SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

JEAN JACOD
VICTOR PÉREZ-ABREU

On martingales which are finite sums of independent random variables with time dependent coefficients

Séminaire de probabilités (Strasbourg), tome 31 (1997), p. 62-68
<http://www.numdam.org/item?id=SPS_1997__31__62_0>

© Springer-Verlag, Berlin Heidelberg New York, 1997, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

On martingales which are finite sums of independent random variables with time dependent coefficients

Jean Jacod
and
Víctor Pérez-Abreu

1 Introduction

We consider the following problem: for a positive integer $n \geq 1$, let $U_1, ..., U_n$ be n independent, integrable, centered, non-degenerate random variables. We are looking for conditions on a family of n càdlàg functions $f_1, ..., f_n$ on $\mathbb{R}_+$ with $f_i(0) = 0$, under which the following process:

$$X_t = \sum_{i=1}^{n} f_i(t) U_i \tag{1}$$

is a martingale, with respect to its own filtration $(\mathcal{F}_t)_{t \geq 0}$.

This (apparently) simple problem has a general solution given in Section 1. However, the answer is not quite satisfactory, since for example it does not allow to recognize whether there is a unique (up to the obvious multiplication by constants and time-changes) set (f_i) meeting our condition.

To get more insight, we specialize in Section 3 to the case where $n = 2$ and (for the most interesting results) with U_1 and U_2 having the same law. In this very particular situation we are able to give a complete description of all martingales of the form (1). This description emphasizes the particular role played by the stable distributions.

For the case $n \geq 3$, we have been unable to provide any interesting result of the same kind as for $n = 2$.

2 A general result

Here is a general theorem solving (in principle) our problem.

Theorem 1. *The process X is a martingale if and only if it satisfies the following:*

Condition [M]: *There are an integer p, $0 \leq p \leq n$, and deterministic times $0 = T_0 < T_1 < ... < T_p < T_{p+1} = \infty$, and p linearly independent vectors $a_j = (a_j^i)_{1 \leq i \leq n}$ in $\mathbb{R}^n$ (when $p \geq 1$), such that, with $V_0 = 0$ and $V_j = \sum_{1 \leq i \leq n} a_j^i U_i$ for $j \geq 1$,*

(M1) $(V_j)_{0\le j\le p}$ *is a discrete-time martingale;*

(M2) $X_t = \sum_{1\le j\le p} V_j 1_{[T_j,T_{j+1})}(t)$.

Before proving this theorem, we state some remarks on the conditions. First, Condition (M2) implies that $f_i(t) = \sum_{1\le j\le p} a_j^i 1_{[T_j,T_{j+1})}(t)$, because of the following property:

$$\alpha_i, \beta_i \in I\!R, \quad \sum_{i=1}^{n} \alpha_i U_i = \sum_{i=1}^{n} \beta_i U_i \quad a.s. \quad \Rightarrow \quad \alpha_i = \beta_i \quad \forall i. \tag{2}$$

Second, Condition (M1) is obviously difficult to verify, except when $p = 0$ (it is void) and $p = 1$ (it is obvious because V_1 is centered). Below we give an equivalent condition based on the characteristic functions φ_i of U_i. We recall that each function φ_i is C^1 with $\varphi_i'(0) = 0$. Then, when $p \ge 2$, (M1) is equivalent to the following:

Condition (M'1). For all $1 \le l \le p-1$ and all v_j in $I\!R$,

$$\sum_{i=1}^{n} (a_{l+1}^i - a_l^i)\varphi_i'(\sum_{j=1}^{l} a_j^i v_j) \prod_{k\ne i} \varphi_k(\sum_{j=1}^{l} a_j^k v_j) = 0. \tag{3}$$

We observe that (3) is the same as $E((V_{l+1} - V_l)\exp i\sum_{j=1}^{l} v_j V_j) = 0$. When the φ_i's do not vanish (so $\varphi_i = \exp\psi_i$ with ψ_i of class C^1 and $\psi_i'(0) = 0$) this condition is also equivalent to:

Condition (M"1). For all $1 \le l \le p-1$ and all v_j in $I\!R$,

$$\sum_{i=1}^{n} (a_{l+1}^i - a_l^i)\psi_i'(\sum_{j=1}^{l} a_j^i v_j) = 0. \tag{4}$$

Proof. The sufficient condition is obvious. For the necessary condition, we suppose that X is a martingale and let $F(t)$ be the vector with components $(f_i(t))_{1\le i\le n}$. Denote by E_t the linear space spanned by $(F(s) : s \le t)$, let $d_t = \dim(E_t)$, $T_{-1} = -1$, $T_j = \inf(t : d_t \ge j)$ for $0 \le j \le n$, and $T_{n+1} = \infty$. Thus $T_{-1} < 0 = T_0 \le T_1 \le ... \le T_p < T_{p+1} = \infty$ for some $0 \le p \le n$, and $d_0 = 0$.

Let $0 \le i \le p$ with $T_i < T_{i+1}$ and consider s, t such that $T_i < s < t < T_{i+1}$. Then $E_t = E_s$ is spanned by the linearly independent vectors $F(s_1), ..., F(s_i)$ with $s_j \le s$ (if $i = 0$, then $E_t = E_s = \{0\}$). Therefore, X_s and X_t are $\sigma(X_{s_1}, ..., X_{s_i})$-measurable and thus $\mathcal{F}_s = \mathcal{F}_t = \sigma(X_{s_1}, ..., X_{s_i})$ (which is the trivial σ-field when $i = 0$). The martingale property $E(X_t|\mathcal{F}_s) = X_s$ yields $X_t = X_s$ a.s., and (2) gives $F(s) = F(t)$. It follows that $F(.)$ is constant on (T_i, T_{i+1}) as well as on $[T_i, T_{i+1})$ by right-continuity. Thus

$$T_i < T_{i+1} \qquad \Rightarrow \qquad d_r = i \quad \forall r \in [T_i, T_{i+1}). \tag{5}$$

In fact $0 < T_1 < ... < T_p$; otherwise we would be in one of the following two situations:

a) $0 = T_j < T_{j+1}$ for some $1 \leq j \leq p$, and therefore $d_{T_j} = d_0 = 0$, which contradicts (5);

b) $T_{i-1} < T_i = T_j < T_{j+1}$ for i, j with $1 \leq i < j \leq p$, in which case $d_r = i - 1$ on $[T_{i-1}, T_j)$ by (3). This implies that $d_{T_j} \leq i$; being also impossible since $d_{T_j} \geq j$.

Since $0 < T_1 < ... < T_p$ holds, we trivially have (M2) with $a_j = F(T_j)$. Finally, (M2) and the martingale property of X yield (M1). □

3 The case n=2

Let φ_i be the characteristic function of U_i, and when φ_i never vanishes we use the notation $\varphi_i = \exp \psi_i$ without further comment. In this section we always assume that $n = 2$.

Theorem 2. *The process X is a martingale if and only if it has one of the following two (mutually exclusive) representations:*

a) For some $\alpha, \beta \in \mathbb{R}$, $S_1, S_2 \in (0, \infty]$

$$X_t = \alpha U_1 1_{[S_1, \infty)}(t) + \beta U_2 1_{[S_2, \infty)}(t). \tag{6}$$

b) For some $0 < T_1 < T_2 < \infty$, $\alpha, \alpha', \gamma, \gamma' \in \mathbb{R}^*$ *with* $\gamma \neq \gamma'$ *and*

$$\varphi_1'(v)\varphi_2(\gamma v) + \gamma' \varphi_1(v)\varphi_2'(\gamma v) = 0 \quad \forall v \in \mathbb{R}, \tag{7}$$

$$X_t = \alpha(U_1 + \gamma U_2)1_{[T_1, \infty)}(t) + \alpha'(U_1 + \gamma' U_2)1_{[T_2, \infty)}(t). \tag{8}$$

Remark. Since the coefficients in (8) do not vanish, the form (8) is indeed symmetric in (U_1, U_2). When φ_1 and φ_2 do not vanish, (7) is equivalent to $\psi_1'(v) + \gamma' \psi_2'(\gamma v) = 0$, which is the same as $\psi_1(v) + \frac{\gamma'}{\gamma} \psi_2(\gamma v) = 0$, which in turn is equivalent to

$$\varphi_1(v) = \varphi_2(\gamma v)^{-\gamma'/\gamma} \quad \forall v \in \mathbb{R}. \tag{9}$$

Proof. Sufficient condition: That (a) gives a martingale is obvious. Condition (b) implies (M2) with $a_1^1 = \alpha$, $a_1^2 = \alpha\gamma$, $a_2^1 = \alpha' + a_1^1$, $a_2^2 = \alpha'\gamma' + a_1^2$ and then (7) gives (M'1).

Necessary condition: We assume (M'1) and (M2). If $T_1 = \infty$, then (a) holds with $\alpha = \beta = 0$ and S_i arbitrary. If $T_1 < T_2 = \infty$, then (a) holds with $\alpha = a_1^1$, $\beta = a_1^2$ and $S_1 = S_2 = T_1$.

Suppose now that $T_1 < T_2 < \infty$. We have $a_1 \neq 0$, and since both (a) and (b) are symmetric in (U_1, U_2), without lost of generality we assume that $a_1^1 \neq 0$. Let $\alpha = a_1^1$ and $\gamma = a_1^2/\alpha$ and write $a_2^i = a_1^i + \beta^i$. Then the linear independence between a_1 and a_2 gives

$$\beta^2 \neq \gamma \beta^1, \tag{10}$$

while (M'1) is

$$\beta^1 \varphi_1'(v)\varphi_2(\gamma v) + \beta^2 \varphi_1(v)\varphi_2'(\gamma v) = 0 \quad \forall v \in \mathbb{R}. \tag{11}$$

We assume first that $\gamma = 0$. Recalling that $\varphi_i(0) = 1$, $\varphi_i'(0) = 0$ and φ_i' is not identically 0 in any neighborhood of 0 (because $P(U_i = 0) < 1$), (11) yields $\beta^1 = 0$, that is, we have (a) with $S_1 = T_1$, $S_2 = T_2$, $\beta = \beta^2$.

Next, assume that $\gamma \neq 0$. Then there exists $\theta \in \mathbb{R}^*$ with $\varphi_1'(\theta) \neq 0$, $\varphi_1(\theta) \neq 0$ and $\varphi_2(\gamma\theta) \neq 0$. Suppose for the time being that $\varphi_2'(\gamma\theta) = 0$. Then (11) yields $\beta^1 = 0$ and since there is another $\theta' \in \mathbb{R}^*$ with $\varphi_1(\theta') \neq 0$ and $\varphi_2(\gamma\theta') \neq 0$, we also have $\beta^2 = 0$, which contradicts (10). Thus $\varphi_2'(\gamma\theta) \neq 0$ and (10) and (11) yield $\beta^1 \neq 0$ and $\beta^2 \neq 0$. Hence we have (b) with $\gamma' = \beta^2/\beta^1$ and $\alpha' = \beta^1$ (note that $\gamma \neq \gamma'$ follows from (10), and (7) is the same as (11)). □

When U_1 and U_2 are arbitrary, it seems there is not much more to say. From now on we concentrate on the case where $U_1 =^d U_2$, i.e. $\varphi_1 = \varphi_2 = \varphi$. In this situation, the existence of a martingale X of the form (b) above depends on the existence of constants $\gamma, \gamma' \in \mathbb{R}^*$ with $\gamma \neq \gamma'$ and

$$\varphi'(v)\varphi(\gamma v) + \gamma'\varphi(v)\varphi'(\gamma v) = 0 \quad \forall v \in \mathbb{R}. \tag{12}$$

Let D denote the set of all $\gamma \in \mathbb{R}^*$ for which (12) holds for some $\gamma' \in \mathbb{R}^*$ with $\gamma' \neq \gamma$. If $\gamma \in D$ there is a unique $\gamma' = \delta(\gamma)$ satisfying (12), because we have seen before that for each $\gamma \neq 0$ there is $v \in \mathbb{R}$ with $\varphi(v) \neq 0$ and $\varphi'(\gamma v) \neq 0$.

Theorem 3. a) *If U_1 is symmetric about 0, then one of the following three cases is satisfied:*

(Cs-1) $D = \{-1, 1\}$.

(Cs-2) $D = \{r^n, -r^n : n \in \mathbb{Z}\}$ *for some* $r > 1$ *and* φ *never vanishes.*

(Cs-3) $D = \mathbb{R}^*$. *This is the case if and only if U_1 is stable with index $\rho \in (1, 2]$, i.e.* $\varphi(u) = e^{-a|u|^\rho}$ *for some* $a > 0$.

b) *If U_1 is not symmetric about 0, we are in one of the following five situations:*

(Ca-1) $D = \{1\}$.

(Ca-2) $D = \{-1, 1\}$. *This is the case if and only if* $\varphi = \rho e^{\eta}$, *where ρ and η are real-valued, $\eta(0) = 0$, and η is constant on each open interval on which ρ (or φ) does not vanish (necessarily φ vanishes somewhere, and η is not identically 0, otherwise we would be in the symmetric case).*

(Ca-3) $D = \{r^n : n \in \mathbb{Z}\}$ *for some* $r > 1$ *and* φ *never vanishes.*

(Ca-4) $D = \{r^n, -r^{n+1/2} : n \in \mathbb{Z}\}$ *for some* $r > 1$ *and* φ *never vanishes.*

(Ca-5) $D = (0, \infty)$. *This is the case if and only if U_1 is asymmetric strictly stable with index $\rho \in (1, 2)$, i.e.,* $\varphi(u) = e^{-a|u|^\rho(1+ib\,sign(u))}$ *for some* $a > 0$, $b \neq 0$, $|b| \leq \tan(\frac{\pi}{2(2-\rho)})$.

c) *There is a constant $\theta \in (1, 2]$ such that $\delta(\gamma) = -\gamma/|\gamma|^\theta$ (so $\delta(1) = -1$, and $\delta(-1) = 1$ if $-1 \in D$), and $\theta = \rho$ in cases (Cs-3) and (Ca-5).*

Therefore the martingales X of the form (8) are indeed represented as

$$X_t = \alpha(U_1 + \gamma U_2)1_{[T_1,\infty)}(t) + \alpha'(U_1 - \gamma U_2/|\gamma|^\theta)1_{[T_2,\infty)}(t), \tag{13}$$

where $\alpha, \alpha' \in \mathbb{R}^*$, $0 < T_1 < T_2 < \infty$, *and* $\gamma \in D$.

Remark. There are of course examples of variables satisfying (Cs-1) or (Cs-3) in the symmetrical case, (Ca-1) in the asymmetrical case. We presume that (Cs-2) and (Ca-3) are not empty, and believe that (Ca-2) is empty (but we have been unable to prove these facts).

Before giving the proof of Theorem 3 we present some useful lemmas. First we note that $\gamma = 1$ and $\gamma' = -1$ always satisfy (12), so $1 \in D$ and $\delta(1) = -1$.

Lemma 4. *We have* $-1 \in D$ *if and only if* $\varphi = \rho e^{\eta}$, *where* ρ *and* η *are real-valued and* $\eta(0) = 0$ *and* η *is constant on each open interval on which* ρ *(or* φ*) does not vanish. Moreover,* $\delta(-1) = 1$.

Proof. Let (x, y) be a maximal interval on which φ does not vanish, so φ does not vanish either on $(-y, -x)$ (we may have $(x, y) = \mathbb{R}$, of course). We can write $\varphi = e^{\psi}$ with ψ of class C^1 on (x, y) and $(-y, -x)$, and since $\psi(-v) = \overline{\psi(v)}$ the property $-1 \in D$ and (12) yield

$$\psi'(v) = \gamma' \overline{\psi'(v)} \quad \forall v \in (x, y).$$

Since $\gamma' \in \mathbb{R}^*$, we deduce that $\psi'(v) \in \mathbb{R}$ and thus $\gamma' = 1$ (because ψ' cannot be identically 0). Therefore, if $v_0 \in (x, y)$, we have $\psi(v) - \psi(v_0) \in \mathbb{R}$ for all $v \in (x, y)$ and hence $\varphi = \rho e^{\eta}$ with $\eta(v) = \eta(v_0) \in \mathbb{R}$ for all $v \in (x, y)$. The converse is obvious.□

Lemma 5. *Let* $\gamma \in \mathbb{R}^*$ *with* $|\gamma| \neq 1$. *Then* $\gamma \in D$ *if and only if* φ *does not vanish, and satisfies for some* $C(\gamma) > 0$

$$\varphi(v) = \varphi(\gamma v)^{C(\gamma)} \quad \forall v \in \mathbb{R}. \tag{14}$$

Moreover,

a) $\mathcal{R}e\psi(v) < 0$ *for all* $v \in \mathbb{R}^*$.

b) $\delta(\gamma) = -\gamma C(\gamma)$.

c) *For all* $n \in \mathbb{Z}$ *we have* $\gamma^n \in D$ *and* $C(\gamma^n) = C(\gamma)^n$.

d) $-\gamma \in D$ *if and only if* φ *is real-valued, and then* $C(-\gamma) = C(\gamma)$.

Proof. The sufficient condition is obvious, as well as (b).

Conversely, assume that $\gamma \in D$. Let $(-x, x)$ be the maximal interval on which φ does not vanish. We have $\varphi = e^{\psi}$ with ψ of class C^1 on $(-x, x)$. For simplicity we set $\psi_r = \mathcal{R}e\psi$, and we have $\psi_r(u) \to -\infty$ as $|u| \uparrow x$ if $x < \infty$. On $(-x, x)$, (12) yields $\psi'(v) + \gamma'\psi'(\gamma v) = 0$, so $\psi(v) + \frac{\gamma'}{\gamma}\psi(\gamma v) = 0$, since $\psi(0) = 0$.

If $|\gamma| > 1$ and $x < \infty$, then $|\psi_r(v)| = |\frac{\gamma'}{\gamma}||\psi_r(\gamma v)| \to \infty$ as $|v| \uparrow x/|\gamma|$, contradicting the fact that ψ is continuous on $(-x, x)$. Similarly, if $|\gamma| > 1$ and $x < \infty$, $|\psi_r(\gamma v)| = |\frac{\gamma}{\gamma'}||\psi_r(v)| \to \infty$ as $|v| \uparrow x$, bringing up the same contradiction; therefore $x = \infty$, and φ does not vanish. It follows that $\varphi = e^{\psi}$ everywhere and, with $C(\gamma) = -\gamma'/\gamma$,

$$\psi(v) = C(\gamma)\psi(\gamma v) \quad \forall v \in \mathbb{R}, \tag{15}$$

that is, we have (14). Since U_1 is non-degenerate, ψ is not identically 0 and thus $C(\gamma) \neq 0$. Note also that (c) is obvious from (14).

We always have that $\psi_r \leq 0$ and that ψ_r is even. Assume that $\psi_r(v) = 0$ for some $v > 0$. Then (15) and (c) imply $\psi_r(v|\gamma|^n) = 0$ for all $n \in \mathbb{Z}$. It follows that the characteristic fonction of the symmetrized random variable $U = U_1 - U_2$ equals 1 for all $v|\gamma|^n$, $n \in \mathbb{Z}$, so U is supported by $\{2k\pi/v|\gamma|^n : k \in \mathbb{Z}\}$, for all $n \in \mathbb{Z}$, which implies that $U = 0$ a.s., contradicting again the non-degeneracy assumption. Thus (a) holds and (15) yields $C(\gamma) > 0$.

Finally, it only remains to prove (d). If φ is real-valued, it is even and (14) is satisfied with $-\gamma$ and $C(-\gamma) = C(\gamma)$. Suppose conversely that $-\gamma \in D$, then (15) gives $\overline{\psi(v)} = C(\gamma)\psi(-\gamma v)$, while $-\gamma \in D$ yields $\psi(v) = C(-\gamma)\psi(-\gamma v)$. Comparing the real parts of these two equalities and using (a) we obtain $C(-\gamma) = C(\gamma)$. Then $\overline{\psi} = \psi$ and φ is real-valued. □

Lemma 6. *With $D_+ = D \cap \mathbb{R}_+$, one of the following three cases is satisfied:*

(C_+1) $D_+ = \{1\}$.

(C_+2) $D_+ = \{r^n : n \in \mathbb{Z}\}$ *for some* $r > 1$.

(C_+3) $D_+ = \mathbb{R}_+^*$.

Moreover, we are in case (C_+3) if and only if either $\varphi(u) = e^{-a|u|^2}$ for some $a > 0$ or $\varphi(u) = e^{-a|u|^\rho(1+ibsign(u))}$ for some $a > 0$, $\rho \in (1,2)$, $|b| \leq \tan(\frac{\pi}{2(2-\rho)})$.

Proof. Due to the fact that $1 \in D$ and to Lemma 5, if we are not in case (C_+1), D_+ contains at least a $\gamma > 0$, $\gamma \neq 1$, and then $\varphi = e^\psi$ satisfies (14). Indeed, D_+ is the set of all $\gamma > 0$ such that (15) holds for some $C(\gamma) > 0$. Then D_+ is clearly a multiplicative group, therefore it is closed since ψ is continuous and thus it is of the form (C_+2) or (C_+3).

Assuming (C_+3), for each $\gamma > 0$ there is $C(\gamma) > 0$ such that, if f denotes either the real or the imaginary part of ψ, we have $f(0) = 0$ and

$$f(v) = C(\gamma)f(\gamma v) \quad \forall v \geq 0.$$

Then f is either identically 0, or everywhere positive, or everywhere negative, on $(0, \infty)$. In the last two cases, $g(u) = \log|f(e^u)/f(1)|$ satisfies $g(u + \log\gamma) = g(u) + g(\log\gamma)$ for all $u \in \mathbb{R}$, $\gamma > 0$, i.e., $g(u + u') = g(u) + g(u')$ for all $u, u' \in \mathbb{R}$. Since g is continuous, we obtain $g(u) = Ku$. Thus, in all cases we have $f(v) = \eta v^\rho$ for some $\eta, \rho \in \mathbb{R}$, and furthermore $\gamma^\rho C(\gamma) = 1$ for all $\gamma > 0$ (hence ρ is the same for both the real and imaginary parts of ψ). We then deduce that $\psi(v) = (\alpha + i\beta)v^\rho$ for some $\alpha, \beta, \rho \in \mathbb{R}$, if $v > 0$. By (a) of Lemma 5 we have $\alpha < 0$ and since $\psi(-v) = \overline{\psi(v)}$, we also have $\psi(v) = (\alpha - i\beta)|v|^\rho$ for $v < 0$. Then $\psi(v) = -a|v|^\rho(1 + ibsign(v))$ for $a > 0$, $b \in \mathbb{R}$, $\rho \in \mathbb{R}$. Conversely, each such ψ satisfies (15) for all $\gamma > 0$, with $C(\gamma 1) = \gamma^{-\rho}$, implying $D_+ = \mathbb{R}_+^*$.

It remains to examine under which conditions on (a, b, ρ) the function $\varphi = e^\psi$ with ψ as above is a characteristic function. Observe that for all $\alpha, \alpha' > 0$ we have $\psi(\alpha v) + \psi(\alpha' v) = \psi(\alpha'' v)$ with $\alpha''^\rho = \alpha^\rho + \alpha'^\rho$. Then, if it is the case, the corresponding distribution will be strictly stable, with a first moment equal to 0. As is well known,

this will be the case if and only if either $\rho = 2$ and $b = 0$ (normal case), or $\rho \in (1,2)$ and $|b| \leq \tan(\frac{\pi}{2(2-\rho)})$. □

Proof of Theorem 3. a) When U_1 is symmetric, so is D, and (Cs-i) = (C_+i). Therefore Lemma 5 yields that one of (Cs-1), (Cs-2) or (Cs-3) is satisfied. Moreover, (Cs-2) implies that φ never vanishes (by Lemma 5), and (Cs-3) holds if and only if $\varphi(v) = e^{-a|v|^\rho}$ (because here φ is real-valued).

b) Now we suppose that U_1 is not symmetric. It suffices to prove that if $D \neq \{1\}$, then we are in one of the cases (Ca-i) for i=2,3,4,5.

First, by Lemma 4, $-1 \in D$ if and only if the necessary and sufficient condition in (Ca-2) is satisfied. Then φ vanishes somewhere, and D contains no γ with $|\gamma| \neq 1$ by Lemma 5. Thus $-1 \in D$ if and only if (Ca-2) holds.

Next, suppose that we are not in any of the cases (Ca-1) and (Ca-2). If $D = D_+$, we are then in cases (Ca-3) or (Ca-5) by Lemma 5. Otherwise there exists $\gamma > 0$ with $\gamma \neq 1$ and $-\gamma \in D$. Then $\gamma^2 \in D$ and $\gamma^2 \neq 1$ and by Lemma 5 either (C_+2) or (C_+3) holds. However, under (C_+3) we also have $\gamma \in D$, hence Lemma 5(d) contradicts the assumption that U_1 is non-symmetric and indeed we have (C_+2) with some $r > 1$. It then follows that $\gamma^2 = r^k$ for some $k \in \mathbb{N}^*$, while Lemma 5(c) gives $C(r^n) = C(r)^n$ and $C(\gamma) = C(r)^{k/2}$. Furthermore if k were even we would have $r^{k/2} \in D$ and $-r^{k/2} = -\gamma \in D$, again a contradiction by Lemma 5(d), so $k = 2p+1$ with $p \in \mathbb{Z}$ and $\gamma = r^{p+1/2}$. In order to obtain (Ca-4), it thus remains to prove that $-r^{n+1/2} \in D$ for all $n \in \mathbb{Z}$. For this, a repeated use of (15) yields

$$\psi(v) = C(r^{n-p})\psi(r^{n-p}v) = C(r^{n-p})C(\gamma)\psi(-\gamma r^{n-p}v) = C(r)^{n+1/2}\psi(-r^{n+1/2}),$$

and the result follows.

c) Since $\delta(1) = -1$ and $\delta(-1) = 1$ (Lemma 4), (c) is obvious in cases (Cs-1), (Ca-1) and (Ca-2). Also, (c) with $\theta = \rho$ follows from Lemma 5(b) and from a comparison between (14) and the explicit form of φ in cases (Cs-3) and (Ca-5).

Under (Ca-4) we have seen that $C(r^n) = C^n$ and $C(-r^{n+1/2}) = C^{n+1/2}$, where $C = C(r)$. Thus, for all $\gamma \in D$, $C(\gamma) = C^{\log(|\gamma|)/\log(r)} = |\gamma|^{-\theta}$ with $\theta = -\log(C)/\log(r)$ (so $\delta(\gamma) = -\gamma/|\gamma|^\theta$). The same holds for (Cs-2) and (Ca-3). Now (15) yields $\psi'(v) = Cr\psi'(rv)$, hence $\psi'(r^{-n}v) = (Cr)^n\psi'(v)$ and since $\psi'(0) = 0$ and $\psi'(v) \neq 0$ for some $v \neq 0$ we must have $Cr < 1$ and thus $\theta > 1$.

Suppose that $\theta > 2$, i.e. $A := Cr^2 < 1$. Then $\psi'(r^{-n}v)/(r^{-n}v) = A^n\psi'(v)/v$ and if $|w| \leq r^{-n}$ there is $m \geq n$ and $v \in (1/r, 1]$ with $w = vr^{-m}$ or $w = -vr^{-m}$. Therefore $\sup_{|w|\leq r^{-n}} |\psi'(w)/w| \leq A^n \sup_{1/r<|v|\leq 1} |\psi'(v)|$. It follows that ψ' is differentiable at 0, with $\psi''(0) = 0$. Hence U_1 is square-integrable, with variance 0, which contradicts once more the non-degeneracy assumption and therefore $\theta \leq 2$. □

J. Jacod: Laboratoire de Probabilités (CNRS, URA 224), Université Paris VI, Tour 56, 4, Place Jussieu, 75252 Paris Cedex 05, France.

V. Pérez-Abreu: Department of Probability and Statistics, Centro de Investigación en Matemáticas A. C., Apdo. Postal 402, Guanajuato, Gto. 36000, México