and $G_t \in L^\infty(\mathcal{G}_t)$, one can write $\mathsf{E}[M_t F_s G_t] = \mathsf{E}[G_t]\,\mathsf{E}[M_t F_s] = \mathsf{E}[G_t]\,\mathsf{E}[M_s F_s] = \mathsf{E}[M_s F_s G_t]$. As products of the form $F_s G_t$ are total in $L^2(\mathcal{F}_s \otimes \mathcal{G}_t)$, $M_t - M_s$ is orthogonal to $L^2(\mathcal{F}_s \otimes \mathcal{G}_t)$. The lemma follows since $\mathcal{H}_s$ is included in $\mathcal{F}_s \otimes \mathcal{G}_t$ by Lemma 2 of Lindvall and Rogers [10]. ∎

Another, very simple, example of immersion is obtained by stopping: if T is an $\mathcal{F}$-stopping time, the stopped filtration $\mathcal{F}^T$ is immersed in $\mathcal{F}$.

The immersion property is in general not preserved when P is replaced with an equivalent probability; we shall sometimes write "P-immersed" to specify the probability. But it is preserved if the density is $\mathcal{F}_\infty$- or $\mathcal{G}_{-\infty}$-measurable:

LEMMA 2. — *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations on $(\Omega, \mathcal{A}, \mathsf{P})$, $\mathcal{F}$ being P-immersed in $\mathcal{G}$; let Q be a probability absolutely continuous with respect to P. If the density $D = d\mathsf{Q}/d\mathsf{P}$ has the form $D = D'D''$, where D' is $\mathcal{F}_\infty$-measurable and D'' is $\mathcal{G}_{-\infty}$-measurable, then $\mathcal{F}$ is also Q-immersed in $\mathcal{G}$.*

PROOF. — By taking absolute values, we may suppose $D' \geqslant 0$ and $D'' \geqslant 0$. It suffices to show that every bounded $(Q, \mathcal{F})$ martingale M is also a $(\mathsf{Q}, \mathcal{G})$-martingale; by adding a constant, we may also suppose M positive.

For $t \in \mathsf{T}$, the following equalities hold Q-a.s. (with $[0,\infty]$-valued conditional expectations; for the last equality, approximate D' by $D' \wedge n$ and use the P-immersion hypothesis):

$$\mathsf{Q}[M_\infty|\mathcal{G}_t] = \frac{\mathsf{P}[DM_\infty|\mathcal{G}_t]}{\mathsf{P}[D|\mathcal{G}_t]} = \frac{D''\mathsf{P}[D'M_\infty|\mathcal{G}_t]}{D''\mathsf{P}[D'|\mathcal{G}_t]} = \frac{\mathsf{P}[D'M_\infty|\mathcal{G}_t]}{\mathsf{P}[D'|\mathcal{G}_t]} = \frac{\mathsf{P}[D'M_\infty|\mathcal{F}_t]}{\mathsf{P}[D'|\mathcal{F}_t]}\,.$$

So $\mathsf{Q}[M_\infty|\mathcal{G}_t]$ is $\mathcal{F}_t$-measurable, hence $\mathsf{Q}[M_\infty|\mathcal{G}_t] = \mathsf{Q}[M_\infty|\mathcal{F}_t] = M_t$. ∎

DEFINITION. — *Let $X = (X_t)_{t\in\mathsf{T}}$ be a process and $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathsf{T}}$ a filtration, both on the same probability space. The process X is* immersed *in the filtration $\mathcal{F}$ if the natural filtration of X is immersed in $\mathcal{F}$.*

A process $X = (X_i)_{1\leqslant i\leqslant n}$ is also said to be immersed in a filtration $(\mathcal{F}_i)_{0\leqslant i\leqslant n}$ if the process $(0, X_1, \ldots, X_n)$ is immersed in $\mathcal{F}$.

Recall that X is adapted to $\mathcal{F}$ if and only if the natural filtration of X is included in $\mathcal{F}$. Saying that X is immersed in $\mathcal{F}$ is much stronger: it further means that for an $\mathcal{F}$-observer, the predictions about the future behaviour of X depend on the past and present of X only.

As for an example, remark that the same name, Brownian motion, is used for two different objects: first, any continuous, centred Gaussian process with covariance $\mathsf{E}[B_s B_t] = s$ for $s \leqslant t$ (this definition characterizes the law of B); second, a process such that $B_0 = 0$ and $B_t - B_s$ is independent of $\mathcal{F}_s$ with law $\mathcal{N}(0, t-s)$ (one often says that B is an $\mathcal{F}$-Brownian motion). The latter definition, involving the filtration, amounts to requiring that B has a Brownian law (former definition) *and is immersed in $\mathcal{F}$.*

Similarly, a Markov process is $\mathcal{F}$-Markov if and only it is immersed in $\mathcal{F}$: as a consequence, if $\mathcal{F}$ is immersed in $\mathcal{G}$, every $\mathcal{F}$-Markov process is also a $\mathcal{G}$-Markov process, with the same transition probabilities.

Lemma 2 can be rephrased in terms of immersed processes:

COROLLARY 1. — *If a process X is $\mathbf{P}$-immersed in a filtration $\mathcal{F}$, and if a probability $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$, with a $\sigma(X)$-measurable, or $\mathcal{F}_{-\infty}$-measurable density $d\mathbf{Q}/d\mathbf{P}$, then X is also $\mathbf{Q}$-immersed in $\mathcal{F}$.*

PROOF. — Apply Lemma 2 and the above definition. ∎

Tsirelson has introduced in [16] the notion of a joining of two filtrations. The next definition is a particular case of joining (we demand that both filtrations are defined on the same Ω).

DEFINITION. — *Two filtrations $\mathcal{F}$ and $\mathcal{G}$ (or two processes X and Y) on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ are* jointly immersed *if there exists a filtration $\mathcal{H}$ on $(\Omega, \mathcal{A}, \mathbf{P})$ such that both $\mathcal{F}$ and $\mathcal{G}$ (or X and Y) are immersed in $\mathcal{H}$.*

To illustrate the immersion property, and also to fix some notations to be used later, here is an easy statement.

LEMMA 3. — *Denote by Ω_n the set $\{-1,1\}^n$, by $\mathbf{P}_n$ the uniform probability on Ω_n (fair coin-tossing), and by $\varepsilon_1, \ldots, \varepsilon_n$ the coordinates on Ω_n; endow Ω_n with the filtration generated by the process $(0, \varepsilon_1, \ldots, \varepsilon_n)$.*

If $\alpha = (\alpha_i)_{1\leqslant i\leqslant n}$ is a predictable process on Ω_n such that $|\alpha| \leqslant 1$ and if $Z = (Z_i)_{0\leqslant i\leqslant n}$ is the $\mathbf{P}_n$-martingale defined by $Z_0 = 1$ and

$$Z_i = Z_{i-1}(1 + \alpha_i\, \varepsilon_i)\ ,$$

the formula $\mathbf{Q} = Z_n \cdot \mathbf{P}_n$ defines a probability law on Ω_n.

Let $X = (X_i)_{1\leqslant i\leqslant n}$ be a process with values ± 1, defined on some $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbf{P}})$ and adapted to some filtration $\mathcal{H} = (\mathcal{H}_i)_{0\leqslant i\leqslant n}$. The process X has law $\mathbf{Q}$ and is immersed in $\mathcal{H}$ if and only if

$$\forall i \in \{1, \ldots, n\} \quad \forall e \in \{-1, 1\} \qquad \bar{\mathbf{P}}[X_i = e | \mathcal{H}_{i-1}] = \tfrac{1}{2}\big(1 + \alpha_i(X_1, \ldots, X_{i-1})\, e\big)\ .$$

REMARK. — Every probability $\mathbf{Q}$ on Ω_n is obtained from such an α, but this correspondence between α and $\mathbf{Q}$ is not a bijection: if $\mathbf{Q}$ neglects some ω's, the martingale Z vanishes from some time on, and modifying α after that time does not change Z nor $\mathbf{Q}$. When restricted to the α's such that $|\alpha| < 1$ and to the $\mathbf{Q}$'s that are equivalent to $\mathbf{P}_n$, this correspondence is a bijection; in this case, α will be called the α-process associated to $\mathbf{Q}$.

PROOF OF LEMMA 3. — Clearly, Z is a positive martingale and $\mathbf{Q}$ is a probability. A process X with values ± 1 has law $\mathbf{Q}$ if and only if

$$\forall i \qquad \bar{\mathbf{P}}[X_1 = e_1, \ldots, X_i = e_i] = 2^{-i}\, Z_i(e_1, \ldots, e_i)\ ,$$

or

$$\forall i \qquad \bar{\mathbf{P}}[X_i = e_i \,|\, X_1 = e_1, \ldots, X_{i-1} = e_{i-1}] = \tfrac{1}{2}\,(Z_i/Z_{i-1})(e_1, \ldots, e_i) = \tfrac{1}{2}\,\big(1 + \alpha_i(e_1, \ldots, e_{i-1})\, e_i\big)\ ,$$

or

$$\forall i \qquad \bar{\mathbf{P}}[X_i = e \,|\, X_1, \ldots, X_{i-1}] = \tfrac{1}{2}\,\big(1 + \alpha_i(X_1, \ldots, X_{i-1})\, e\big)\ .$$

Now if X has law $\mathbb{Q}$ and is immersed in $\mathcal{H}$,

$$\bar{\mathsf{P}}[X_i = e\,|\mathcal{H}_{i-1}] = \bar{\mathsf{P}}[X_i = e\,|X_1,\dots,X_{i-1}] = \tfrac{1}{2}\left(1+\alpha_i(X_1,\dots,X_{i-1})\,e\right).$$

Conversely, for a ± 1-valued process X adapted to $\mathcal{H}$ such that, for each i in $\{1,\dots,n\}$ and each e in $\{-1,1\}$, $\bar{\mathsf{P}}[X_i = e\,|\mathcal{H}_{i-1}] = \left(1+\alpha_i(X_1,\dots,X_{i-1})\,e\right)$, one has on the one hand $\bar{\mathsf{P}}[X_i = e\,|\mathcal{H}_{i-1}] = \bar{\mathsf{P}}[X_i = e\,|X_1,\dots,X_{i-1}]$ and on the other hand $\bar{\mathsf{P}}[X_i = e\,|X_1,\dots,X_{i-1}] = \left(1+\alpha_i(X_1,\dots,X_{i-1})\,e\right)$; so X is immersed in $\mathcal{H}$ and has law $\mathbb{Q}$. ∎

3. — Separate σ-fields

DEFINITION. — *Given a probability space $(\Omega,\mathcal{A},\mathsf{P})$, two sub-$\sigma$-fields $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ are* separate *if $\mathsf{P}[B=C]=0$ for all random variables $B\in \mathrm{L}^0(\mathcal{B})$ and $C\in\mathrm{L}^0(\mathcal{C})$ with diffuse laws. Two filtrations $\mathcal{F}$ and $\mathcal{G}$ on $(\Omega,\mathcal{A},\mathsf{P})$ are* separate *if the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_\infty$ are.*

Two independent sub-σ-fields are always separate; this can be seen by taking $a=p=1$ in Proposition 1 below. But observe that separation depends only on the null sets of P, whereas independence, and more generally hypercontractivity, is not preserved by equivalent changes of probability.

PROPOSITION 1. — *Let $\mathcal{B}$ and $\mathcal{C}$ be two sub-σ-fields in a probability space $(\Omega,\mathcal{A},\mathsf{P})$. Suppose that, for some $p\in[1,2)$ and some $a<\infty$, the following inequality holds:*

$$\forall B\in\mathrm{L}^p(\mathcal{B})\quad \forall C\in\mathrm{L}^p(\mathcal{C})\qquad \mathsf{E}[BC]\leqslant a\,\|B\|_{\mathrm{L}^p}\,\|C\|_{\mathrm{L}^p}\,.$$

The σ-fields $\mathcal{B}$ and $\mathcal{C}$ are separate.

PROOF. — Let $B\in\mathrm{L}^0(\Omega,\mathcal{B},\mathsf{P})$ and $C\in\mathrm{L}^0(\Omega,\mathcal{C},\mathsf{P})$ have diffuse laws μ and ν; the measure $\lambda=\mu+\nu$ is positive, diffuse, with mass 2. For each $n\geqslant 1$ it is possible to partition $\mathbb{R}$ into $2n$ Borel sets $E_1,\dots,E_{2n}$, each with measure $\lambda(E_i)=1/n$. When applied to the r.v.'s $\mathbb{1}_{E_i}\circ B$ and $\mathbb{1}_{E_i}\circ C$, the hypothesis entails

$$\begin{aligned}\mathsf{P}\big[B\in E_i \text{ and } C\in E_i\big] &\leqslant a\,\mathsf{P}[B\in E_i]^{\frac1p}\,\mathsf{P}[C\in E_i]^{\frac1p}\\ &= a\,\mu(E_i)^{\frac1p}\,\nu(E_i)^{\frac1p}\leqslant a\,\lambda(E_i)^{\frac2p}=a\,n^{-\frac2p}\,.\end{aligned}$$

Summing over i gives

$$\begin{aligned}\mathsf{P}[B=C]&\leqslant\mathsf{P}\big[B \text{ and } C \text{ are in the same } E_i\big]\\ &=\sum_i \mathsf{P}\big[B\in E_i \text{ and } C\in E_i\big]\leqslant 2n\,a\,n^{-\frac2p}=2a\,n^{\frac{p-2}{p}}\,.\end{aligned}$$

Since $\dfrac{p-2}{p}<0$, letting n tend to infinity now yields $\mathsf{P}[B=C]=0$. ∎

REMARK. — When $a=1$, the inequality featuring in Proposition 1 is called hypercontractivity (because it means that the conditional expectation operator $\mathsf{E}[\;\,|\mathcal{B}]$ is not only a contraction from $\mathrm{L}^p(\mathcal{C})$ to $\mathrm{L}^p(\mathcal{B})$, but also a contraction from $\mathrm{L}^p(\mathcal{C})$ to the smaller space $\mathrm{L}^q(\mathcal{B})$, where q is the conjugate exponent of p).

The next proposition is one of Tsirelson's tools in [16]. It uses Proposition 1 to show separation for some Gaussianly generated σ-fields; Thouvenot [15] has another proof, via ergodic theory, that does not use hypercontractivity.

PROPOSITION 2. — *Let X' and X'' be two independent, centred Gaussian processes with the same law. For each $\theta \in \mathbf{R}$ the process $X^\theta = X'\cos\theta + X''\sin\theta$ has the same law as X' and X''; for $\theta \neq 0 \bmod \pi$ the σ-fields $\sigma(X^\theta)$ and $\sigma(X')$ satisfy the hypercontractivity property of Proposition 1 with $p = 1 + |\cos\theta| < 2$ and $a = 1$, and are therefore separate.*

PROOF. — The first property is well known—it is the very definition of 2-stability!—and can be readily verified by computing the covariance of X^θ.

Hypercontractivity is a celebrated theorem of Nelson [11]. When X' is just a normal r.v., a proof by stochastic calculus is given by Neveu [12]; a straightforward extension gives the case when X' is a normal random vector in $\mathbf{R}^n$ (see Dellacherie, Maisonneuve and Meyer [3]); and the general case follows by approximating $B \in L^p(\sigma(X^\theta))$ and $C \in L^p(\sigma(X'))$ in L^p with r.v.'s of the form $B' = f(X^\theta_{t_1}, \ldots, X^\theta_{t_n})$ and $C' = g(X'_{t_1}, \ldots, X'_{t_n})$, where f and g are Borel in n variables.

Separation stems from hypercontractivity, as shown by Proposition 1. ∎

Non probabilistic proofs of Gaussian hypercontractivity, as well as references to the literature, can be found in Gross [7] and Janson [8].

4. — Cosiness

Cosiness was invented by Tsirelson in [16], as a necessary condition for a filtration to be Brownian. There is a whole range of possible variations on his original definition; the one we choose below is taylor-made for Proposition 4.

DEFINITION. — *A filtered probability space $(\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$ is* cosy *if for each $\varepsilon > 0$ and each $U \in L^0(\Omega, \mathcal{F}_\infty, \mathbf{P})$, there exists a probability space $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}})$ with two filtrations $\mathcal{F}'$ and $\mathcal{F}''$ such that*

(i) *$(\bar\Omega, \mathcal{F}'_\infty, \bar{\mathbf{P}}, \mathcal{F}')$ and $(\bar\Omega, \mathcal{F}''_\infty, \bar{\mathbf{P}}, \mathcal{F}'')$ are isomorphic to $(\Omega, \mathcal{F}_\infty, \mathbf{P}, \mathcal{F})$;*

(ii) *$\mathcal{F}'$ and $\mathcal{F}''$ are jointly immersed;*

(iii) *$\mathcal{F}'$ and $\mathcal{F}''$ are separate;*

(iv) *the copies $U' \in L^0(\mathcal{F}'_\infty)$ and $U'' \in L^0(\mathcal{F}''_\infty)$ of U by the isomorphisms in condition* (i) *are ε-close in probability: $\bar{\mathbf{P}}\big[|U'-U''| > \varepsilon\big] \leqslant \varepsilon$.*

When there is no ambiguity on the underlying space $(\Omega, \mathcal{A}, \mathbf{P})$, we shall often simply say that the filtration $\mathcal{F}$ is cosy.

This definition is not equivalent to Tsirelson's original one. In his definition, the notion of separation featuring in condition (iii) is a bound on the joint bracket of any two martingales in the two filtrations, in terms of their quadratic variations. Other possible choices would be for instance a hypercontractivity inequality, as in Proposition 1, or a bound on the correlation coefficient for two r.v.'s in $L^2(\mathcal{F}'_\infty)$ and $L^2(\mathcal{F}''_\infty)$, or, when time is discrete, a bound on the predictable covariation $\bar{\mathbf{E}}[(X'_{n+1}-X'_n)(X''_{n+1}-X''_n)|\bar{\mathcal{F}}_n]$ of any two martingales in the two filtrations. In any case, the basic idea is always the same: we have two jointly immersed copies of the given filtration, that are close to each other as expressed by (iv), but nonetheless separate in some sense. As for an example, with the weak notion of separation we are using, a filtration $\mathcal{F}$ such that $\mathcal{F}_\infty$ has an atom is always cosy. Indeed, it suffices to take $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}}, \mathcal{F}') = (\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}}, \mathcal{F}'') = (\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$; (i), (ii) and (iv) are trivial, and (iii) is vacuously satisfied since no $\mathcal{F}_\infty$-measurable r.v. has a diffuse law! When dealing with such filtrations, other definitions of separation are more appropriate.

Another, more superficial, difference with Tsirelson's definition is that, instead of fixing U and ε, he deals with a whole sequence of joint immersions, and condition (iv) becomes the convergence to 0 for every U of the distance in probability $d(U', U'')$.

The use of a real random variable U could also be extended to random elements in an arbitrary separable metric space, for instance some space of functions; by Slutsky's lemma, it would not be more general, but it may be notationally more convenient.

Notice that conditions (i) and (ii) involve $\mathbf{P}$ and the whole filtration $\mathcal{F}$ in an essential way, whereas conditions (iii) and (iv) act only on the end-σ-field $\mathcal{F}_\infty$ and on the equivalence class of $\mathbf{P}$. Cosiness is not always preserved when $\mathbf{P}$ is replaced by an equivalent probability (see Theorems 1 and 2 below); but it is an instructive exercise to try proving this preservation by means of Lemma 2—and to see why it does not work.

LEMMA 4. — *Let $(\mathcal{F}_t)_{t\in\mathbf{T}}$ be a cosy filtration. If $(t_n)_{n\leqslant 0}$ is a sequence in $\mathbf{T}$ such that $t_{n-1} < t_n$ for all $n \leqslant 0$, the filtration $\mathcal{H} = (\mathcal{H}_n)_{n\leqslant 0}$ defined by $\mathcal{H}_n = \mathcal{F}_{t_n}$ is cosy too.*

PROOF. — It suffices to remark that if a filtration $(\mathcal{F}_t)_{t\in\mathbf{T}}$ is immersed in a filtration $(\mathcal{G}_t)_{t\in\mathbf{T}}$, then $(\mathcal{F}_{t_n})_{n\leqslant 0}$ is immersed in $(\mathcal{G}_{t_n})_{n\leqslant 0}$. The lemma follows then immediately from the definition of cosiness and the transitivity of immersions. ∎

LEMMA 5. — *A filtration immersed in a cosy filtration is itself cosy.*

PROOF. — If $(\Omega, \mathcal{A}, \mathbf{P})$ is endowed with two filtrations $\mathcal{F}$ and $\mathcal{G}$, if $\mathcal{F}$ is immersed in $\mathcal{G}$ and if Ψ is an embedding of $(\Omega, \mathcal{G}_\infty, \mathbf{P})$ into some probability space, then the filtration $\Psi(\mathcal{F})$ is immersed in $\Psi(\mathcal{G})$. The lemma follows immediately from this remark, the definition of cosiness and the transitivity of immersions. ∎

COROLLARY 2. — *Let $(\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$ and $(\Omega', \mathcal{A}', \mathbf{P}', \mathcal{F}')$ be filtered probability spaces and Ψ be an embedding of $(\Omega, \mathcal{A}, \mathbf{P})$ into $(\Omega', \mathcal{A}', \mathbf{P}')$, such that the filtration $\Psi(\mathcal{F})$ is immersed in $\mathcal{F}'$. If $(\Omega', \mathcal{F}'_\infty, \mathbf{P}', \mathcal{F}')$ is cosy, so is also $(\Omega, \mathcal{F}_\infty, \mathbf{P}, \mathcal{F})$.*

PROOF. — This filtered probability space is isomorphic to $\big(\Omega', \Psi(\mathcal{F}_\infty), \mathbf{P}', \Psi(\mathcal{F})\big)$, which is cosy by Lemma 5. ∎

We now turn to a very important sufficient condition for cosiness, or conversely a necessary condition for a filtration to be Gaussianly generated. It is due to Tsirelson [16], who devised cosiness to have this necessary condition at his disposal.

PROPOSITION 3. — *The natural filtration of a Gaussian process is cosy. More generally, let $X = (X_i)_{i\in I}$ be a Gaussian process and $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ a filtration. If there are subsets $I_t \subset I$ such that $\mathcal{F}_t = \bigcap_{\varepsilon>0} \sigma(X_i, i \in I_{t+\varepsilon})$ for each $t \in \mathbf{T}$, the filtration $\mathcal{F}$ is cosy.*

(In this statement, $I_{t+\varepsilon}$ should be taken equal to I_t when the time $\mathbf{T}$ is discrete.)

PROOF. — We may suppose X centred. Call $(\Omega, \mathcal{A}, \mathbf{P})$ the sample space where X is defined. On the filtered product space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbf{P}}, \bar{\mathcal{F}}) = (\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F}) \otimes (\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$, the processes $X^0(\bar{\omega}) = X^0(\omega_1, \omega_2) = X(\omega_1)$ and $X^{\pi/2}(\bar{\omega}) = X^{\pi/2}(\omega_1, \omega_2) = X(\omega_2)$ are independent copies of X. For $\theta \in \mathbf{R}$, the process $X^\theta = X^0 \cos\theta + X^{\pi/2} \sin\theta$ is yet another copy of X; notice that X^θ and $X^{\theta+\pi/2}$ are independent.

Every r.v. $U \in \mathrm{L}^0(\mathcal{F}_\infty)$ has the form $u(X_{i_1}, \ldots, X_{i_k}, \ldots)$, where u is Borel and $(i_1, \ldots, i_k, \ldots)$ is a sequence in I. This makes it possible to define an embedding Φ^θ

of $(\Omega, \mathcal{F}_\infty, \mathsf{P})$ into $(\bar\Omega, \bar{\mathcal{F}}_\infty, \bar{\mathsf{P}})$ by $\Phi^\theta(U) = u(X^\theta_{i_1}, \dots, X^\theta_{i_k}, \dots)$. When $\theta \to 0$, X^θ tends to X^0 almost surely, hence also in probability, and $\Phi^\theta(U) \to \Phi^0(U)$ by Slutsky's lemma (see Théorème 1 of [1]); so, given U and ε, $\Phi^\theta(U)$ is ε-close to $\Phi^0(U)$ in probability if θ is close enough to 0.

To establish cosiness, we shall take $\mathcal{F}' = \Phi^0(\mathcal{F})$ and $\mathcal{F}'' = \Phi^\theta(\mathcal{F})$ with θ close (but not equal) to 0. By the preceding remark, condition (iv) is fulfilled for some such θ; as $\theta \neq 0$, condition (iii) stems from Proposition 2, with $X' = X^0$ and $X'' = X^{\pi/2}$.

To prove (ii), we shall establish that each filtration $\mathcal{F}^\theta = \Phi^\theta(\mathcal{F})$ is immersed in the product filtration $\bar{\mathcal{F}}$. The latter is the smallest right-continuous filtration such that $\bar{\mathcal{F}}_t \supset \mathcal{F}^0_t \vee \mathcal{F}^{\pi/2}_t$. Now $\mathcal{F}^\theta_t = \bigcap_{\varepsilon>0} \sigma(X^\theta_i, i \in I_{t+\varepsilon}) \subset \mathcal{F}^0_{t+\varepsilon} \vee \mathcal{F}^{\pi/2}_{t+\varepsilon}$, whence $\mathcal{F}^\theta_t \subset \mathcal{F}^0_t \vee \mathcal{F}^{\pi/2}_t$ by applying twice Lemma 2 of Lindvall and Roger [10]; this yields the inclusion $\mathcal{F}^\theta_t \vee \mathcal{F}^{\theta+\pi/2}_t \subset \mathcal{F}^0_t \vee \mathcal{F}^{\pi/2}_t$. The inversion formulae

$$X^0 = X^\theta \cos\theta - X^{\theta+\pi/2} \sin\theta \qquad \text{and} \qquad X^{\pi/2} = X^\theta \sin\theta + X^{\theta+\pi/2} \cos\theta$$

give the reverse inclusion, so $\mathcal{F}^\theta_t \vee \mathcal{F}^{\theta+\pi/2}_t = \mathcal{F}^0_t \vee \mathcal{F}^{\pi/2}_t$, and $\bar{\mathcal{F}}$ is also the smallest right-continuous filtration such that $\bar{\mathcal{F}}_t$ contains $\mathcal{F}^\theta_t \vee \mathcal{F}^{\theta+\pi/2}_t$. As $\mathcal{F}^\theta$ and $\mathcal{F}^{\theta+\pi/2}$ are independent filtrations, $\mathcal{F}^\theta$ is immersed in $\bar{\mathcal{F}}$ by Lemma 1. ∎

COROLLARY 3. — *A Brownian filtration $(\mathcal{F}_t)_{t\geqslant 0}$, a standard filtration $(\mathcal{F}_n)_{n\in -\mathbb{N}}$, are always cosy.*

REMARKS. — Proposition 3 becomes false if it is only supposed that $\mathcal{F}_\infty$ is Gaussianly generated, without assuming the same for each $\mathcal{F}_t$. In that case, the same construction as in the above proof can be performed, yielding filtrations enjoying (ii), (iii) and (iv). But the immersion property (i) may fail, because $\mathcal{F}^0_t \vee \mathcal{F}^{\pi/2}_t$ and $\mathcal{F}^\theta_t \vee \mathcal{F}^{\theta+\pi/2}_t$ are no longer equal. For instance, the σ-field of any Lebesgue space is always generated by a normal random variable, and yet there exist non-cosy filtrations $\mathcal{F}$ such that $(\Omega, \mathcal{F}_\infty, \mathsf{P})$ is a Lebesgue space. Some examples are the filtered probability spaces constructed by Dubins, Feldman, Smorodinsky and Tsirelson (Theorems 1 and 2 below); other examples are the natural filtration of a Walsh process, shown by Tsirelson [16] to be non-cosy, and of a standard Poisson process (by the same argument as in the next paragraph).

Proposition 3 also becomes false if "Gaussian" is replaced with "α-stable for some $\alpha < 2$", because the separation property (iii) is not suited to those processes. For instance, let X be a Lévy α-stable process; call $\Psi(\lambda)$ its characteristic exponent (so $\exp[i\lambda X_t + t\Psi(\lambda)]$ is a martingale for each λ), call T the first time when $|\Delta X_T| \geqslant 1$ (so $0 < T < \infty$), and let $h > 0$ be such that $\mathsf{P}[T<h] \leqslant 1/3$. Suppose we have two copies X' and X'' of X, jointly immersed in some filtration $\mathcal{H}$, with separate filtrations.

The $\mathcal{H}$-stopping times $T' = \inf\{t : |\Delta X'| \geqslant 1\}$ and $T'' = \inf\{t : |\Delta X''| \geqslant 1\}$ verify $T' \neq T''$ a.s. by separation. Let $Y_t = X'_{T''+t} - X'_{T''}$. As T'' is an $\mathcal{H}$-stopping time and X' is immersed in $\mathcal{H}$, the processes $\exp[i\lambda Y_t + t\Psi(\lambda)]$ are $\mathcal{H}$-martingales, and Y has the same law as X. But on the event $\{T''<T'<T''+h\}$, a jump larger than 1 occurs for Y at time $T'-T''$, that is, between the times 0 and h. So, by definition of h, one has $\mathsf{P}[T''<T'<T''+h] \leqslant 1/3$; and similarly, by exchanging X' and X'' in the definition of Y, $\mathsf{P}[T'<T''<T'+h] \leqslant 1/3$. Taking the union of these two events and using $T' \neq T''$ gives $\mathsf{P}\big[|T'-T''|<h\big] \leqslant 2/3$. This bounds below the distance in probability between T' and T'', and condition (iv) in the definition of cosiness cannot be satisfied.

The next proposition, a sufficient condition for non-cosiness, summarizes the strategy of Dubins, Feldman, Smorodinsky and Tsirelson in [4].

PROPOSITION 4. — *Let* $(\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$ *be a filtered probability space and* $(\varepsilon_n)_{n<0}$ *a sequence in* $[0,1)$ *such that* $\sum_{n<0} \varepsilon_n < \infty$. *Suppose given a strictly increasing sequence* $(t_n)_{n\leqslant 0}$ *in* $\mathbb{T}$ *(that is,* $t_{n-1} < t_n$*), and an* $\mathbf{R}^{-\mathbf{N}}$*-valued random vector* $(U_n)_{n\leqslant 0}$*, with diffuse law, such that* U_n *is* $\mathcal{F}_{t_n}$*-measurable for each* n *and* U_0 *takes only finitely many values.*

Assume that for any filtered probability space $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbf{P}}, \bar{\mathcal{F}})$ *and for any two filtrations* $\mathcal{F}'$ *and* $\mathcal{F}''$ *isomorphic to* $\mathcal{F}$ *and jointly immersed in* $\bar{\mathcal{F}}$, *one has for each* $n < 0$

$$\bar{\mathbf{P}}[U'_{n+1} = U''_{n+1} | \bar{\mathcal{F}}_{t_n}] \leqslant \varepsilon_n \quad \text{on the event } \{U'_n \neq U''_n\}$$

(where U'_n *and* U''_n *denote the copies of* U_n *in the* σ*-fields* $\mathcal{F}'_\infty$ *and* $\mathcal{F}''_\infty$*).*

Then $\mathcal{F}$ *is not cosy.*

The hypothesis that the whole process $(U_n)_{n\leqslant 0}$ has a diffuse law is of course linked to the separation condition we are using. But notice that each U_n taken separately is not necessarily diffuse; in the situation considered later (in the proof of Theorem 1), each U_n can take only finitely many values.

PROOF. — By Lemma 4, we may suppose $\mathbb{T} = -\mathbb{N}$ and $t_n = n$ without loss of generality.

If $\mathcal{F}'$ and $\mathcal{F}''$ are isomorphic to $\mathcal{F}$ and immersed in $\bar{\mathcal{F}}$, we know that

$$\bar{\mathbf{E}}[\mathbb{1}_{\{U'_{n+1} \neq U''_{n+1}\}} | \bar{\mathcal{F}}_n] \geqslant (1-\varepsilon_n) \quad \text{on the event } \{U'_n \neq U''_n\} .$$

By induction on n, this implies

$$\mathbb{1}_{\{U'_n \neq U''_n\}} \bar{\mathbf{E}}[\mathbb{1}_{\{U'_{n+1} \neq U''_{n+1}\}} \cdots \mathbb{1}_{\{U'_0 \neq U''_0\}} | \bar{\mathcal{F}}_n] \geqslant \mathbb{1}_{\{U'_n \neq U''_n\}} (1-\varepsilon_n) \ldots (1-\varepsilon_{-1})$$

for $n < 0$, and a fortiori

$$(*) \qquad \forall n \leqslant 0 \qquad \bar{\mathbf{P}}[U'_0 \neq U''_0 | \bar{\mathcal{F}}_n] \geqslant \varepsilon \quad \text{on the event } \{U'_n \neq U''_n\} ,$$

where $\varepsilon > 0$ denotes the value of the convergent infinite product $\prod_{n<0} (1-\varepsilon_n)$.

To establish non-cosiness, consider any two isomorphic copies of $\mathcal{F}$, jointly immersed in $\bar{\mathcal{F}}$ and separate. As the law of $(U_n)_{n\leqslant 0}$ is diffuse, the separation assumption gives $\bar{\mathbf{P}}[\exists n \leqslant 0 \; U'_n \neq U''_n] = 1$, and there exists an $m < 0$ such that

$$\bar{\mathbf{P}}[\exists n \in \{m, m+1, \ldots, 0\} \; U'_n \neq U''_n] \geqslant \tfrac{1}{2} .$$

The $\bar{\mathcal{F}}$-stopping time $T = \inf\{n : m \leqslant n \leqslant 0 \text{ and } U'_n \neq U''_n\}$ verifies $\bar{\mathbf{P}}[T<\infty] \geqslant \frac{1}{2}$ and $U'_n \neq U''_n$ on $\{T = n\}$. The minoration $(*)$ gives $\bar{\mathbf{P}}[U'_0 \neq U''_0 | \bar{\mathcal{F}}_T] \geqslant \varepsilon$ on $\{T<\infty\}$, whence $\bar{\mathbf{P}}[U'_0 \neq U''_0] \geqslant \bar{\mathbf{P}}[U'_0 \neq U''_0, T<\infty] \geqslant \frac{1}{2}\varepsilon$. As U'_0 and U''_0 assume only finitely many values, their distance in probability is bounded below, and condition (iv) in the definition of cosiness is not satisfied. ∎

5. — The main results

The following two theorems are the rewriting, in the language of cosiness, of the amazing results of Dubins, Feldman, Smorodinsky and Tsirelson [4] and [5]; see also [14].

The canonical space for a coin-tossing game indexed by the time $\mathbb{T} = -\mathbb{N}$ will be denoted by $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$, where $\Omega = \{-1, 1\}^{-\mathbb{N}}$ is endowed with the coordinates ε_n, $\mathcal{F}_n$ is generated by $\sigma(\varepsilon_m, m \leqslant n)$ and the null events, $\mathcal{A} = \mathcal{F}_0 = \mathcal{F}_\infty$ and $\mathbb{P}$ is the fair coin-tossing probability, making the ε_n's independent and uniformly distributed on $\{-1, 1\}$.

THEOREM 1. — *Given $\delta > 0$, there exists on $(\Omega, \mathcal{A})$ a probability $\mathbb{Q}$ such that*
(i) $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$ *and* $\left|\frac{d\mathbb{Q}}{d\mathbb{P}} - 1\right| < \delta$;
(ii) $\mathcal{F} = (\mathcal{F}_n)_{n \leqslant 0}$ *is not cosy on* $(\Omega, \mathcal{A}, \mathbb{Q})$.

By Corollary 3, (ii) implies that $(\Omega, \mathcal{A}, \mathbb{Q}, \mathcal{F})$ is not standard.

If $X = (X_n)_{n \leqslant 0}$ is a process with law $\mathbb{Q}$ (defined on some probability space), its natural filtration is isomorphic to $\mathcal{F}$ under $\mathbb{Q}$; by Theorem 1 and Corollary 2, X cannot be immersed in any cosy filtration whatsoever, nor a fortiori in any standard filtration (Corollary 3).

Let $(W, \mathcal{B}, \lambda, \mathcal{G})$ denote the one-dimensional Wiener space: On $W = C(\mathbb{R}_+, \mathbb{R})$, w_t are the coordinates, λ makes w a Brownian motion started at the origin, $\mathcal{G} = (\mathcal{G}_t)_{t \geqslant 0}$ is the natural filtration of w, and $\mathcal{B} = \mathcal{G}_\infty$.

THEOREM 2. — *Given $\delta > 0$, there exists on $(W, \mathcal{B})$ a probability μ such that*
(i) μ *is equivalent to* λ *and* $\left|\frac{d\mu}{d\lambda} - 1\right| < \delta$;
(ii) $\mathcal{G} = (\mathcal{G}_t)_{t \geqslant 0}$ *is not cosy on* $(W, \mathcal{B}, \mu)$.

By Corollary 3, (ii) implies that the filtration $\mathcal{G}$ on $(W, \mathcal{B}, \mu)$ is not Brownian.

If $X = (X_t)_{t \geqslant 0}$ is a process with law μ (defined on some probability space), its natural filtration is isomorphic to $\mathcal{G}$ under μ; by Theorem 2 and Corollary 2, X cannot be immersed in any cosy filtration whatsoever, nor a fortiori in any Brownian filtration (Corollary 3).

PROOF OF THEOREM 2, ASSUMING THEOREM 1. — Given $\delta > 0$, Theorem 1 yields a probability $\mathbb{Q}$ on Ω, such that $(\Omega, \mathcal{A}, \mathbb{Q}, \mathcal{F})$ is not cosy, and whose density $D = D(\varepsilon_n, n \leqslant 0) = d\mathbb{Q}/d\mathbb{P}$ verifies $|D - 1| < \delta$. Denote by $\mathcal{H}_n$ the σ-field $\mathcal{G}_{2^n}$ on W and by $\mathcal{H}$ the filtration $(\mathcal{H}_n)_{n \leqslant 0}$.

Define a mapping $S : W \to \Omega$ by $S = (S_n)_{n \leqslant 0}$ with $S_n(w) = \operatorname{sgn}(w_{2^n} - w_{2^{n-1}})$; the law $\lambda \circ S^{-1}$ of S under λ is $\mathbb{P}$. Define μ on W by $d\mu/d\lambda = D' = D \circ S$. On $(W, \mathcal{B}, \lambda)$, the vector $(S_{n+1}, \dots, S_0)$ and the σ-field $\mathcal{H}_n$ are independent; hence, S is λ-immersed in $\mathcal{H}$. By Corollary 1, S is also μ-immersed in $\mathcal{H}$. Now the law $\mu \circ S^{-1}$ of S under μ is $\mathbb{Q}$ since, for $A \in \mathcal{A}$,

$$\mu[S \in A] = \mu[\mathbb{1}_A \circ S] = \lambda[\mathbb{1}_A \circ S\, D'] = \lambda[(D \mathbb{1}_A) \circ S] = \mathbb{P}[D \mathbb{1}_A] = \mathbb{Q}(A) .$$

So the mapping $\Psi : L^0(\Omega, \mathcal{A}, \mathbb{Q}) \to L^0(W, \mathcal{B}, \mu)$ defined by $\Psi(X) = X \circ S$ is an embedding of $(\Omega, \mathcal{A}, \mathbb{Q})$ into $(W, \mathcal{B}, \mu)$. The filtration $\Psi(\mathcal{F})$ is the natural filtration of S, so it is μ-immersed in $\mathcal{H}$. Since $(\Omega, \mathcal{F}_\infty, \mathbb{Q}, \mathcal{F})$ is not cosy by definition of $\mathbb{Q}$, neither is $(W, \mathcal{H}_0, \mu, \mathcal{H})$ by Corollary 2, nor $(W, \mathcal{G}_\infty, \mu, \mathcal{G})$ by Lemma 4. ∎

6. — Proof of Theorem 1

From now on, we follow closely Schachermayer's simplified exposition [14] of the construction by Dubins, Feldman, Smorodinsky and Tsirelson.

DEFINITION. — Let $p \geqslant 1$ be an integer and $\mathcal{M}$ the set of all rectangular matrices $\tau = (\tau_i^j)_{1\leqslant i\leqslant p, 1\leqslant j\leqslant 2^p}$ with entries τ_i^j in $\{-1,1\}$. Two matrices τ' and τ'' in $\mathcal{M}$ are *close* if for at least $2^p/p$ values of j, there are at most $p^{12/13}$ values of i such that $\tau'^j_i \neq \tau''^j_i$.

LEMMA 6 (Schachermayer's Combinatorial Lemma). — *For each p large enough, there exist 2^{4p} matrices in $\mathcal{M}$ that are pairwise not close.*

Schachermayer's proof below uses very rough combinatorial estimates and can give $2^{2^p/p}$ pairwise not close matrices, instead of the mere 2^{4p} needed in the sequel: this overabundance is already present in the original proof by Dubins, Feldman, Smorodinsky and Tsirelson.

PROOF. — Let π denote the uniform probability on $\mathcal{M}$; π chooses the entries τ_i^j by tossing a fair coin. It suffices to show that for each $\tau \in \mathcal{M}$, the "neighbourhood" $C_\tau = \{\sigma \in \mathcal{M} : \sigma \text{ and } \tau \text{ are close}\}$ has probability $\pi(C_\tau) \leqslant 2^{-4p}$. So fix τ and let $s_i^j = -\tau_i^j$. As

$$C_\tau = \{\sigma \in \mathcal{M} : \text{ for at least } 2^p/p \text{ values of } j, \text{ there are at most } p^{12/13} \text{ values of } i \text{ such that } \sigma_i^j = s_i^j\},$$

one has

$$C_\tau = \bigcup_{\substack{J\subset\{1,\dots,2^p\}\\ |J|=\lceil 2^p/p\rceil}} \bigcap_{j\in J} C_\tau^j ,$$

where

$$C_\tau^j = \{\sigma \in \mathcal{M} : \text{ for at most } p^{12/13} \text{ values of } i,\ \sigma_i^j = s_i^j\}$$

and

$$\pi(C_\tau^j) = \frac{1}{2^p}\left\{\binom{p}{0} + \binom{p}{1} + \ldots + \binom{p}{[p^{12/13}]}\right\}.$$

For p large enough, the sum has less than $p/2$ terms and the last one is the largest, giving

$$\pi(C_\tau^j) \leqslant \frac{1}{2^p}\frac{p}{2}\binom{p}{[p^{12/13}]} \leqslant \frac{1}{2^p}\frac{p}{2}p^{p^{12/13}} ;$$

$$\ln \pi(C_\tau^j) \leqslant -p\ln 2 + \ln\frac{p}{2} + p^{12/13}\ln p .$$

This is equivalent to $-p\ln 2$ when p tends to infinity, so, for p large enough, $\ln\pi(C_\tau^j) \leqslant -\frac{1}{2}p\ln 2$ and $\pi(C_\tau^j) \leqslant 2^{-p/2}$.

The columns of a random matrix in $\mathcal{M}$ are independent for π; so the C_τ^j's are independent events, and, setting $q = \lceil 2^p/p\rceil$,

$$\pi(C_\tau) = \pi\Big[\bigcup_{\substack{J\subset\{1,\dots,2^p\}\\ |J|=q}} \bigcap_{j\in J} C_\tau^j\Big] \leqslant \sum_{\substack{J\subset\{1,\dots,2^p\}\\ |J|=q}} \pi\Big[\bigcap_{j\in J} C_\tau^j\Big] \leqslant \binom{2^p}{q}(2^{-p/2})^q .$$

Using $\binom{a}{q} \leqslant \frac{a^q}{q!} \leqslant \frac{a^q}{(q/3)^q}$, one gets

$$\pi(C_\tau) \leqslant \Big(\frac{2^p}{q/3}2^{-\frac{p}{2}}\Big)^q \leqslant \big(3p\,2^{-p/2}\big)^q$$

and

$$\frac{1}{4p}\ln\pi(C_\tau) \leqslant \frac{1}{4p}\left\lceil\frac{2^p}{p}\right\rceil\big(\ln(3p) - \frac{p}{2}\ln 2\big) .$$

This tends to $-\infty$ when p tends to infinity, so $\pi(C_\tau) \leqslant 2^{-4p}$ for p large enough. ∎

The next lemma is the Fundamental Lemma of Dubins, Feldman, Smorodinsky and Tsirelson [4]; we borrow it from [14]. Recall the notations of Lemma 3: P_n is the uniform law (fair coin-tossing) on $\Omega_n = \{-1,1\}^n$.

LEMMA 7. — *For every p large enough, there exists a set $\mathcal{Q}_{2p}$ of probabilities on Ω_{2p} with the following three properties:*

(i) *$\mathcal{Q}_{2p}$ has 2^{4p} elements;*

(ii) *each $\mathsf{Q} \in \mathcal{Q}_{2p}$ satisfies*

$$\left|\frac{d\mathsf{Q}}{d\mathsf{P}_{2p}} - 1\right| \leqslant p^{-1/4} ;$$

(iii) *for any two different probabilities Q' and Q'' in $\mathcal{Q}_{2p}$ and any two $\{-1,1\}$-valued processes X' and X'', indexed by $\{1,\ldots,2p\}$, with laws Q' and Q'', defined on the same probability space $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathsf{P}})$ and jointly immersed, one has*

$$\bar{\mathsf{P}}\big[X'_i = X''_i \text{ for all } i \in \{1,\ldots,2p\}\big] \leqslant p^{-1/4} .$$

Since Q' and Q'' are not far from P_{2p} by (ii), hence not far from each other, it is always possible to find two processes X' and X'' with laws Q' and Q'' and such that $\bar{\mathsf{P}}[X' = X'']$ is close to 1. What the lemma says, is that if X' and X'' have these laws *and are jointly immersed,* then $\bar{\mathsf{P}}[X' = X'']$ must be small.

In [16], Tsirelson shows that a Walsh process cannot be immersed in a cosy filtration. A key step in his method is a lower estimate of the expected distance $\mathsf{E}[d(X',X'')]$, where X' and X'' are two Walsh processes embedded in a common filtration. The majoration of $\bar{\mathsf{P}}[X' = X'']$ in Lemma 7 is a discrete analogue of this minoration.

PROOF. — The 2^{4p} probabilities to be constructed on Ω_{2p} will be defined through their α-processes (notations of Lemma 3). More precisely, to each matrix $\tau \in \mathcal{M}$ (notations of Lemma 6), we shall associate an α^τ and a Q^τ such that $|d\mathsf{Q}^\tau/d\mathsf{P}_{2p} - 1| \leqslant p^{-1/4}$. Then we shall prove that if τ' and τ'' are not close, any two processes X' and X'', having laws $\mathsf{Q}^{\tau'}$ and $\mathsf{Q}^{\tau''}$ and jointly immersed, verify $\bar{\mathsf{P}}[X' = X''] \leqslant p^{-1/4}$. As Lemma 6 gives 2^{4p} such matrices τ, the 2^{4p} associated probabilities will pairwise have property (iii), and Lemma 7 will be established.

As was already the case for Lemma 6, the proof works for $p \geqslant p_0$ where p_0 is an unspecified constant. The symbol $\overset{\bullet}{\leqslant}$ will be used for inequalities valid for p large enough.

<u>Step one</u>: *Definition of a probability Q^τ for each matrix τ, and two estimates on Q^τ.*

We shall slightly change the notations: a matrix $\tau \in \mathcal{M}$ will not be written (τ_i^j) with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant 2^p$ as in Lemma 6, but $(\tau_i^{e_1,\ldots,e_p})$, where i ranges from $p{+}1$ to $2p$ and $e_1,\ldots,e_p$ are in $\{-1,1\}$ (use an arbitrary bijection between $\{1,\ldots,2^p\}$ and $\{-1,1\}^p$).

The matrix $\tau \in \mathcal{M}$ is fixed. The coordinates on Ω_{2p} are $\varepsilon_1,\ldots,\varepsilon_{2p}$. Define a predictable process on Ω_{2p} by

$$\beta_i = \begin{cases} 0 & \text{for } 1 \leqslant i \leqslant p \\ \eta\,\tau_i^{\varepsilon_1,\ldots,\varepsilon_p} & \text{for } p{+}1 \leqslant i \leqslant 2p, \end{cases}$$

where $\eta = p^{-11/12}$ is a small positive number. (This β is not the promised α^τ yet; be patient!) A P_{2p}-martingale $Z = (Z_i)_{1\leqslant i\leqslant 2p}$ is defined on Ω_{2p} by $Z_0 = 1$ and

$$Z_i = Z_{i-1}(1 + \beta_i\varepsilon_i) \qquad \text{for } 1 \leqslant i \leqslant 2p ;$$

it verifies $Z_i = 1$ for $1 \leqslant i \leqslant p$. Set $\gamma = p^{-2/7}$, introduce the stopping time

$$T = 2p \wedge \inf\{i : |Z_i - 1| > \gamma\} ,$$

and remark that $p < T \leqslant 2p$. The probability $\mathbb{Q}^\tau$ will be $Z_T \cdot \mathbb{P}_{2p}$; in other words, the martingale associated with $\mathbb{Q}^\tau$ is the stopped Z^T, and the corresponding α-process (notations of Lemma 3) is β up to time T and 0 after T.

Now, by definition of T, $1-\gamma \leqslant Z_{T-1} \leqslant 1+\gamma \leqslant 2$, so

$$|Z_T-1| \leqslant |Z_T-Z_{T-1}| + |Z_{T-1}-1| \leqslant \eta\, Z_{T-1} + |Z_{T-1}-1| \leqslant 2\eta + \gamma \overset{\bullet}{\leqslant} p^{-1/4} ,$$

yielding property (ii).

Observe that

$$\begin{aligned} \|Z_T-1\|^2_{L^2(\mathbb{P}_{2p})} &= \sum_{i=p}^{2p-1} \|Z^T_{i+1}-Z^T_i\|^2_{L^2(\mathbb{P}_{2p})} \leqslant \sum_{i=p}^{2p-1} \|Z^T_{i+1}-Z^T_i\|^2_{L^\infty} \\ &\leqslant \sum_{i=p}^{2p-1} \eta^2 \,\|Z^T_i \mathbb{1}_{\{T>i\}}\|^2_{L^\infty} \leqslant \sum_{i=p}^{2p-1} 4\eta^2 = 4p\eta^2 ; \end{aligned}$$

this gives the estimate

$$\mathbb{P}_{2p}[T<2p] \leqslant \mathbb{P}_{2p}\big[|Z_T-1| > \gamma\big] \leqslant \frac{4p\eta^2}{\gamma^2} ,$$

whence, using property (ii),

$$\mathbb{Q}^\tau[T<2p] \leqslant \big(1+p^{-1/4}\big)\,\mathbb{P}_{2p}[T<2p] \leqslant 8p\eta^2\gamma^{-2} \overset{\bullet}{\leqslant} \tfrac{1}{4}\,p^{-1/4} .$$

<u>Step two</u>: *If two matrices $\tau' \in \mathcal{M}$ and $\tau'' \in \mathcal{M}$ are not close, any two jointly immersed processes X' and X'' with laws $\mathbb{Q}^{\tau'}$ and $\mathbb{Q}^{\tau''}$ verify $\bar{\mathbb{P}}[X' = X''] \leqslant p^{-1/4}$.*

We are given two matrices τ' and τ'', not close to each other; the construction of Step one, performed for τ' and τ'', yields on Ω_{2p} two martingales Z' and Z'', two stopping times T' and T'', and two laws $\mathbb{Q}^{\tau'}$ and $\mathbb{Q}^{\tau''}$.

Let X' and X'' be two processes indexed by $\{1,\dots,2p\}$, defined on some $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbb{P}})$, jointly immersed in some filtration $(\mathcal{H}_i)_{1\leqslant i\leqslant 2p}$, with respective laws $\mathbb{Q}^{\tau'}$ and $\mathbb{Q}^{\tau''}$. The processes X' and X'' can be considered as Ω_{2p}-valued random variables; as X' and X'' are $\mathcal{H}$-adapted, $S' = T'\circ X'$ and $S'' = T''\circ X''$ are $\mathcal{H}$-stopping times, as well as $S = S' \wedge S''$.

One has $\bar{\mathbb{P}}[S'<2p] = \mathbb{Q}^{\tau'}[T'<2p] \leqslant \frac{1}{4}\,p^{-1/4}$ and similarly for S''; hence $\bar{\mathbb{P}}[S<2p] \leqslant \frac{1}{2}\,p^{-1/4}$, and to establish the claim of Step two, it suffices to show that $\bar{\mathbb{P}}[X'=X''$ and $S = 2p] \leqslant \frac{1}{2}\,p^{-1/4}$.

Fix $e_1,\dots,e_p \in \{-1,1\}^p$. For i such that $p{+}1 \leqslant i \leqslant 2p$, set $t'_i = {\tau'}_i^{e_1,\dots,e_p}$ and $t''_i = {\tau''}_i^{e_1,\dots,e_p}$. On the event $E_{e_1,\dots,e_p} = \{X'_1 = X''_1 = e_1, \dots, X'_p = X''_p = e_p\}$, Lemma 3 and the definitions of $\mathbb{Q}^{\tau'}$ and $\mathbb{Q}^{\tau''}$ yield for $p{+}1 \leqslant i \leqslant 2p$ and $e = \pm 1$

$$\begin{aligned} \bar{\mathbb{P}}[X'_i = e \,|\, \mathcal{H}_{i-1}] &= \tfrac{1}{2}(1+\eta t'_i e \mathbb{1}_{\{S'>i-1\}}) \\ \bar{\mathbb{P}}[X''_i = e \,|\, \mathcal{H}_{i-1}] &= \tfrac{1}{2}(1+\eta t''_i e \mathbb{1}_{\{S''>i-1\}}) ; \end{aligned}$$

hence, on the event $E_{e_1,\dots,e_p} \cap \{S > i-1\}$, one has

$$\bar{\mathbb{P}}[X'_i = e \,|\, \mathcal{H}_{i-1}] = \tfrac{1}{2}(1+\eta t'_i e) \qquad \text{and} \qquad \bar{\mathbb{P}}[X''_i = e \,|\, \mathcal{H}_{i-1}] = \tfrac{1}{2}(1+\eta t''_i e) .$$

Consequently, on the same event, if $t'_i \neq t''_i$, that is, if $t'_i t''_i = -1$, one can write

$$\begin{aligned}\bar{\mathbf{P}}[X'_i = X''_i \mid \mathcal{H}_{i-1}] &\leqslant \bar{\mathbf{P}}[X'_i = t''_i \text{ or } X''_i = t'_i \mid \mathcal{H}_{i-1}] \\ &\leqslant \bar{\mathbf{P}}[X'_i = t''_i \mid \mathcal{H}_{i-1}] + \bar{\mathbf{P}}[X''_i = t'_i \mid \mathcal{H}_{i-1}] \\ &= \tfrac{1}{2}(1+\eta t'_i t''_i) + \tfrac{1}{2}(1+\eta t''_i t'_i) = \tfrac{1}{2}(1-\eta) + \tfrac{1}{2}(1-\eta) = 1-\eta\,;\end{aligned}$$

and on $E_{e_1,\ldots,e_p}$, one has $\mathbb{1}_{\{S>i-1\}}\,\bar{\mathbf{P}}[X'_i = X''_i \mid \mathcal{H}_{i-1}] \leqslant \mathbb{1}_{\{S>i-1\}}\,(1-\eta)^{\mathbb{1}_{t'_i \neq t''_i}}$. Since the events

$$A_i = E_{e_1,\ldots,e_p} \cap \{X'_{p+1} = X''_{p+1}, \ldots, X'_i = X''_i, S > i-1\}$$

verify $A_i \in \mathcal{H}_i$ and $A_i = A_{i-1} \cap \{S > i-1\} \cap \{X'_i = X''_i\}$, one can write for $p+1 \leqslant i \leqslant 2p$

$$\begin{aligned}\bar{\mathbf{P}}[A_i \,|\mathcal{H}_{i-1}] &= \mathbb{1}_{A_{i-1}} \mathbb{1}_{\{S>i-1\}}\,\bar{\mathbf{P}}[X'_i = X''_i \,|\mathcal{H}_{i-1}] \\ &\leqslant \mathbb{1}_{A_{i-1}} \mathbb{1}_{\{S>i-1\}}\,(1-\eta)^{\mathbb{1}_{t'_i \neq t''_i}} \leqslant \mathbb{1}_{A_{i-1}}\,(1-\eta)^{\mathbb{1}_{t'_i \neq t''_i}}\,.\end{aligned}$$

By induction, this gives for $i \geqslant p$

$$\bar{\mathbf{P}}[A_i \,|\mathcal{H}_p] \leqslant \mathbb{1}_{E_{e_1,\ldots,e_p}}\,(1-\eta)^{\mathbb{1}_{t'_{p+1} \neq t''_{p+1}} + \cdots + \mathbb{1}_{t'_i \neq t''_i}}\,;$$

taking $i = 2p$, we finally get

$$\mathbb{1}_{E_{e_1,\ldots,e_p}}\,\bar{\mathbf{P}}[X' = X'', S = 2p \mid \mathcal{H}_p] \leqslant \mathbb{1}_{E_{e_1,\ldots,e_p}}\,(1-\eta)^{\operatorname{Card}\{i\,:\, t'_i \neq t''_i\}}.$$

Unfix $e_1, \ldots, e_p$ and write

$$\bar{\mathbf{P}}[X' = X'', S = 2p] = \bar{\mathbf{E}}\Big[\sum_{e_1,\ldots,e_p} \mathbb{1}_{E_{e_1,\ldots,e_p}} \bar{\mathbf{P}}[X' = X'', S = 2p \mid \mathcal{H}_p]\Big] \leqslant \bar{\mathbf{E}}\big[(1-\eta)^N\big]\,,$$

where N denotes the number of i's such that ${\tau'}_i^{X'_1,\ldots,X'_p} \neq {\tau''}_i^{X'_1,\ldots,X'_p}$. Since the α-process associated to $\mathbf{Q}^{\tau'}$ is zero on the interval $\{1, \ldots, p\}$, $X'_1, \ldots, X'_p$ are independent and uniformly distributed on $\{-1, 1\}$. Now τ' and τ'' are not close, so for less than $2^p/p$ values of $(e_1, \ldots, e_p)$, there are at most $p^{12/13}$ values of i such that ${\tau'}_i^{e_1,\ldots,e_p} \neq {\tau''}_i^{e_1,\ldots,e_p}$, and

$$\bar{\mathbf{P}}\big[N \leqslant p^{12/13}\big] \leqslant \frac{1}{p}\,.$$

This implies

$$\begin{aligned}\bar{\mathbf{P}}[X' = X'', S = 2p] &\leqslant \bar{\mathbf{E}}[(1-\eta)^N] \leqslant \frac{1}{p} + (1-\eta)^{p^{12/13}} \\ &\leqslant p^{-1} + e^{-\eta p^{12/13}} = p^{-1} + e^{-p^{1/156}} \overset{\bullet}{\leqslant} \tfrac{1}{2} p^{-1/4}\,.\end{aligned}$$

The proof of step two and of Lemma 7 is complete. ∎

As we shall use Lemma 7 only in the case when p has the form 2^{k-1}, it is convenient to re-state it in this case:

Corollary 4. — *For every k large enough, there exists a set $\mathcal{Q}_{2^k}$ of probabilities on Ω_{2^k} with the following three properties:*

(i) *$\mathcal{Q}_{2^k}$ has $2^{2^{k+1}}$ elements;*

(ii) *each $\mathbf{Q} \in \mathcal{Q}_{2^k}$ satisfies*

$$\left|\frac{d\mathbf{Q}}{d\mathbf{P}_{2^k}} - 1\right| \leqslant 2^{-(k-1)/4}\,;$$

(iii) *for any two different probabilities $\mathbf{Q}'$ and $\mathbf{Q}''$ in $\mathcal{Q}_{2^k}$ and any two $\{-1,1\}$-valued processes X' and X'', indexed by $\{1, \ldots, 2^k\}$, with laws $\mathbf{Q}'$ and $\mathbf{Q}''$, defined on the same probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbf{P}})$ and jointly immersed, one has*

$$\bar{\mathbf{P}}\big[X'_i = X''_i \text{ for all } i \in \{1, \ldots, 2^k\}\big] \leqslant 2^{-(k-1)/4}\,.$$

PROOF OF THEOREM 1. — We may and shall suppose that $\delta < \frac{1}{2}$. Let k_0 be a number such that Corollary 4 holds for $k \geqslant k_0$ and that

$$\prod_{k \geqslant k_0} \left(1 - 2^{-(k-1)/4}\right) > 1-\delta \qquad \text{and} \qquad \prod_{k \geqslant k_0} \left(1 + 2^{-(k-1)/4}\right) < 1+\delta .$$

Instead of working with the sample space $\{-1, 1\}^{-\mathbb{N}}$, we shall consider

$$\Omega = \prod_{k=\infty}^{k_0} \Omega_{2^k} = \ldots \times \Omega_{2^k} \times \ldots \times \Omega_{2^{k_0}}$$

endowed with the product probability $\mathbf{P} = \ldots \times \mathbf{P}_{2^k} \times \ldots \times \mathbf{P}_{2^{k_0}}$. The projection of Ω on the factor Ω_{2^k} will be called X_{-k}. The coordinates on Ω_{2^k} will not be denoted by $\varepsilon_1, \ldots, \varepsilon_{2^k}$, but by $\varepsilon_{-2^{k+1}+1}, \varepsilon_{-2^{k+1}+2}, \ldots, \varepsilon_{-2^k}$. This identifies Ω with the canonical space of a coin-tossing game indexed by the integers $\leqslant -2^{k_0}$ (this is $\{-1, 1\}^{-\mathbb{N}}$ up to a translation of the time-axis). The factor space Ω_{2^k} corresponds to time ranging from $-2^{k+1}+1$ to -2^k. The filtration $\mathcal{F} = (\mathcal{F}_i)_{i \leqslant -2^{k_0}}$ is the one generated by the coordinates $(\varepsilon_i)_{i \leqslant -2^{k_0}}$.

For each $k \geqslant k_0$, notice that the sets $\Omega_{2^{k+1}}$ and $\mathcal{Q}_{2^k}$ have the same cardinality by condition (i) of Corollary 4, and choose a bijection M_k from $\Omega_{2^{k+1}}$ to $\mathcal{Q}_{2^k}$. For $x_{-k-1} \in \Omega_{2^{k+1}}$, $M_k(x_{-k-1})$ is a probability on Ω_{2^k}; this defines a Markov transition matrix $M_k(x_{-k-1}, x_{-k})$ from $\Omega_{2^{k+1}}$ to Ω_{2^k}. Its density

$$m_k(x_{-k-1}, x_{-k}) = \frac{M_k(x_{-k-1}, x_{-k})}{\mathbf{P}_{2^k}(x_{-k})} = \frac{M_k(x_{-k-1}, x_{-k})}{2^{-2^k}}$$

verifies $\mathbf{E}\left[m_k(X_{-k-1}, X_{-k})\right] = 1$, and also $|m_k(x_{-k-1}, x_{-k}) - 1| \leqslant 2^{-(k-1)/4}$ by condition (ii) of Corollary 4. The infinite product

$$D(\omega) = \prod_{k \geqslant k_0} m_k\big(X_{-k-1}(\omega), X_{-k}(\omega)\big)$$

satisfies

$$\prod_{k \geqslant k_0} (1-2^{-(k-1)/4}) \leqslant D(\omega) \leqslant \prod_{k \geqslant k_0} (1+2^{-(k-1)/4}) ,$$

whence $|D(\omega)-1| < \delta$ and $\mathbf{E}[D] = 1$ by dominated convergence. This defines a probability $\mathbf{Q} = D \cdot \mathbf{P}$ on Ω, satisfying condition (i) of Theorem 1. For $\mathbf{Q}$, the process $X = (\ldots, X_{-k}, \ldots, X_{-k_0})$ is Markov, with the M_k's as transition matrices, as can readily be checked by setting $D_k = \mathbf{P}[D|\mathcal{F}_{-2^k}] = \mathbf{P}\big[\prod_{\ell \geqslant k-1} m_\ell(X_{-\ell-1}, X_{-\ell})\big]$ and writing

$$\mathbf{Q}\Big[\bigcap_{\ell=k_0}^{k} \{X_{-\ell} = x_{-\ell}\}\Big] = \prod_{\ell=k_0}^{k-2} M_\ell(x_{-\ell-1}, x_{-\ell})\, \mathbf{P}\big[D_k\, \mathbb{1}_{\{X_{-k}=x_{-k}\}}\big]$$
$$= \prod_{\ell=k_0}^{k-2} M_\ell(x_{-\ell-1}, x_{-\ell})\, \mathbf{Q}[X_{-k} = x_{-k}] .$$

To complete the proof, it remains to establish that $(\Omega, \mathcal{F}_\infty, \mathbf{Q}, \mathcal{F})$ is not cosy; this will be done by applying Proposition 4 to $(\Omega, \mathcal{F}_\infty, \mathbf{Q}, \mathcal{F})$, with $t_\ell = -2^{-\ell}$ and $U_\ell = X_\ell$. The random vector $X = (X_\ell)_{\ell \leqslant -k_0}$ can be identified with $(\varepsilon_i)_{i \leqslant -2^{k_0}}$; its law $\mathbf{Q}$ is diffuse, for it is equivalent to $\mathbf{P}$. So, to apply Proposition 4 (with the index $-k_0$ instead of 0; this is irrelevant), it suffices to establish the following lemma.

LEMMA 8. — *On some* $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbf{P}})$, *let* $\mathcal{F}'$ *and* $\mathcal{F}''$ *be two filtrations isomorphic to* $\mathcal{F}$, *jointly immersed in some filtration* $\bar{\mathcal{F}}$; *call* X' *and* X'' *the copies of* X *in the* σ*-fields* $\mathcal{F}'_{-2^{k_0}}$ *and* $\mathcal{F}''_{-2^{k_0}}$. *For every* $k > k_0$, *on has*

$$\bar{\mathbf{P}}[X'_{-k+1} = X''_{-k+1}|\bar{\mathcal{F}}_{-2^k}] \leqslant 2^{-(k-2)/4} \quad \textit{on the event } \{X'_{-k} \neq X''_{-k}\}\ .$$

PROOF. — In the filtration $\mathcal{F}$, X is a Markov process with transition probabilities the M_k's. By isomorphic transfer and immersion, so are also X' and X'' in $\bar{\mathcal{F}}$.

Fix $k \geqslant k_0$, fix x' and x'' in Ω_{2^k} such that $x' \neq x''$. Take an arbitrary $\bar{\mathcal{F}}_{-2^k}$-measurable event A included in $\{X'_{-k} = x', X''_{-k} = x''\}$, introduce the new probability $\bar{\mathbf{P}}^A = \bar{\mathbf{P}}[\ \ |A]$, and observe that the density $\bar{\mathbf{P}}^A/d\bar{\mathbf{P}}$ is $\bar{\mathcal{F}}_{-2^k}$-measurable. By Corollary 1, $X'_{-k+1}=(\varepsilon'_{-2^k+1},\dots,\varepsilon'_{-2^{k-1}})$ and $X''_{-k+1}=(\varepsilon''_{-2^k+1},\dots,\varepsilon''_{-2^{k-1}})$ are $\bar{\mathbf{P}}^A$-immersed in the filtration $(\bar{\mathcal{F}}_{-2^k+1},\bar{\mathcal{F}}_{-2^k+1},\dots,\bar{\mathcal{F}}_{-2^{k-1}})$. Since their respective laws under $\bar{\mathbf{P}}^A$ are $M_{k-1}(x')$ and $M_{k-1}(x'')$, two different probabilities in $\mathcal{Q}_{2^{k-1}}$, property (iii) of Corollary 4 gives $\bar{\mathbf{P}}^A[X'_{-k+1} = X''_{-k+1}] \leqslant 2^{-(k-2)/4}$. As A is an arbitrary $\bar{\mathcal{F}}_{-2^k}$-measurable event included in $\{X'_{-k} = x', X''_{-k} = x''\}$, this implies

$$\bar{\mathbf{P}}[X'_{-k+1} = X''_{-k+1}|\bar{\mathcal{F}}_{-2^k}] \leqslant 2^{-(k-2)/4} \quad \text{on } \{X'_{-k} = x', X''_{-k} = x''\}\ . \qquad \blacksquare$$

REFERENCES

[1] M.T. Barlow. M. Émery, F.B. Knight. S. Song & M. Yor. Autour d'un théorème de Tsirelson sur des filtrations browniennes et non browniennes. *Séminaire de Probabilités XXXII.* Lecture Notes in Mathematics 1686. Springer 1998.

[2] P. Brémaud & M. Yor. Changes of filtrations and of probability measures. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 45*, 269–295, 1978.

[3] C. Dellacherie, B. Maisonneuve & P. A. Meyer. Probabilités et Potentiel. Chapitres XVII à XXIV : Processus de Markov (fin), Compléments de calcul stochastique. Hermann. 1992.

[4] L. Dubins, J. Feldman, M. Smorodinsky & B. Tsirelson. Decreasing sequences of σ-fields and a measure change for Brownian motion. *Ann. Prob. 24*, 882–904. 1996.

[5] J. Feldman & B. Tsirelson. Decreasing sequences of σ-fields and a measure change for Brownian motion. II *Ann. Prob. 24*, 905–911, 1996.

[6] R.K. Getoor & M.J. Sharpe. Conformal martingales. *Inventiones math. 16*, 271–308, 1972.

[7] L. Gross. Logarithmic Sobolev inequalities and contractive properties of semigroups. *Dirichlet Forms,* Lecture Notes in Mathematics 1563, Springer 1993.

[8]. S. Janson. Gaussian Hilbert Spaces. Cambridge Tracts in Mathematics 129. Cambridge University Press 1997.

[9] T. Jeulin & M. Yor. Nouveaux résultats sur le grossissement des tribus. *Ann. Scient. E.N.S. 11*, 429–443, 1978.

[10] T. Lindvall & L.C.G. Rogers. Coupling of multidimensional diffusions by reflexion. *Ann. Prob. 14*, 860–872, 1986.

[11] E. Nelson. The free Markov field. *J. Funct. Anal. 12*, 211–227, 1973.

[12] J. Neveu. Sur l'espérance conditionnelle par rapport à un mouvement brownien. *Ann. Inst. Henri Poincaré, Section B, 12*, 105–109, 1976.

[13] D. Revuz & M. Yor. Continuous Martingales and Brownian Motion. Springer, 1991.

[14] W. Schachermayer. On certain probabilities equivalent to Wiener measure, d'après Dubins, Feldman, Smorodinsky and Tsirelson. In this volume.

[15] J.-P. Thouvenot. Private communication.

[16] B. Tsirelson. Triple points: From non-Brownian filtrations to harmonic measures. *GAFA, Geom. funct. anal. 7*, 1096–1142, 1997.