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A HORIZONTAL LEVY PROCESS ON THE BUNDLE OF
ORTHONORMAL FRAMES OVER A COMPLETE
RIEMANNIAN MANIFOLD

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ABSTRACT

We establish an existence theorem for a class of SDE's driven by Lévy processes on a manifold. As an application we consider an SDE driven by horizontal vector fields on the orthonormal frame bundle of a Riemannian manifold. The canonical projection of the solution of this equation on to the base is considered as a candidate for a "Lévy process on a Riemannian manifold".

1) Introduction

A Lévy process in $\mathbb{R}^n$ is essentially a stochastic process with independent and stationary increments. All the random variables comprising such a process are infinitely divisible. Conversely, as was shown by Itô, any infinitely divisible random variable can be embedded in a Lévy process (see [Itô], theorem 3.1). Hence Lévy processes can be characterised by the Lévy-Khintchine formula, through their characteristic functions. Alternatively, at the level of

random variables we have the Lévy-Itô decomposition which exhibits every Lévy process as a combination of a Brownian motion, a Poisson point process (suitably renormalised) and a drift. Itô's formula then extends this decomposition to C^2 functions of the process (see e.g. [IkWa] for details). An obvious generalisation of the above would be to replace $\mathbb{R}^n$ by an arbitrary Lie group G . A major advance in this direction was the work of Hunt in 1956. He showed that there was a one to one correspondence between convolution semigroups of probability measures p on G and a class of linear operators on $C^2(G)$, the correspondence being that each such operator generates a Markov semigroup with kernel p [Hun]. This can be shown to be equivalent to the Lévy-Khintchine formula when $G = \mathbb{R}^n$. More recently, H.Kunita and the present author have obtained an analogue of the Lévy-Itô decomposition for smooth functions of such processes. As in the abelian case the decomposition is obtained with the aid of a Brownian motion, a Poisson random measure and a drift ([ApKu]).

The aim of the present paper is to begin the work of generalising the above ideas to a Riemannian manifold M . The procedure we adopt herein is to imitate the well-known Eels-Elworthy construction for obtaining Brownian motion on a manifold M by canonical projection of a suitable process in the bundle of orthonormal frames $O(M)$ (see e.g. [Elw], [IkWa]). An existence theorem for solutions of SDE's in compact manifolds has been established by Fujiwara however as $O(M)$ is not compact we cannot use this result herein. §2 of this paper is then devoted to proving a general existence result for a class of SDE's driven by Lévy processes on not-necessarily compact manifolds. In §3 we specialise to the case of $O(M)$ and construct a process which we call a horizontal Lévy process on $O(M)$ which satisfies an SDE driven by a Lévy process taking values in the horizontal vector fields. We note that there is some similarity here with recent work by A.Estrade and M.Pontier who have constructed the horizontal lift of a manifold-valued càdlàg semimartingale [EsPo]. We wish to go in the opposite direction and obtain a

Lévy process on the manifold as the canonical projection of the horizontal Lévy process.

We make two observations

(a) Intuitively a Lévy process on M is a combination of a drift, a Brownian motion on M and a Poisson point process which is constrained to jump along geodesics of arbitrary length. In order to ensure that there is a rich supply of the latter we will assume that M is geodesically complete.

(b) Brownian motion on a manifold is obtained by projection of the appropriate frame bundle-valued process on to the base manifold and is characterised by its generator (see e.g. [Eme] p. 62) which is of course the Laplace-Beltrami operator. In our case, the operator which is our natural candidate to be the generator of a "Lévy process on a manifold" exhibits a manifest time dependence which indicates that our process is not, in general, Markovian.

Note: - After writing this paper, it was brought to my attention that the existence and uniqueness result of §2 is in fact a special case of a more general construction given in [Coh]. I have however retained my original proof as I think there is some value in showing that the elegant method of [Elw] extends to the case of SDE's with jumps.

Notation: If M is a manifold, $\text{Diff}(M)$ is the group of all diffeomorphisms of M with identity id . Every complete vector field Y on M generates a one-parameter subgroup of $\text{Diff}(M)$ which we denote as $\{\text{Exp}(tY), t \in \mathbb{R}\}$. If S is a topological space, $\mathcal{B}(S)$ will denote the Borel σ -algebra of S and $C_0(S)$ is the space of continuous functions on S which vanish at ∞ . Einstein summation convention will be used throughout.

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2. Existence of Solutions to an SDE on A Manifold Driven by a Lévy Process

Let V be a d -dimensional connected paracompact smooth manifold and let $Y_1, \dots, Y_n$ be smooth complete vector fields on V . We denote by $\mathcal{T}$ the linear span of $\{Y_1, \dots, Y_n\}$ and make the assumption that every element of $\mathcal{T}$ is complete. We note that this condition is automatically satisfied if the Lie algebra $\mathcal{L}$ generated by $\mathcal{T}$ is finite-dimensional, but we do not make this latter assumption here. For each $x \in \mathbb{R}^n$, we denote by $\xi(x)$ the diffeomorphism of V defined by

$$\xi(x) = \text{Exp}(x^j Y_j)$$

Let X be an n -dimensional Lévy process on some complete probability space $(\Omega, \mathfrak{F}, P)$ equipped with a filtration $(\mathfrak{F}_t, t \in \mathbb{R}^+)$. Hence there exists an m -dimensional Brownian motion $B = (B(t), t \in \mathbb{R}^+)$ where $m \leq n$ and a Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^n - \{0\})$ which is independent of B and has associated Lévy measure ν on $\mathbb{R}^n - \{0\}$ given by $E(N(t, G)) = t \nu(G)$ for all $t \in \mathbb{R}^+, G \in \mathcal{B}(\mathbb{R}^n - \{0\})$, such that $X = (X^1, \dots, X^n)$ has Lévy-Itô decomposition

$$\begin{aligned} X^j(t) = & c^j t + \sigma_k^j B^k(t) + \int_0^{t+} \int_{|x| \geq 1} x^j N(dt, dx) \\ & + \int_0^{t+} \int_{|x| < 1} x^j \tilde{N}(dt, dx) \end{aligned} \quad \dots (2.1)$$

for $1 \leq j \leq n, t \in \mathbb{R}^+$

Here $c = (c^1, \dots, c^n) \in \mathbb{R}^n$, $\sigma = (\sigma_k^j)$ is a real $(n \times m)$ matrix and $\tilde{N}$ is the compensated process $\tilde{N}(t, G) = N(t, G) - t \nu(G)$.

We introduce the $\mathcal{T}$ -valued Lévy process $X_g = (X_g(t), t \in \mathbb{R}^+)$ given by

$$X_{\mathcal{G}}(t) = X^j(t) Y_j \quad \text{for } t \in \mathbb{R}^+ \quad \dots(2.2)$$

We aim to study SDE's on V driven by $X_{\mathcal{G}}$.

To the extent that we are thus attempting to construct a "stochastic exponential" we might write such an SDE as

$$d\phi(t) = dX_{\mathcal{G}}(t)(\phi(t-)) \quad \dots(2.3)$$

with $\phi(0) = p$ a.s.

More precisely we are seeking a unique càdlàg adapted process $\phi = (\phi(t), 0 \leq t \leq \sigma)$ taking values in V with explosion time $\sigma \leq \infty$ which satisfies the stochastic integro-differential equation

$$\begin{aligned} f(\phi(t)) &= f(p) + \int_0^t (Z_0 f)(\phi(s-)) ds + \int_0^t (Z_k f)(\phi(s-)) \cdot dB^k(s) \\ &\quad + \int_0^{t+} \int_{|x| \geq 1} [f(\xi(x)\phi(s-)) - f(\phi(s-))] N(ds, dx) \\ &\quad + \int_0^{t+} \int_{|x| < 1} [f(\xi(x)\phi(s-)) - f(\phi(s-))] \tilde{N}(ds, dx) \\ &\quad + \int_0^{t+} \int_{|x| < 1} [f(\xi(x)\phi(s-)) - f(\phi(s-)) - x^j Y_j f(\phi(s-))] \nu(dx) ds \end{aligned} \quad \dots(2.4)$$

for each $f \in C^{\infty}(V)$, $t \in \mathbb{R}^+$. Note that we have, for convenience, introduced the notation

$$Z_0 = c^j Y_j, \quad Z_k = \sigma_k^j Y_j \quad (1 \leq k \leq m)$$

For further examination of the relationship between (2.3) and (2.4), see pages 1105-6 of [ApKu].

Note: One gains a nice understanding of how (2.4) arise from (2.3) from the discretisation result of S.Cohen (§III.3 of [Coh]). Taking $0 = T_0^n < T_1^n < \dots < T_k^n \rightarrow \infty$ to be a sequence of stochastic partitions whose mesh tends to zero, it is shown that the solution of (2.4), up to its explosion time, is obtained by taking limits of the sequence defined by

$$\begin{aligned}\phi_0(t) &= p \\ \phi_n(t) &= \xi((X(t) - X(T_k^n-))\phi_n(T_k^n-)) \quad \text{for } T_k^n \leq t < T_{k+1}^n\end{aligned}$$

In the case where $\dim \mathcal{L} < \infty$, (2.4) was solved in [ApKu] and it was shown that the solution defines a Lévy process on the transformation Lie group associated to $\mathcal{L}$ with $\sigma = \infty$ a.s. (see also [Est]). When V is compact equations of a similar type to (2.4) were studied in [Fuj] and the flat case $V = \mathbb{R}^d$ can be found in [FuKu]. Further examples of classes of SDE's with jumps are investigated in [Rog] and [Coh].

Now consider (2.4) in the case $V = \mathbb{R}^d$ and write

$Y_j(x) = a_j^i(x)\partial_i$ for $1 \leq j \leq n$, $1 \leq i \leq d$ where each $a_j^i \in C^\infty(\mathbb{R}^d)$. The following result is established in §2.2 of [ApKu].

Theorem 2.1 Suppose that a_j^i , $\partial_k(a_j^i)$ and $\partial_k\partial_l(a_j^i)$ are bounded on $\mathbb{R}^d$ for all $1 \leq j \leq n$, $1 \leq i, k, l \leq d$, then (2.4) has a unique solution on $\mathbb{R}^d$.

The main result in this section is the following,

Theorem 2.2 There exists a unique maximal solution to (2.4) on V .

Proof By Whitney's embedding theorem we can smoothly embed V into $\mathbb{R}^b$ where $b = 2d + 1$. Consider the corresponding extended version of (2.4) as an SDE on $\mathbb{R}^b$. Provided that the extension can be carried out in such a way that the hypothesis of theorem 2.1 is satisfied, we immediately have existence and uniqueness for the extended equation. It must now be shown that

a solution of the extended SDE which has initial condition in V never leaves it. To do this we must construct the extension in a careful way. Our method is very closely based on that used by Elworthy in [Elw].

We begin by considering an equation closely related to (2.4)

$$\begin{aligned}
 f(\zeta(t)) &= f(p) + \int_0^t (Z_0 f)(\zeta(s-)) ds + \int_0^t (Z_j f)(\zeta(s-)) \circ dB^j(s) \\
 &\quad + \int_0^{t+} \int_{|x| < 1} [f(\xi(x)\zeta(s-)) - f(\zeta(s-))] \tilde{N}(ds, dx) \\
 &\quad + \int_0^{t+} \int_{|x| < 1} [f(\xi(x)\zeta(s-)) - f(\zeta(s-)) - x^j Y_j f(\zeta(s-))] \nu(dx) ds \\
 &\hspace{15em} \dots(2.5)
 \end{aligned}$$

for $f \in C^\infty(V)$, $t \in \mathbb{R}^+$.

We will first show that (2.5) has a unique maximal solution on V . We recall some notation from [Elw]. Let $a: V \rightarrow \mathbb{R}^+$ be smooth and let $N(V)$ denote the normal bundle in $\mathbb{R}^b$ with base V and canonical projection π . If S is an open set in V , we define

$$M_a(S) = \bigcup_{q \in S} \{ y \in \mathbb{R}^b; \|y - q\| < a(q) \}$$

Note that $\{ y \in N(V), \|y\| < a(\pi(y)) \}$ is diffeomorphic to the tubular neighborhood $M_a(V)$ in $\mathbb{R}^b$.

Now let G_0 be an open neighborhood of $p \in V$ with compact closure in $\mathbb{R}^b$. We denote as $B_R(0)$ the open ball of radius R about the origin in $\mathbb{R}^b$.

Choose $R > 0$ and (following [Elw]) define

$$a_R = \inf\{a(q), q \in G_0 \cap B_{R+1}(0)\},$$

Let $\lambda_R \in C^\infty(\mathbb{R}^b, \mathbb{R}^+)$ be such that $\text{supp}(\lambda_R) \subseteq B_{R+1}(0)$ and $\lambda_R = 1$ in $B_R(0)$ and let $\mu_R \in C^\infty(\mathbb{R}^+)$ be such that $\mu_R(y) = 1$ if $|y| \leq \frac{1}{2} a_R^2$ and $\mu_R(y) = 0$ if $|y| > a_R^2$.

Let $\gamma : M_a(V) \rightarrow V$ be defined by $\gamma(y) = q$ where

$$||y - q|| = \inf\{||y - r||, r \in V\}.$$

Again as in [Elw], we extend the vector fields Y_j on G_0 to smooth vector fields $\overline{Y}_j$ on the whole of $\mathbb{R}^b$ ($1 \leq j \leq n$) by

$$\overline{Y}_j(p) = 0, \quad p \notin M_a(G_0)$$

$$\overline{Y}_j(p) = \lambda_R(p) \mu(d(p, V)^2) Y_j(\gamma(p)), \quad p \in M_a(G_0)$$

where d is the usual (Euclidean) metric in $\mathbb{R}^b$.

Define the extended diffeomorphisms $\overline{\xi}(x) = \text{Exp}(x^j \overline{Y}_j)$. (For ease of notation, we have suppressed the dependence of $\overline{Y}_j$ and $\overline{\xi}(x)$ on R). We may now use theorem 2.1 to assert the existence and uniqueness of the extended SDE on $\mathbb{R}^b$.

Now choose $S > R$ such that

$$S > \sup_{|x| < 1} \sup_{q \in G_0} |\xi(x) - G_0|$$

and let $g_S \in C^\infty(\mathbb{R}^b)$ be given by

$$g_S(y) = \lambda_S(y) \mu_S(d(y, V)^2) d(y, V)^2 \quad ^1$$

Now consider equation (2.5) with $f = g_S$. As in [Elw], we have

$$\overline{Y}_j(q) g_S(q) = 0 \quad \text{for all } q \in B_R(0), 1 \leq j \leq n.$$

Define for $0 \leq t \leq 1$, $|x| < 1$ the automorphisms $j_t(x)$ of $C^\infty(\mathbb{R}^b)$ by

$$j_t(x)(f) = f \circ \overline{\xi}(tx)$$

¹I am grateful to David Elworthy for this correction to [Elw].

then we have that

$$\frac{dj_t(x)f}{dt} = x^j j_t(x)(\overline{Y}_j(f))$$

However each $j_t(x)(\overline{Y}_j(g_s))(q) = \overline{Y}_j(\xi(tx)q) g_s(\xi(tx)q) = 0$ for $q \in B_R(0)$, since g_s is constant on the level sets of the map $q \rightarrow d(\xi(tx)q, V)^2$ for each $|x| < 1$, $0 \leq t \leq 1$, thus

$$g_s(\xi(tx)q) = g_s(q) \quad \text{for each } q \in B_R(0).$$

Hence from (2.5) we obtain $g_s(\zeta(t)) = 0$ whenever $p \in G_0$. As R is arbitrary, we find that $\zeta(t)$ remains in G_0 for all $0 \leq t < \sigma_0$ where $\sigma_0 \leq \sigma$.

Suppose that $\sigma_0 < \sigma$, then take G_1 to be an open neighborhood with compact closure of $\zeta(\sigma_0)$ and repeat the above argument wherein $\zeta(\sigma_0)$ replaces p in (2.5). We thus obtain a new extension of the equation on $\mathbb{R}^b$ which yields a solution of (2.5) which lies in G_1 for $\sigma_0 \leq t < \sigma_1 \leq \sigma$. We continue in this fashion to obtain $\zeta(t) \in M$ for all $0 \leq t < \sigma$.

This solution is clearly maximal. Uniqueness follows from that of the extended equation on each open neighborhood. We conclude by constructing the unique maximal solution of (2.4).

Let $\rho = (\rho^1, \dots, \rho^n)$ be the Poisson point process defined by

$$\rho^j(t) = \Delta \left(\int_{|x| \geq 1} x^j N(dt, dx) \right) \quad \text{for } 1 \leq j \leq n \text{ and let}$$

$\xi(\rho(t)) = \text{Exp}(\rho^j(t) Y)$ for $t \in \mathbb{R}^+$. Define the random diffeomorphisms $\chi(t) = \xi(\rho(t))$ of V for $t \in \mathbb{R}^+$ and let

$\{\tau_n; 0 \leq n \leq N\}$ be the jump times of ρ . We now proceed to construct ϕ as in [FuKu] p.84. Hence for $0 \leq t < \tau_1$, define $\phi(t) = \zeta(t)$; for $t = \tau_1$, define $\phi(\tau_1) = \chi(\tau_1)(\zeta(\tau_1-))$ and for $\tau_1 < t < \tau_2$ we define $\phi(t) = \hat{\zeta}(t)$ where $\hat{\zeta}(t)$ is the solution of (2.5) with initial condition $\phi(\tau_1)$. We thus proceed inductively to define $\phi(t)$ for all $0 \leq t \leq \sigma$ □

Note: As indicated in the introduction, theorem 2.2 may also be proved by appealing to corollary 2 in §III.2 of [Coh] and

taking the map Φ therein from $\mathbb{R}^n \times V \times \mathbb{R}^n \rightarrow V$ as

$$\Phi(a, p, b) = \xi(b - a)(p)$$

3 Horizontal Lévy Processes and Lévy Processes on Manifolds

We begin this section by collecting some geometrical facts which can all be found in [KoNo].

Let M be an n -dimensional connected, paracompact Riemannian manifold and denote by $O(M)$ the bundle of orthonormal frames over M with canonical projection $\pi: O(M) \rightarrow M$. Let $r = (r_1, \dots, r_n) \in O(M)$ with $\pi(r) = p$ then we will also denote by π the induced linear map from $T_r(O(M))$ onto $T_p(M)$. Let $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, then r may be regarded as a linear map from $\mathbb{R}^n$ onto $T_p(M)$ with the action

$$r(x) = x^j r_j$$

We equip M with its unique Riemannian connection so that at each $r \in O(M)$ we have the decomposition

$$T_r(O(M)) = H_r(O(M)) \oplus V_r(O(M))$$

where H_r and V_r comprise the horizontal and vertical vectors at r (respectively). Note that each $\dim(H_r(O(M))) = n$. For each $x \in \mathbb{R}^n$ there exists a canonical horizontal vector field $L(x)$ on $O(M)$ which has the following properties

- (i) L is smooth and each $L(x)(r) \in H_r(O(M))$,
- (ii) $\pi(L(x)(r)) = r(x)$

We assume from now on that each $L(x)$ is complete. M is then said to be geodesically complete. For each $p \in M$, let $\exp$ denote the exponential mapping from $T_p(M)$ into M , then if $\pi(r) = p$ we have

$$\pi(\exp(t L(x)(r))) = \exp(t r(x))(p) \quad \dots (3.1)$$

for all $t \in \mathbb{R}$. The right hand side of (3.1) is the unique geodesic through p in the direction $r(x) \in T_p(M)$.

We fix an orthonormal basis $e_1, \dots, e_n$ in $\mathbb{R}^n$ and write $L_j = L(e_j)$ for $1 \leq j \leq n$ so that $L(x) = x^j L_j$ for each $x \in \mathbb{R}^n$. We now study the SDE (2.4) in the following context: take V to be $O(M)$ and each $Y_j = L_j$ ($1 \leq j \leq n$). From the above discussion we see immediately that each member of $\mathcal{T}$ is complete as is required.

By theorem 2.2 we can assert the existence of a unique maximal solution to the SDE

$$\begin{aligned} f(r(t)) &= f(r) + \int_0^t (Z_0 f)(r(s-)) ds + \int_0^t (Z_j f)(r(s-)) \cdot dB^j(s) \\ &\quad + \int_0^{t+} \int_{|x| \geq 1} [f(\text{Exp}(L(x))r(s-)) - f(r(s-))] N(ds, dx) \\ &\quad + \int_0^{t+} \int_{|x| < 1} [f(\text{Exp}(L(x))r(s-)) - f(r(s-))] \tilde{N}(ds, dx) \\ &\quad + \int_0^{t+} \int_{|x| < 1} [f(\text{Exp}(L(x))r(s-)) - f(r(s-)) - L(x)f(r(s-))] \nu(dx) ds \end{aligned}$$

...(3.2)

for each $f \in C^\infty(O(M))$, $r \in O(M)$, $t \in \mathbb{R}^+$.

We call the solution $(r(t), 0 \leq t \leq \sigma)$ of (3.2) a horizontal Lévy process in $O(M)$. It is interesting to compare the form of (3.2) with the formula obtained by A.Estrade and M.Pontier for the horizontal lift of a manifold-valued càdlàg semimartingale in proposition 4.3 of [EsPo].

We note that if $\nabla T = 0$ where ∇ is the covariant derivative and T is the torsion tensor field then $\dim(\mathcal{L}) < \infty$ ([KoNo] p.137). In this case we have $\sigma = \infty$ a.s. by the results of [ApKu].

Now let $(T_t, t \in \mathbb{R}^+)$ be the Markov semigroup on $C_0(O(M))$

defined by

$$T_t(f)(r) = E(f(r(t)) / r(0) = r) \quad \dots(3.3)$$

for $f \in C_0(O(M))$, $r \in O(M)$.

If N denotes its infinitesimal generator we have $C^\infty(O(M)) \subset \text{Dom}(N)$ and a standard calculation yields

$$\begin{aligned} N(f)(r) &= m^j L_j(f)(r) + \frac{1}{2} a^{ij} L_i L_j(f)(r) \\ &+ \int_{\mathbb{R}^n - \{0\}} [f(\text{Exp}(L(x)(r))) - f(r) - \frac{x^j}{1 + |x|^2} L_j f(r)] \nu(dx) \end{aligned} \quad \dots(3.4)$$

for $f \in C^\infty(O(M))$, $r \in O(M)$, where $a = (a^{ij})$ is the non-negative definite matrix $\sigma \sigma^T$ and for $1 \leq j \leq n$

$$m^j = c^j - \int_{|x| \geq 1} \frac{x^j}{1 + |x|^2} \nu(dx) + \int_{|x| < 1} \frac{x^j |x|^2}{1 + |x|^2} \nu(dx)$$

We may regard (3.4) as a horizontal Lévy-Khintchine-Hunt formula on $O(M)$ (see [Hun], [ApKu]).

We now consider the càdlàg process $X = (X(t), 0 \leq t < \sigma)$ on M defined by $X(t) = \pi(r(t))$. To investigate X , we define the linear operator $\mathcal{A}(r)$ on $C^\infty(M)$ by

$$\mathcal{A}(r) g(p) = N(g \circ \pi)(r) \quad \dots(3.5)$$

where $g \in C^\infty(M)$, $r \in O(M)$ and $p = \pi(r)$.

Using (3.1), we then obtain the following for $g \in C^\infty(M)$, $p = \pi(r)$,

$$\begin{aligned} \mathcal{A}(r)(g)(p) &= m^j R_j(g)(p) + \frac{1}{2} \Delta_a(g)(p) \\ &+ \int_{\mathbb{R}^n - \{0\}} [g(\exp(x^j R_j)(p)) - g(p) - \frac{x^j}{1 + |x|^2} R_j(g)(p)] \nu(dx) \end{aligned} \quad \dots(3.6)$$

where $R_j = r(e_j) \in T_p(M)$ and Δ_a is the linear operator on $C^\infty(M)$ defined by

$$\Delta_a \circ \pi = \pi \circ a^{ij} L_i L_j \quad \dots (3.7)$$

We call (3.6) the Lévy-Khintchine-Hunt formula on the Riemannian manifold M. $\mathcal{A}(r)$ is clearly independent of choice of orthonormal basis $(R_1, R_2, \dots, R_n)$ for $T_p(M)$ and hence is independent of the choice of lift r of p to $O(M)$ in (3.5), however we still retain the frame r in the notation for reasons that will become clear below.

To get a clearer insight into the nature of Δ_a , we work in local co-ordinates in $O(M)$. Let $r(p) = (p^i, e_j^i)$, and ∇_i be the covariant derivative in the direction ∂_i then a straightforward calculation yields (c.f [IkWa] p.260 - 274),

$$\Delta_a = g_a^{ij} \nabla_i \nabla_j$$

where $g_a^{ij} = a^{kl} e_k^i e_l^j$.

If the matrix a is positive-definite, $g_a = (g_a^{ij})$ is an (inverse) metric tensor on M . In this case we say that the generator $\mathcal{A}(r)$ is non-degenerate. We note that if σ is the matrix of an orthogonal transformation (so a is the identity matrix) g_a is the original (inverse) Riemannian metric on M and Δ_a is the Laplace-Beltrami operator.

The frame dependence of the operator $\mathcal{A}$ indicates that the semigroup $(T_t, t \in \mathbb{R}^+)$ on $C_0(O(M))$ does not, in general, project to a semigroup on $C_0(M)$ so that X is not, in general, a Markov process. Clearly one could define X to be a "Lévy process on a manifold" whenever it is indeed a Markov process. Alternatively, it might be argued that this is not a natural generalisation of the Euclidean case. Clearly further work on this question is required.

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