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ANTICIPATIVE CALCULUS FOR THE POISSON PROCESS BASED ON THE FOCK SPACE

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1. Introduction.

The stochastic anticipative calculus for the Brownian motion has been developed recently by several authors [**6**],[**7**]. This stochastic calculus is based on the Skorohod integral δ, which is known to be the adjoint operator of the derivative operator D on the Wiener space [**3**]. There are some basic properties of the Skorohod integral and of the derivative operator which can be expressed in terms of the Wiener chaos expansion. This fact leads in a natural way, to study the behaviour of those operators on a different context like the Poisson case. More generally, these operators can be defined on an arbitrary Fock space associated with a Hilbert space H.

The aim of this note is to present some properties of these operators D and δ on a general Fock space, and to analyze their behaviour when the Fock space is interpreted as a Poisson space.

Sections 2 to 4 are devoted to the study of the operators D and δ on a general Fock space. The particular case of the Poisson process is considered in sections 5 to 7. The main results are the interpretation of the derivative operator as a translation given in section 6, and the representation of the operator δ as a Stieltjes integral on the predictable processes, obtained in sections 4 and 7.

For related works see the papers [**11**] by L.Wu and [**2**] by A.Dermoune and al.

2. The Fock Space.

Let H be a real separable Hilbert space. Consider the n-th tensorial product $H^{\otimes n}$. Let $\mathcal{S}_n$ be the set of permutations of $\{1,2,\ldots,n\}$. Any permutation $\sigma \in \mathcal{S}_n$ induces an automorphism over $H^{\otimes n}$, given by

$$U_\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

We denote by $H^{\odot n}$ the Hilbert space of symmetric tensors, that means, which are invariant under any automorphism U_σ.
In $H^{\odot n}$ we consider the modified norm,

$$\| f \|^2_{H^{\odot n}} = n! \| f \|^2_{H^{\otimes n}}.$$

DEFINITION 2.1. *The Fock space associated to H is the Hilbert space*

$$\Phi(H) = \bigoplus_{n=0}^{\infty} H^{\odot n},$$

equipped with the inner product

$$\langle h, g \rangle_{\Phi(H)} = \sum_{n=0}^{\infty} \langle h_n, g_n \rangle_{H^{\odot n}},$$

if $h = \sum_{n=0}^{\infty} h_n$ *and* $g = \sum_{n=0}^{\infty} g_n$. *Here we take* $H^{\odot 0} = \mathbf{R}$ *and* $H^{\odot 1} = H$.

The most interesting case is $H = L^2(T)$, where $(T, \mathcal{B}, \lambda)$ is a separable, σ-finite and atomless measure space. In this case $H^{\otimes n}$ is isometric to $L^2(T^n)$. Moreover, $H^{\odot n}$ is the space of square-integrable symmetric functions $L_s^2(T^n)$ with the modified norm $\| \cdot \|_{H^{\odot n}}$.

There are different representations of the Fock space $\Phi(L^2(T))$ as an L^2-space, which produce several useful interpretations of the elements of the Fock space. For example:

a) Let $W = \{W(B), B \in \mathcal{B}, \lambda(B) < +\infty\}$ be a Brownian measure on $(T, \mathcal{B}, \lambda)$, that means a zero-mean Gaussian process with covariance given by $E[W(B_1)W(B_2)] = \lambda(B_1 \cap B_2)$, defined in some probability space $(\Omega, \mathcal{F}, P)$, and assume that $\mathcal{F}$ is generated by W. In this case, $\Phi(L^2(T))$ is the L^2-space of the Wiener functionals via the Wiener-chaos decomposition:

$$F \in L^2(\Omega, \mathcal{F}, P) \Rightarrow F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L_s^2(T^n).$$

b) Similar representations can be obtained using a Poisson process.

3. The Derivative Operator on the Fock Space.

Let $H = L^2(T, \mathcal{B}, \lambda)$ be as in the previous section, and consider the associated Fock space $\Phi(H)$. For every $F \in \Phi(H)$, $F = \sum_{n \geq 0} f_n$, we define the derivative of F, DF as the element of $\Phi(H) \bigotimes H \cong L^2(T; \Phi(H))$ given by

$$D_t F = \sum_{n=1}^{\infty} n f_n(\cdot, t), \quad \text{for a.e. } t.$$

provided that sum converges in $L^2(T; \Phi(H))$. That means DF exists if

$$\| DF \|_{L^2(T;\Phi(H))}^2 = \int_T \| D_t F \|_{\Phi(H)}^2 \; \lambda(dt)$$

$$= \sum_{n=1}^{\infty} n^2 (n-1)! \int_T \| f_n(\cdot, t) \|_{L^2(T^{n-1})}^2 \; \lambda(dt)$$

$$= \sum_{n=1}^{\infty} n\, n! \; \| f_n \|_{L^2(T^n)}^2 < +\infty.$$

It is easy to check that D is an unbounded and closed operator on $\Phi(H)$. We will denote the domain of D by $\mathcal{D}$, which is a dense subspace of $\Phi(H)$.

For any h$\in L^2(T)$, we can also define a closed and unbounded operator D_h from $\Phi(H)$ to $\Phi(H)$, by

$$D_h F = \sum_{n=1}^{\infty} n \int_T f_n(\cdot, t)\, h(t)\, \lambda(dt),$$

provided that this series converges in $\Phi(H)$.

The domain $\mathcal{D}_h$ of D_h is the set of elements $F = \sum_{n=0}^{\infty} f_n$ such that

$$\sum_{n=0}^{\infty} \int_T n^2 \parallel f_n(\cdot, t)h(t) \parallel^2_{L^2(T^{n-1})} (n-1)!\, \lambda(dt) < +\infty.$$

By the Schwarz inequality, it is clear that $\mathcal{D}_h \supset \mathcal{D}$ for all h.
Although in a general Fock space we don't have a sample space or σ-fields, we can introduce a generalized notion of adaptability, and also, a generalized conditional expectation. These notions are intrinsic in the sense that they do not depend on the particular representation of the Fock space.

DEFINITION 3.1. *Let $F \in \Phi(L^2(T, \mathcal{B}, \lambda))$ be given by $F = \sum_{n=0}^{\infty} f_n$, and consider a subset of T, $A \in \mathcal{B}$. We will say that F is $\mathcal{F}_A$-measurable if for any $n \geq 1$ we have $f_n(t_1, \dots t_n) = 0, \quad \lambda^n -$ a.e. unless $t_i \in A$, $\forall i = 1, \dots, n$.*

Note that in the above definition $\mathcal{F}_A$ is not defined as a σ-field because we don't have a sample space Ω.
In the particular case of a Brownian measure $W(B)$ on $(T, \mathcal{B}, \lambda)$ this definition means that F is measurable w.r.t. the σ-field $\mathcal{F}_A = \sigma\{W(C), C \subset A, C \in \mathcal{B}\}$.
As in the case of Brownian measure we have the following result

LEMMA 3.1. *Let $F \in \mathcal{D}$ be $\mathcal{F}_A$-measurable. Then $D_t F = 0$, for a.e. $t \in A^c$.*

This lemma will help us to generalize the notion of conditional expectation.

DEFINITION 3.2. *For any $F \in \Phi(H)$ and $A \in \mathcal{B}$, we define*

$$E[F|\mathcal{F}_A] = \sum_{n=0}^{\infty} f_n(t_1, \dots, t_n) \cdot 1_A(t_1) \cdots 1_A(t_n).$$

LEMMA 3.2. *For any $F \in \mathcal{D}$, and $A \in \mathcal{B}$ we have*

$$D_t E[F|\mathcal{F}_A] = E[D_t F|\mathcal{F}_A] \cdot 1_A(t), \quad \text{for a.e. } t.$$

4. The operator δ.

Consider the Hilbert space $L^2(T; \Phi(H)) \cong L^2(T) \otimes \Phi(H)$. This Hilbert space can be decomposed into the orthogonal sum $\bigoplus_{n=0}^{\infty} \sqrt{n!} \cdot \hat{L}^2(T^{n+1})$, where $\hat{L}^2(T^{n+1})$ is the subspace of $L^2(T^{n+1})$ formed by all square integrable functions on T^{n+1} which are symmetric in the first n variables.

Let $u \in L^2(T; \Phi(H))$ be given by

$$u = \sum_{n \geq 0} u_n, \quad u_n \in \hat{L}^2(T^{n+1}).$$

We will denote by $\tilde{u}_n$ the symmetrization of u_n with respect to its $n+1$ variables .

We can define the Skorohod integral of u as the element of $\Phi(H)$ given by,

$$\delta(u) = \sum_{n\geq 0} \tilde{u}_n,$$

assuming that

$$\sum_{n\geq 0}(n+1)! \parallel \tilde{u}_n \parallel^2_{L^2(T^{n+1})} < +\infty.$$

We denote by Dom δ the set of elements $u \in L^2(T;\Phi(H))$ verifying the above property. The notion of predictability can also be defined in the general context of the Fock space. This fact has been pointed out to us by P.A. Meyer.

DEFINITION 4.1. *An element $u \in L^2(T;\Phi(H))$ is called an elementary predictable process if*

(1) $$u(t) = F \otimes 1_{A^c}(t),$$

where $F \in \Phi(L^2(T))$ is $\mathcal{F}_A$-measurable, and $A \in \mathcal{B}$.

PROPOSITION 4.1. *If u is an elementary predictable process of the form (1), and $\lambda(A^c) < +\infty$ then $u \in$ Dom δ and*

(2) $$\delta(u) = \sum_{n=0}^{\infty} f_n \tilde{\otimes} 1_{A^c},$$

where $F = \sum_{n\geq 0} f_n$, and $\tilde{\otimes}$ denotes the symmetric tensor product.

PROOF: The above series converge because

$$\sum_{n\geq 0}(n+1)! \parallel f_n \tilde{\otimes} 1_{A^c} \parallel^2_{L^2(T^{n+1})} = \sum_{n\geq 0} n! \parallel f_n \otimes 1_{A^c} \parallel^2_{L^2(T^{n+1})}$$

$$= \parallel F \parallel^2_{\Phi(H)} \lambda(A^c) < +\infty. \quad \blacksquare$$

The right hand side of (2) could be used as an intrinsic definition of the product of F by $\delta(1_{A^c})$, which coincides with the usual product in the Brownian and the Poisson case. This follows from the product formulas for the multiple stochastic integrals. See [4] for the Poisson process. In that sense the Skorohod integral is equal to the ordinary stochastic integral on elementary predictable processes. We will continue this discussion in section 7.

The following results provide the duality relation between the operators D and δ.

PROPOSITION 4.2. *Let $u \in$ Dom δ, and $F \in \mathcal{D}$, then,*

$$\langle u, DF\rangle_{L^2(T;\Phi(H))} = \langle F, \delta(u)\rangle_{\Phi(H)}.$$

PROOF: Suppose that $u = \sum_{n\geq 0} u_n$, and $F = \sum_{n\geq 0} f_n$. Then

$$\langle u, DF\rangle_{L^2(T;\Phi(H))} = \int_T \langle u(\cdot,t), D_t F\rangle_{\Phi(H)}\,\lambda(dt)$$

$$= \sum_{n\geq 0} n! \int_T \langle u_n(\cdot,t), (n+1)\, f_{n+1}(\cdot,t)\rangle_{L^2(T^n)}\,\lambda(dt)$$

$$= \sum_{n\geq 0}(n+1)! \int_{T^{n+1}} u_n(\cdot,t)\, f_{n+1}(\cdot,t)\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt)$$

and using the fact that f_{n+1} is a symmetric function,

$$= \int_{T^{n+1}} \sum_{n\geq 0}(n+1)!\,\tilde{u}_n(\cdot,t)\, f_{n+1}(\cdot,t)\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt) = \langle F, \delta(u)\rangle_{\Phi(H)}. \quad \blacksquare$$

Consequently, δ is the dual operator of D and it is clear that $\operatorname{Dom}\delta$ is dense in $L^2(T;\Phi(H))$, and δ is a closed operator.

We are going to introduce some subsets of $\operatorname{Dom}\delta$: We denote by $\mathcal{L}^2$ the class of elements $u \in L^2(T;\Phi(H))$ such that $u_t \in \mathcal{D}$, for a.e. t. and $D_s u_t \in L^2(T^2;\Phi(H))$. In terms of the Fock expansion this is equivalent to

$$\sum_{n\geq 1} n\,(n+1)!\ \| \tilde{u}_n \|^2_{L^2(T^{n+1})} < +\infty.$$

It is clear that this implies $\mathcal{L}^2 \subset \operatorname{Dom}\delta$.

THEOREM 4.1. *Let u,v be elements of* $\mathcal{L}^2$. *Then we have*

$$\langle \delta(u), \delta(v)\rangle_{\Phi(H)} = \langle u, v\rangle_{L^2(T;\Phi(H))} + \int_{T^2} \langle D_s u_t, D_t v_s\rangle_{\Phi(H)}\,\lambda(dt)\,\lambda(ds).$$

PROOF: We have

$$\langle \delta(u), \delta(v)\rangle_{\Phi(H)} = \sum_{n\geq 0}(n+1)! \int_{T^{n+1}} \tilde{u}_n(\cdot,t)\,\tilde{v}_n(\cdot,t)\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt).$$

On the other hand

$$\langle u, v\rangle_{L^2(T;\Phi(H))} = \int_T \langle u_t, v_t\rangle_{\Phi(H)}\,\lambda(dt)$$

$$= \int_{T^{n+1}} \sum_{n\geq 0} n!\, u_n(\cdot,t)\, v_n(\cdot,t)\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt).$$

The difference between these two terms is

$$\int_{T^{n+1}} \sum_{n\geq 0}[(n+1)!\,\tilde{u}_n(\cdot,t)\,\tilde{v}_n(\cdot,t) - n!\, u_n(\cdot,t)\, v_n(\cdot,t)]\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt)$$

$$= \sum_{n\geq 0}(n+1)! \int_{T^{n+1}} [\tilde{u}_n(\cdot,t)\,\tilde{v}_n(\cdot,t) - \frac{u_n(\cdot,t)\, v_n(\cdot,t)}{n+1}]\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt)$$

$$= \sum_{n\geq 0}(n+1)! \int_{T^{n+1}} u_n(\cdot,t)\,[\tilde{v}_n(\cdot,t) - \frac{v_n(\cdot,t)}{n+1}]\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt)$$

$$= \sum_{n\geq 0} n! \sum_{i=0}^{n} \int_{T^{n+1}} v_n(\cdot,t_i)\,u_n(\cdot,t)\,\lambda(dt_1)\cdots\lambda(dt_n)\,\lambda(dt),$$

then using the symmetry of v_n with respect to the first n variables and putting $s = t_n$ we obtain

$$\sum_{n\geq 1} n\,n! \int_{T^{n+1}} v_n(\cdot,t,s)\,u_n(\cdot,s,t)\,\lambda(dt_1)\cdots\lambda(dt_{n-1})\lambda(ds)\lambda(dt)$$

$$= \int_{T^2} \langle D_s u_t, D_t v_s\rangle_{\Phi(H)}\,\lambda(ds)\,\lambda(dt). \quad \blacksquare$$

THEOREM 4.2. *Let $u \in \mathcal{L}^2$, $D_t u \in \operatorname{Dom}\delta$ and $\delta(D_t u) \in L^2(T;\Phi(H))$. Then $\delta(u) \in \mathcal{D}$ and*

$$D_t\delta(u) = u_t + \delta(D_t u).$$

PROOF: Suppose that $u = \sum_{n\geq 0} u_n$. Then $\delta(u) = \sum_{n\geq 0} \tilde{u}_n$, and

$$D_t\delta(u) = \sum_{n\geq 0}(n+1)\,u_n(\cdot,t) = \sum_{n\geq 0} u_n(\cdot;t) + \sum_{n\geq 1} n\,u_n(\cdot,t) = u_t + \delta(D_t u). \quad \blacksquare$$

The notions and properties we have introduced so far, depend only on the underlying Hilbert space H. There are other concepts which are related to the particular representation of the Fock space, like the product of two elements, the composition of a function with an element of the Fock space or the notion of positivity [**9**]. In the papers [**6**], and [**7**] these notions are developed for the case of the Gaussian representation. In this paper we are going to investigate the behaviour of this notions in the Poisson case.

5. The Poisson space.

Let $(T,\mathcal{B},\lambda)$, be a measure space such that T is a locally compact space with a countable basis and λ is a Radon-measure that charges all the open sets, and that is diffuse over the σ-field $\mathcal{B}$.
Following [**11**], and [**5**] we can define the Poisson space over this measure space by taking

$$\Omega = \{\omega = \sum_{j=0}^{n} \delta_{t_j},\, n \in \mathbb{N}\cup\{\infty\},\, t_j \in T\},$$

$$\mathcal{F}_0 = \sigma\{p_A : p_A(\omega) = \omega(A),\, A \in \mathcal{B}\}.$$

and P the probability measure defined over $(\Omega,\mathcal{F}_0)$ in such a way that

i) $P(p_A = k) = e^{-\lambda(A)}\frac{\lambda(A)^k}{k!}$, where $k \geq 0$, and $A \in \mathcal{B}$.

ii) $\forall A, B \in \mathcal{B}$ with $A \cap B = \emptyset$, p_A and p_B are P-independent.

Finally we denote by $\mathcal{F}$ the completion of $\mathcal{F}_0$ with respect to P.
Note that:

$$P(\{\omega \in \Omega : \omega \text{ is a Radon-measure}\}) = 1,$$

and

$$P(\{\omega \in \Omega : \exists t \in T : \omega(\{t\}) > 1\}) = 0.$$

It is well-known that we can define a multiple stochastic integral with respect to a compensated Poisson measure. If $f \in L^2(T^n)$, $n \geq 1$ we denote the multiple stochastic integral of f by

$$I_n(f) = \int_{T_*^n} f(t_1, \cdots, t_n)(\omega - \lambda)(dt_1)\cdots(\omega - \lambda)(dt_n),$$

where $\omega - \lambda$ is the compensated Poisson measure, and

$$T_*^n = \{(t_1, \cdots, t_n) \in T^n : t_i \neq t_j \forall i \neq j\}.$$

This integral has the following properties:

1. $I_n(f) = I_n(\tilde{f})$, where $\tilde{f}$ is the symmetrization of f.

2. $I_n(f) \in L^2(\Omega, \mathcal{F}, P)$

3. $\langle I_n(f), I_m(g)\rangle_{L^2(\Omega)} = n! \cdot \langle \tilde{f}, \tilde{g}\rangle_{L^2(T^n)} \cdot 1_{\{n=m\}}$.

It is also possible to define this integral via the Charlier polynomials [8]

Consider the Hilbert space $H = L^2(T)$. Then we have

$$\| I_n(f) \|_{L^2(\Omega)} = \| f \|_{\odot n},$$

where $\| \cdot \|_{\odot n} = \sqrt{n!} \, \| f \|_{L^2(T^n)}$ is the modified norm of $H^{\odot n}$ introduced in section 1 . We also set $I_0(c) = c$, $\forall c \in R$. Then we have the orthogonal decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n \geq 0} C_n$$

with

$$C_n = \{I_n(f), f \in H^{\odot n}\},$$

which provides an isometry between $L^2(\Omega, \mathcal{F}, P)$ and the Fock space $\Phi(L^2(T))$. We have therefore an interpretation of the Fock space as a space of L^2-functionals over the Poisson space. This interpretation is different from that obtained in the Wiener case. We will study in this setting, the operators D and δ.

6. A Translation Operator.

Consider the following application on the set Ω of point measures over $(T, \mathcal{B}, \lambda)$

$$\Psi_t(\omega) = \omega + \delta_t.$$

This application is well-defined from Ω to Ω, and if $\Omega_0 = \{\omega : \omega(t) \leq 1, \forall t \in T\}$, it is clear that $\Psi_t(\Omega_0) \subset \Omega_0$ a.s. for every fixed t, because $P\{\omega : \omega(t) = 1\} = 0$. Therefore the mapping Ψ induces a transformation of the Poisson functionals defined by

$$(\Psi_t(F))(\omega) = F(\omega + \delta_t) - F(\omega).$$

The next result gives a product formula for the translation operator Ψ_t, and it will be useful later:

LEMMA 6.1. *Let F, G be functionals over Ω. Then*

$$\Psi_t(F \cdot G) = F \cdot \Psi_t(G) + G \cdot \Psi_t(F) + \Psi_t(F) \cdot \Psi_t(G).$$

PROOF:

$$\Psi_t(F \cdot G) = F(\omega + \delta_t)\, G(\omega + \delta_t) - F(\omega)\, G(\omega)$$

$$= F(\omega + \delta_t)\, G(\omega + \delta_t) - F(\omega + \delta_t)\, G(\omega) + F(\omega + \delta_t)\, G(\omega) - F(\omega)\, G(\omega)$$

$$= F(\omega + \delta_t)\, \Psi_t(G) + \Psi_t(F)\, G(\omega)$$

$$= (F(\omega + \delta_t) - F(\omega))\, \Psi_t(G) + \Psi_t(F)\, G + F\, \Psi_t(G)$$

$$= F\, \Psi_t(G) + G\, \Psi_t(F) + \Psi_t(F)\, \Psi_t(G). \quad \blacksquare$$

The next result shows that the operators Ψ_t and D_t coincide.

THEOREM 6.2. *For every $F \in \mathcal{D}$, $\Psi_t(F) = D_t F$ a.s., for a.e. t.*

PROOF: We will do the proof by induction, using a formula of Yu. M. Kabanov [4]

a) Suppose first that F is an element of the first chaos, that means

$$F = I_1(f) = \int_T f(t)(\omega - \lambda)(dt) = \sum_i f(t_i) - \int_T f(t)\, \lambda(dt)$$

where $f \in L^2(T)$. Then

$$\Psi_t(F) = \sum_i f(t_i) + f(t) - \int_T f(t)\, \lambda(dt) - \sum_i f(t_i) + \int_T f(t)\, \lambda(dt)$$

$$= f(t) = D_t F.$$

b) We recall the following formula proved by Yu. M. Kabanov in [4], which is the Poisson version of the product formula for the multiple stochastic integrals

$$(3) \qquad I_{k+1}(\varphi \otimes g) = I_k(\varphi) \cdot I_1(g) - \sum_{j=1}^{k} I_k(\varphi *_{(j)} g) - \sum_{j=1}^{k} I_{k-1}(\varphi \times_{(j)} g),$$

where $\varphi \in L^2(T^k)$, $g \in L^2(T)$ and

$$(\varphi \otimes g)(t_1, \cdots t_k, t) = \varphi(t_1, \cdots, t_k)\, g(t)$$

$$(\varphi *_{(j)} g)(t_1, \cdots t_k) = \varphi(t_1 \cdots, t_j, \cdots, t_k)\, g(t_j),$$

and

$$(\varphi \times_{(j)} g)(t_1, \cdots, \hat{t}_j, \cdots, t_k) = \int_T \varphi(t_1, \cdots, t_k)\, g(t_j)\, \lambda(dt_j).$$

By induction on k we will show that

$$(4) \qquad D_t I_k(f_1 \otimes \cdots \otimes f_k) = \Psi_t I_k(f_1 \otimes \cdots \otimes f_k), \quad \forall k \geq 1,$$

assuming that $f_1, f_2, \cdots, f_k$ are orthogonal elements of $L^2(T)$. By (a) this formula is true if $k = 1$. Suppose that it holds up to k, and let us compute, using the orthogonality of the f_k, Lemma 6.1, formula (3) and the induction hypothesis

$$\Psi_t I_{k+1}(f_1 \otimes \cdots \otimes f_{k+1}) = \Psi_t(I_k(f_1 \otimes \cdots \otimes f_k) I_1(f_{k+1}))$$

$$- \sum_{j=1}^{k} \Psi_t(I_k([f_1 \otimes \cdots \otimes f_k] *_{(j)} f_{k+1}))$$

$$= I_k(f_1 \otimes \cdots \otimes f_k) f_{k+1}(t) + D_t I_k(f_1 \otimes \cdots \otimes f_k) I_1(f_{k+1}) + D_t I_k(f_1 \otimes \cdots \otimes f_k) f_{k+1}(t)$$

$$- \sum_{j=1}^{k} D_t I_k([f_1 \otimes \cdots \otimes f_k] *_{(j)} f_{k+1})$$

$$= I_k(f_1 \otimes \cdots \otimes f_k) f_{k+1} + I_1(f_{k+1}) \sum_{j=1}^{k} f_j(t) I_{k-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_k)$$

$$- \sum_{j=1}^{k} \sum_{l=1, l \neq j}^{k} f_l(t) \cdot I_{k-1}(f_1 \otimes \cdots \otimes \hat{f}_l \otimes \cdots \otimes (f_j f_{k+1}) \otimes \cdots \otimes f_k).$$

On the other hand,

$$D_t I_{k+1}(f_1 \otimes \cdots \otimes f_{k+1}) = \sum_{j=1}^{k+1} f_j(t) I_k(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_{k+1}).$$

Then it suffices to show that

$$\sum_{j=1}^{k} f_j(t) \cdot I_k(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_{k+1})$$

$$= I_1(f_{k+1}) \sum_{j=1}^{k} f_j(t) I_{k-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_k)$$

$$- \sum_{j=1}^{k} \sum_{l=1, l \neq j}^{k} f_l(t) I_{k-1}(f_1 \otimes \cdots \otimes \hat{f}_l \otimes \cdots \otimes (f_j f_{k+1}) \otimes \cdots \otimes f_k),$$

and this follows from

$$I_k(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_{k+1}) = I_{k-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_k) I_1(f_{k+1})$$

$$- \sum_{l=1, l \neq j} I_{k-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes (f_l f_{k+1}) \otimes \cdots \otimes f_{k+1}).$$

c) Formula (4) holds for every k and for every function $f_k \in L^2_S(T^k)$ by a continuous argument.

d) Finally if F is an L^2-limit of a sequence of F_n and everyF_n is a sum of stochastic integrals, then F is the limit almost surely of a partial subsequence F_{n_j}. Consequently $\Psi_t(F)$ is the limit almost surely of $\Psi_t(F_{n_j})$, for a.e. t. On the other hand $D_t F_{n_j}$ converges weakly to $D_t F$ because D is a closed operator. Then $D_t F = \Psi_t F$ (a.s.). ∎

As an application we will compute the derivative of the discontinuity times of a Poisson process. Let $T = [0,1]$, and let $S_1, S_2, \ldots, S_n$, be the jump times of the standard Poisson process over T. We are going to calculate the transformation $\Phi_t S_i = S_i(\omega + \delta_t) - S_i(\omega)$

Then we have $\Phi_t S_i = 0$ if $t > S_i$, $\Phi_t S_i = t - S_i(\omega)$, if $S_{i-1} < t < S_i$ and $\Phi(S_i) = S_{i-1} - S_i$ if $t < S_{i-1}$.

Therefore, $\Phi_t S_i(\omega) = S_{i-1} 1_{\{t < S_i\}} + t 1_{\{S_{i-1} < t < S_i\}} - S_i 1_{\{t < S_i\}}$, and for example for $i = 1$, $\Phi_t S_1 = (t - S_1) 1_{[0,S_1]}(t)$.

Note that this expression is completely different from the results of [1]. Remember that in [1] the operator D is introduced as a real derivative operator with respect to some scale parameter, and this gives the possibility to stablish a Malliavin calculus on this Poisson space. Note also that this operator Ψ is not local as it follows from the following example. Suppose that F and G are functionals that coincide and are equal to zero over the subset $\{N(1/2) = N(1)\}$ and take the values 1 and 2 respectively on the complementary subset. For all $t > 1/2$ we have $D_t F = 1$ and $D_t G = 2$.

7.The Skorohod integral over the Poisson space.

Let u be a process of $L^2(T; L^2(\Omega))$, i.e. taking values in the Poisson space. Clearly, by the Poisson-Wiener expansion we have

$$u_t = \sum_{n \geq 0} I_n(f_n(t_1, \ldots, t_n; t)),$$

for a.e. t. where $f_n(\cdot; t)$ is a function of $L^2_s(T)$.

We define

$$\delta(u) = \sum_{n\geq 0} I_{n+1}(\tilde{f}_n(t_1,\ldots,t_n;t))$$

provided that

$$\sum_{n\geq 0}(n+1)! \parallel \tilde{f}_n(t_1,\ldots,t_n,t) \parallel^2_{L^2(T^{n+1})} < +\infty$$

where

$$\tilde{f}_n(t_1,\ldots,t_n,t) = \frac{1}{n+1}\{\sum_{i=1}^{n} f_n(t_1,\ldots,t,\ldots t_n,t_i) + f_n(t_1\ldots,t_n,t)\}.$$

In order to compute the Skorohod integral of elementary processes, we will use the next result.

THEOREM 7.1. *Let* $u \in L^2(T;L^2(\Omega)) \approx L^2(T\times\Omega)$, $u \in Dom\,\delta$ *and* $F \in \mathcal{D} \subset L^2(\Omega)$. *Suppose also that* $DF\cdot u \in Dom\,\delta$. *Then,*

$$Fu_t \in Dom\,\delta,$$

and

$$\delta(Fu_t) = F\delta(u) - \int_T u_t\, D_tF\lambda(dt) - \delta(DFu).$$

PROOF: If $G \in \mathcal{D}$, is a test variable, and using that $\mathcal{D}$ is dense in $L^2(\Omega)$,

$$\begin{aligned} E\int_T Fu_tD_tG\lambda(dt) &= E\int_T u_t\{D_t(FG) - GD_tF - D_tFD_tG\}\lambda(dt) \\ &= E[FG\delta(u)] - E[G\int_T u_tD_tF\lambda(dt)] - E[G\delta(uDF)] \\ &= E[G\{F\delta(u) - \int_T u_tD_tF\lambda(dt) - \delta(uDF)\}] \\ &= E[G\delta(Fu)]. \quad \blacksquare \end{aligned}$$

As a consequence of Proposition 4.1, if $T = [0,1]$, every square-integrable predictable process u is in the Domδ, and $\delta(u)$ is equal to the Poisson-Wiener integral.

In fact, every square-integrable predictable process can be approximated in $L^2(\Omega)$ by finite linear combinations of elementary predictable processes like $F\cdot 1_{(s,t]}$, with $F, \mathcal{F}_{[0,s]}$-measurable.

Then, from Theorem 7.1, or directly from Proposition 4.1 and Kabanov formula, we have

$$\delta(F\cdot 1_{(s,t]}) = F\cdot\delta(1_{(s,t]}).$$

Finally the isometry of the Poisson-Wiener integral implies the equivalence between the integrals.

For a non-predictable process we can interpret δ as the Poisson- Wiener integral minus a corrective term. This corrective term can be expressed in terms of the whole derivatives of u on the jump times of the Poisson process. This interpretation coincides with the results of [**2**].

We have the following theorem.

THEOREM 7.2. *If u is a process over $[0,1]$ such that $u_t \in \mathcal{D}_{2,\infty}$, for a.e. t. then there exists a process A_s which depends on u_s such that*

$$\delta(u) = \int_0^1 u_s d\tilde{N}_s - \int_0^1 A_s dN_s.$$

PROOF: The proof can be done in two steps. First of all for simple processes like $I_k(f_1 \otimes \cdots \otimes f_k)1_{[0,t]}(s)$, we can show by induction on k that the theorem holds with

$$A_s = D_s u_s - D_s D_s u_s + \cdots + (-1)^{k-1} D_s \cdots^{k)} \cdots D_s u_s.$$

Finally by means of a limit argument we show the result, first for a simple process of the form $F1_{[0,t]}(s)$ and then for a general process u_s verifying the conditions of the theorem.

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