SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

LESZEK SLOMINSKI

# Approximation of predictable characteristics of processes with filtrations

*Séminaire de probabilités (Strasbourg)*, tome 21 (1987), p. 447-478

<http://www.numdam.org/item?id=SPS_1987__21__447_0>

© Springer-Verlag, Berlin Heidelberg New York, 1987, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme*
*Numérisation de documents anciens mathématiques*
http://www.numdam.org/

# APPROXIMATION OF PREDICTABLE CHARACTERISTICS OF PROCESSES WITH FILTRATIONS

Leszek Słomiński

## 1. Introduction.

Let $(\Omega, F, P)$ be a complete probability space and let $S$ be a Polish space. Let $\mathfrak{F} = \{\mathfrak{F}(t)\}_{t\in\mathbb{R}^+}$ be a filtration on $(\Omega, F, P)$ i.e. a nondecreasing family of sub-$\sigma$-algebras of $F$. In the sequel we will consider $\mathfrak{F}$ adapted processes $X$ such that :

(1) $X(t)$ is a random $S$ element on $(\Omega, F, P)$, $t\in\mathbb{R}^+$,

(2) almost all trajectories $\Omega \ni \omega \longmapsto (X(\omega) : \mathbb{R}^+ \longrightarrow S)$ are right-continuous and admit left hand limits, i.e. belong to $D(S)$,

(3) the filtration $\mathfrak{F}$ is right-continuous and complete.

We will denote by $\mathfrak{B}(S)$ and $\mathfrak{P}(S)$ the $\sigma$-algebra of Borel subsets of $S$ and the space of probability measures on $\mathfrak{B}(S)$, respectively. It is well known that if $\mathfrak{P}(S)$ is equipped with the topology of weak convergence and $D(S)$ is endowed with the Skorokhod topology $J_1$ then both spaces are metrisable as Polish spaces (see e.g. [2], [14]).

Let $T = \{T_n\}_{n\in\mathbb{N}}$, $T_n = \{t_{nk}\}_{k\in\mathbb{N}\cup\{0\}}$, $0 = t_{n0} < t_{n1} < \ldots$, $\lim_{k\to\infty} t_{nk} = +\infty$, $n\in\mathbb{N}$ be a sequence of partitions of $\mathbb{R}^+$ such that:

(4) $T_n \subset T_{n+1}$, $n\in\mathbb{N}$,

(5) $\max_{k\leqslant r_n(t)} (t_{nk} - t_{n,k-1}) \downarrow 0$, $t\in\mathbb{R}^+$,

where $r_n(t) \stackrel{df}{=} \max[k : t_{nk} \leqslant t]$, $t\in\mathbb{R}^+$, $n\in\mathbb{N}$. For the array $\{t_{nk}\}$ we define the sequence of summation rules $\{\rho_n\}_{n\in\mathbb{N}}$ by the equality $\rho_n(t) \stackrel{df}{=} \max[t_{nk} : t_{nk} \leqslant t]$, $t\in\mathbb{R}^+$, $n\in\mathbb{N}$. Notice that $\{r_n\}_{n\in\mathbb{N}} \subset D(\mathbb{R})$ and $\{\rho_n\}_{n\in\mathbb{N}} \subset D(\mathbb{R})$.

Let $x$ be an element of $D(S)$. Having the sequence of summation rules $\{\varrho_n\}_{n\in\mathbb{N}}$ we may introduce a sequence $\{x\circ\varrho_n\}_{n\in\mathbb{N}}$ of elements from $D(S)$ by the equality $x\circ\varrho_n(t) \stackrel{df}{=} x(\varrho_n(t))$, $t\in\mathbb{R}^+$, $n\in\mathbb{N}$. The Skorokhod convergence is such that :

$$(6) \qquad x\circ\varrho_n \longrightarrow x \quad \text{in} \quad D(S) .$$

Let $S = \mathbb{R}$ and let $X$ be an $\mathcal{F}$ adapted real semimartingale, $X(0)= 0$, with the triplet of local predictable characteristics $(B^h, \sigma^2, \nu)$ (see Section 3). Let us fix $\omega\in\Omega$. By Theorem 1 of Grigelionis [5] there exists a semimartingale with independent increments $X^\omega$ such that its law $\mathcal{L}(X^\omega)$ is uniquely determined by the triplet $(B^h(\omega), \sigma^2(\omega), \nu(\omega))$ (see Section 3).

Let us denote by $\Pi^X(\omega)$ the distribution of $X^\omega$ considered as a random element with values in $\mathcal{P}(D(\mathbb{R}))$ :

$$(7) \qquad \Pi^X(\omega) \stackrel{df}{=} \mathcal{L}(X^\omega) .$$

Hence $\Pi^X$ is a random measure with values in the set (denoted by PII) of distributions of processes with independent increments and with trajectories in $D(\mathbb{R})$. The set PII is a closed subset of $\mathcal{P}(D(\mathbb{R}))$.

Now, let $X$ be an $\mathcal{F}$ adapted real process, $X(0)= 0$. We will consider a sequence $\{X\circ\varrho_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}\circ\varrho_n$ adapted processes, which is in fact a sequence of discretization of $X$ according to $\{\varrho_n\}_{n\in\mathbb{N}}$ :

$$(8) \qquad X\circ\varrho_n(t) \stackrel{df}{=} X(\varrho_n(t)) = \sum_{k=1}^{r_n(t)} \Delta_k^n X$$

$$(9) \qquad \mathcal{F}\circ\varrho_n(t) \stackrel{df}{=} \mathcal{F}(\varrho_n(t))$$

$t\in\mathbb{R}^+$, $n\in\mathbb{N}$ where $\Delta_k^n X \stackrel{df}{=} X(t_{nk}) - X(t_{n,k-1})$ $k,n\in\mathbb{N}$ .

By (1) and (6) we may trivially obtain

$$(10) \qquad X\circ\varrho_n \longrightarrow X \quad \text{in} \quad D(\mathbb{R})$$

almost surely.

Since for every $n\in\mathbb{N}$ $X\circ\varrho_n$ is a process with bounded variation, $X\circ\varrho_n$ is a semimartingale. Therefore there exists a random measure $\Pi^{X\circ\varrho_n}$ defined by (7). Moreover the special form of $X\circ\varrho_n$ and $\mathcal{F}\circ\varrho_n$ implies that :

$$(11) \qquad \Lambda^{X\circ\varrho_n}(t) = \prod_{k=1}^{r_n(t)} \ast \; \lambda(t_{nk}, t_{n,k-1}) \qquad t\in\mathbb{R}^+, n\in\mathbb{N}$$

where "$\ast$" denotes the convolution taken pointwise for the random measures $\lambda(t_{nk}, t_{n,k-1})$ $n,k\in\mathbb{N}$ and $\lambda(t_{nk}, t_{n,k-1})$ is a regular version of the conditional distribution of the increment $\Delta_k^n X$ given $\mathcal{F}(t_{n,k-1})$ $n,k\in\mathbb{N}$.

Now, we are ready to introduce our main notion.

Definition 1. Let $X$ be an $\mathcal{F}$ adapted real process, $X(0)=0$, and let $T=\{T_n\}_{n\in\mathbb{N}}$ be a sequence of discretizations satisfying (4), (5). We will say that $X$ is $T$ tangent to the family of processes with independent increments or for simplicity $X$ is $T$ tangent to PII iff there exists a random measure $\Lambda_g^X : \Omega \longrightarrow \mathrm{PII} \subset \mathcal{P}(D(\mathbb{R}))$ such that

$$(12) \qquad \Lambda^{X\circ\varrho_n} \xrightarrow[P]{} \Lambda_g^X \quad \text{in} \quad \mathcal{P}(D(\mathbb{R})).$$

In the sequel we will denote the class of processes $T$ tangent to PII by $S_g(T,D)$.

In our paper we characterise the class of processes $T$ tangent to PII and we formulate limit theorems for processes from $S_g(T,D)$. Main theorems are contained in Section 2. We defer the proofs to Section 5.

It is clear by using the counter example from Dellacherie, Doleans-Dade [4] that it is possible to construct a process $X$ (even a semimartingale) and two sequences of discretizations $T=\{T_n\}_{n\in\mathbb{N}}$, $T^1=\{T_n^1\}_{n\in\mathbb{N}}$ for which $X$ is $T$ tangent to PII but $X$ is not $T^1$ tangent to PII. Hence in this case $S_g(T,D)\neq S_g(T^1,D)$ and the property "$X$ is $T$ tangent to PII" should be checked for fixed $T=\{T_n\}_{n\in\mathbb{N}}$.

Since the random measures $\Lambda^X$ and $\Lambda_g^X$ associated to the semimartingale $X$ and to the element of $S_g(T,D)$, respectively, have some different properties (for more detail see Section 3) we reserve the notion $\Lambda^X$ only for semimartingales.

Recently Jacod [9] examined a particular case of the theorems considered in our paper. Jacod characterised in detail the class of processes $T$ tangent to PII such that for every $\omega\in\Omega$ $\Lambda_g^X(\omega)$ is additionally the law of continuous in probability process with

independent increments.

Below we give Jacod's results. In fact we change slightly the form and notation in those theorems. Let $S_g(T,C)$ denote the subspace of $S_g(T,D)$ examined in [9].

Theorem J1 ([9]). (i) Every continuous in probability process with independent increments $X$, $X(0)=0$ belongs to $S_g(T,C)$. Then $\mathcal{A}_g^X = \mathcal{L}(X)$.

(ii) Every quasileft-continuous semimartingale $X$, $X(0)=0$ belongs to $S_g(T,C)$. In this case $\mathcal{A}_g^X = \mathcal{A}^X$.

In order to give a characterisation of processes from $S_g(T,C)$ it is necessary to define the following family of processes.

Definition J2 ([9]). (i) We say that the bounded and predictable process $B$, $B(0)=0$ with continuous trajectories belongs to the class $B(T,C)$ iff

$$\sup_{t\le q} \Big| \sum_{k=1}^{r_n(t)} E^n_{k-1}\Delta^n_k B - B(t) \Big| \xrightarrow{P} 0 , \quad q\in\mathbb{R}^+ \tag{13}$$

$$\sum_{k=1}^{r_n(t)} \Big[ E^n_{k-1}(\Delta^n_k B)^2 - (E^n_{k-1}\Delta^n_k B)^2 \Big] \xrightarrow{P} 0 , \quad t\in\mathbb{R}^+ \tag{14}$$

where $E^n_{k-1}(\cdot) = E(\cdot)\,|\,\mathcal{F}(t_{n,k-1})$ $n,k\in\mathbb{N}$.

(ii) We say that the process $B$ belongs to $B_{loc}(T,C)$ iff there exists a localizing sequence $\{\tau_k\}_{k\in\mathbb{N}}$, $\tau_k \uparrow +\infty$ a.s. of $\mathcal{F}$ stopping times for which $B^{\tau_k} \in B(T,C)$, $k\in\mathbb{N}$.

We will also use the characteristics $\sigma^2$, $\nu$ such that

(15) $\sigma^2$ is a process with continuous and nondecreasing trajectories, $\sigma^2(0)=0$

(16) $\nu$ is a random measure on $\mathcal{B}(\mathbb{R}^+\times\mathbb{R})$, $\nu(\{t\}\times\mathbb{R})=0$, $t\in\mathbb{R}^+$, $\nu(\mathbb{R}^+\times\{0\})=0$, $\int_{\mathbb{R}} x^2\wedge 1\ \nu((0,t]\times dx) < +\infty$, $t\in\mathbb{R}^+$.

Theorem J3 ([9]). (i) $B_{loc}(T,C)$ and $S_g(T,C)$ are two vector spaces.

(ii) The sum of a quasileft-continuous semimartingale and a process from $B_{loc}(T,C)$ belongs to $S_g(T,C)$.

(iii) The process $X$ belongs to $S_g(T,C)$ iff there exists the triplet $(B,\sigma^2,\nu)$ with $B\in B_{loc}(T,C)$ and $\sigma^2$, $\nu$ satisfying (15) and (16) respectively such that $X-B$ is a quasileft-continuous semimartingale with triplet of predictable characteristics $(0,\sigma^2,\nu)$. In this case the triplet $(B,\sigma^2,\nu)$

is uniquely determined.

(iv) The space $B_{loc}(T,C)$ contains : all the processes $B$, $B(0)=0$ with continuous trajectories and bounded variation, all the continuous elements from $D(\mathbb{R})$ equal null in 0 .

It can be observed (see [9] , Remark 1.16 ) that the technique used for the characterisation of the class $S_g(T,C)$ can not to be extended to the class $S_g(T,D)$ . Our method is more general and we hope that it is slightly simpler to the one mentioned above.

We end this section with a simple example of a family of processes from $S_g(T,D)$ not necessary belonging to $S_g(T,C)$ .

Example. Every process with independent increments $X$ , $X(0)=0$ is $T$ tangent to PII .

In order to explain this fact let us note that for each $n\in\mathbb{N}$ $X\circ\varrho_n$ is a semimartingale with independent increments for which :

$$\Lambda^{X\circ\varrho_n} = \mathcal{L}(X\circ\varrho_n) \quad .$$

By (10) the conclusion follows and $\Lambda_g^X = \mathcal{L}(X)$ .

In the following sections we restrict our attention to the real $\mathcal{F}$ adapted processes $X$ satisfying the assumption

(17) $\qquad X(0)=0$ .

## 2. Main results.

### 2.1 The semimartingales $T$ tangent to PII .

Let $T=\{T_n\}_{n\in\mathbb{N}}$ be a sequence of discretizations satisfying (4) , (5) with the accompanying sequence of summation rules $\{\varrho_n\}_{n\in\mathbb{N}}$. Let us fix $t\in\mathbb{R}^+$ , $n\in\mathbb{N}$ and let $\sigma$ be a $\mathcal{F}\circ\varrho_n$ stopping time . Since for $k\leq r_n(t)$ , $[t_{nk}\leq\sigma<t_{n,k+1}]\in\mathcal{F}\circ\varrho_n(t_{n,k+1}-) = \mathcal{F}(t_{nk}) \subset \mathcal{F}\circ\varrho_n(t)$ so by simple calculations we have

$$[\varrho_n(\sigma)\leq t] = \bigcup_{k=0}^{r_n(t)}[\varrho_n(\sigma)=t_{nk}]$$

$$= \bigcup_{k=0}^{r_n(t)}[t_{nk}\leq\sigma<t_{n,k+1}] \in \mathcal{F}\circ\varrho_n(t) .$$

But if $\sigma$ is an $\mathcal{F}$ stopping time only we do not know whether $\varrho_n(\sigma)$ is an $\mathcal{F}\circ\varrho_n$ stopping time or not. This implies the

existence of examples of semimartingales which are not T tangent to PII .

Theorem 1. Let X be a semimartingale with the predictable characteristics $\Lambda^X$ defined by (7) . The semimartingale X is T tangent to PII i.e. $X \in S_g(T,D)$ iff the following condition (T) is satisfied :

(T) for every predictable $\mathcal{F}$ stopping time $\sigma$ there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}\cdot\rho_n$ stopping times such that

$$\lim_{n\to\infty} P\left[\rho_n(\sigma) \neq \sigma_n \ , \ A_\sigma\right] = 0$$

where $A_\sigma \overset{df}{=} [\nu(\{\sigma\},\mathbb{R}) > 0 \ , \ \sigma < +\infty]$ .

In this case $\Lambda^X_g = \Lambda^X$ .

Due to Theorem 1 it is possible to give a nontrivial example of a semimartingale from $S_g(T,D)$ .

Corollary 1. Let X be a semimartingale of which every predictable jump $\sigma$ has one of the two following forms :

(18) $\sigma = \sum_{k=1}^{\infty} s_k I(\sigma = s_k)$ on the set $A_\sigma$ for some sequence of positive constants $\{s_k\}_{k\in\mathbb{N}}$

(19) $\sigma = \tau + c$ on the set $A_\sigma$ for some $\mathcal{F}$ stopping time $\tau$ and for some positive constant $c$ .

Then $X \in S_g(T,D)$ .

## 2.2 The characterisation of processes from $S_g(T,D)$ .

First we introduce a new class of processes appropriate to $B_{loc}(T,C)$ .

Definition 2. (i) We say that a bounded and predictable process B , $B(0) = 0$ belongs to the class $B(T,D)$ iff

(20) $$\sup_{t\leq q} \left| \sum_{k=1}^{r_n(t)} E^n_{k-1}\Delta^n_k B - \sum_{k=1}^{r_n(t)} \Delta^n_k B \right| \xrightarrow{P} 0 \ , \quad q\in\mathbb{R}^+ .$$

(ii) We say that the process B belongs to $B_{loc}(T,D)$ iff there exists a localizing sequence $\{\tau_k\}_{k\in\mathbb{N}}$ , $\tau_k \uparrow +\infty$ a.s. of $\mathcal{F}$ stopping times and $B^{\tau_k} \in B(T,D)$ , $k\in\mathbb{N}$ .

Let us assume that $B \in B(T,D)$ . Since an $\mathcal{F}\cdot\rho_n$ adapted process

$\left\{ \sum_{k=1}^{r_n(t)} \left[ E^n_{k-1} \Delta^n_k B - \Delta^n_k B \right] \right\}_{t \in \mathbb{R}^+}$ is for fixed $n \in \mathbb{N}$ a local martingale it follows by the Davis-Burkholder-Gundy inequality (see [7]) that (20) implies (14) . Therefore :

$$B(T,C) = B(T,D) \cap \{ B \text{ with continuous trajectories} \} .$$

We can easily extend the above equality to the classes $B_{loc}(T,C)$ and $B_{loc}(T,D)$ . It is clear that in general $B_{loc}(T,D) \neq B_{loc}(T^1,D)$ for two different sequences of discretizations $T$ , $T^1$ .

Now, let us observe that it is possible to express (20) in terms of convergence in $D(\mathbb{R})$. By (71) $B$ , $B(0)=0$ belongs to $B(T,D)$ iff

$$(21) \qquad \widetilde{B \circ \rho_n} \xrightarrow{P} B \qquad \text{in} \quad D(\mathbb{R}) ,$$

where above and in the next sections for every special semimartingale $X$ , $\widetilde{X}$ denotes its predictable compensator , $\widetilde{X}(0) = 0$ .

Let $(B^h_g, \sigma^2_g, \nu_g)$ be a triplet of characteristics such that :

(22) $B^h_g$ is a predictable process, $\sup_t |\Delta B^h_g(t)| \leq 1$ , $B^h_g(0) = 0$ ,

(23) $\sigma^2_g$ is a process with continuous and nondecreasing trajectories $\sigma^2_g(0) = 0$ ,

(24) $\nu_g$ is a random measure on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ for which

$$\nu_g(\{0\} \times \mathbb{R}) = 0 , \quad \nu_g(\mathbb{R}^+ \times \{0\}) = 0 , \quad \nu_g((0,t] \times [|x| > \varepsilon]) < +\infty$$

$$\nu_g(\{t\} \times \mathbb{R}) \leq 1 ,$$

$$\int_0^t \int_{\mathbb{R}} (h(x) - \Delta B^h_g(s))^2 \, \nu_g(ds \times dx) + \sum_{s \leq t} (1 - \nu_g(\{s\} \times \mathbb{R})) (\Delta B^h_g(s))^2 < +\infty$$

$$\Delta B^h_g(t) = \int_{\mathbb{R}} h(x) \, \nu_g(\{t\} \times dx) , \quad \text{for every} \quad t \in \mathbb{R}^+ , \ \varepsilon > 0 .$$

**Theorem 2.** (i) $B_{loc}(T,D)$ and $S_g(T,D)$ are two vector spaces.

(ii) The sum of a $T$ tangent to PII semimartingale and a process from $B_{loc}(T,D)$ belongs to $S_g(T,D)$ .

(iii) The process $X$ belongs to $S_g(T,D)$ iff there exists a system of characteristics $(B^h_g, \sigma^2_g, \nu_g)$ satisfying (22) - (24) such that $B^h_g \in B_{loc}(T,D)$ and $X - B^h_g$ is a semimartingale

from $S_g(T,D)$ with the triplet of predictable characteristics $(B^h, C^{2g}, \nu^h)$ given by :

$$C^2(t) \stackrel{df}{=} C_g^2(t) \quad , \quad t \in \mathbb{R}^+ \ ,$$

$$\nu^h(A) \stackrel{df}{=} \iint_{\mathbb{R}^+\mathbb{R}} I(x \neq \Delta B_g^h(s), (s, x - \Delta B_g^h(s)) \in A)\, \nu_g(ds \times dx)$$
$$+ \sum_s (1 - \nu_g(\{s\} \times \mathbb{R}))\, I(0 \neq \Delta B_g^h(s), (s, -\Delta B_g^h(s)) \in A)$$
$$A \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}) \ ,$$

$$B^h(t) \stackrel{df}{=} \sum_{s \leq t} \int_{\mathbb{R}} h(x)\, \nu^h(\{s\} \times dx) \quad , \quad t \in \mathbb{R}^+ \ .$$

In this case the triplet $(B_g^h, C_g^2, \nu_g)$ is uniquely determined.

(iv) The space $B_{loc}(T,D)$ contains : all predictable processes $B$, $B(0) = 0$ with bounded variation, satisfying the condition $(T)$, and all $F(0)$ measurable processes equal null in $0$ .

Corollary 2. Let $X$ be a process with conditionally independent increments. Then $X \in S_g(T,D)$ .

## 2.3 Functional limit theorems for processes tangent to PII .

It is interesting that limit theorems for the processes tangent to PII can be formulated in the same way as for semimartingales (functional limit theorems for semimartingales can be found in [6] , [10] ). In order to study those theorems we will use an approach of Aldous [1] .

Let $X$ be an $\mathcal{F}$ adapted real process. Aldous has shown that there exists a unique $\mathcal{F}$ adapted process $Z$ with trajectories in the space $D(\mathcal{P}(D(\mathbb{R}))$ such that for every $t \in \mathbb{R}^+$ and $A \in \mathcal{B}(D(\mathbb{R}))$ we have :

$$Z(t,A) = P\,(X \in A \mid \mathcal{F}(t))$$

i.e. $Z(t) : \Omega \times \mathcal{B}(D(\mathbb{R})) \longrightarrow [0,1]$ is a regular version of the conditional distribution of $X$ given $\mathcal{F}(t)$ .

For every $\omega \in \Omega$ the trajectory

$$\mathbb{R}^+ \ni t \longmapsto (X(t,\omega), Z(t,\omega)) \in \mathbb{R}^+ \times \mathcal{P}(D(\mathbb{R}))$$

is an element of the space $D(\mathbb{R} \times \mathcal{P}(D(\mathbb{R})))$ so we can define the extended distribution of the process $X$ as the distribution of the random element

$$\Omega \ni \omega \longmapsto (X(\omega), Z(\omega)) \in D(\mathbb{R} \times \mathcal{P}(D(\mathbb{R}))) \quad .$$

Let $\{X^n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a sequence of $\mathcal{F}^n$ adapted processes. We say that the sequence $\{X^n\}_{n \in \mathbb{N}}$ converges extendedly to $X^\infty$ and write $X^n \xrightarrow[E]{} X^\infty$ iff the extended distributions of $\{X^n\}_{n \in \mathbb{N}}$ are weakly convergent to the extended distribution of $X^\infty$ .

Some necessary and sufficient conditions for extended convergence of semimartingales have been given in [11] and [17] . It is proved by Kubilius [12] that the theorems from [11] and [17] can be extended to the case where the limit process is a semimartingale but not necessarily with independent increments.

In the present paper we propose another way of generalization. We apply the method from [11] and [17] to the processes tangent to PII .

Theorem 3. Let $\{X^n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{F}^n$ adapted processes, $T^n = \{T_k^n\}_{k \in \mathbb{N}}$ tangent to PII , and let $X^\infty$ be a continuous in probability process with independent increments. Under the condition

$$(\text{Sup } B_g) \qquad \sup_{t \leqslant q} \left| B_g^{h,n}(t) - B_g^{h,\infty}(t) \right| \xrightarrow[P]{} 0 \quad , \quad q \in \mathbb{R}^+ ,$$

the following two conditions are equivalent :

$$\text{(i)} \qquad \Lambda_g^{X^n} \xrightarrow[P]{} \mathcal{L}(X^\infty) \quad ,$$

$$\text{(ii)} \qquad X^n \xrightarrow[E]{} X^\infty \quad .$$

Similarly we could formulate a version of Theorem 3 from [11] where the condition $(\text{Sup } B_g)$ is also necessary in some special sense.

## 3. Preliminary remarks.

### 3.1 Convergence in the Skorokhod topology.

The space $D(S)$ with the Skorokhod topology $J_1$ has been discussed in detail by several authors : Lindvall [14], Billingsley [2] and Aldous [1] . In the present paper we will use frequently the results from [1] .

Let $x$ be an element of $D(S)$. Let us denote by ${}^s x$ the element $x$ stopped at $s$, $s \in \mathbb{R}^+$, i.e.

$$ {}^s x(t) \stackrel{\text{def}}{=} \begin{cases} x(t) & t \leqslant s \\ x(s) & t > s \end{cases} . $$

Remark 1. Let $\{x_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a sequence of elements from $D(S)$ such that $x_n \longrightarrow x_\infty$. Then by Proposition 26.8 from [1] for each $s \in \mathbb{R}^+$ there exists a sequence $\{s_n\}_{n \in \mathbb{N}}$, $s_n \longrightarrow s$ for which ${}^{s_n}x_n \longrightarrow {}^s x_\infty$. Moreover if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence satisfying $u_n \geqslant s_n$, $n \in \mathbb{N}$ and $u_n \longrightarrow s$ then also ${}^{u_n}x_n \longrightarrow {}^s x_\infty$.

Suppose that $S$, $S^1$ are two Polish spaces. In Section 2.3 and in other sections of our paper we often use the convergence in the Skorokhod topology in $D(S \times S^1)$. By Proposition 29.2 from [1] we obtain following simple characterisation of the convergence in $D(S \times S^1)$.

Let $\{x_n\}_{n \in \mathbb{N} \cup \{\infty\}}$, $\{y_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be two sequences of elements from $D(S)$ and $D(S^1)$ respectively. Then $(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$ in $D(S \times S^1)$ iff $x_n \longrightarrow x_\infty$ in $D(S)$ $y_n \longrightarrow y_\infty$ in $D(S^1)$ and for every $t \in \mathbb{R}^+$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \longrightarrow t$ such that $x_n(t_n) \longrightarrow x_\infty(t)$ $x_n(t_n-) \longrightarrow x_\infty(t-)$, $y_n(t_n) \longrightarrow y_\infty(t)$, $y_n(t_n-) \longrightarrow y_\infty(t-)$.

Remark 2. The above result is simpler in the case $S = \mathbb{R}$, $S^1 = \mathbb{R}$. Then $(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$ in $D(\mathbb{R}^2)$ iff $x_n \longrightarrow x_\infty$, $y_n \longrightarrow y_\infty$ in $D(\mathbb{R})$ and for every $t \in \{t \in \mathbb{R}^+ : \Delta x_\infty(t) \neq 0 \text{ and } \Delta y_\infty(t) \neq 0\}$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \longrightarrow t$, such that $\Delta x_n(t_n) \longrightarrow \Delta x_\infty(t)$ and $\Delta y_n(t_n) \longrightarrow \Delta y_\infty(t)$. Consequently $(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$ in $D(\mathbb{R}^2)$ iff $x_n \longrightarrow x_\infty$, $y_n \longrightarrow y_\infty$ and $x_n - y_n \longrightarrow x_\infty - y_\infty$ in $D(\mathbb{R})$.

Now, assume that $x \in D(\mathbb{R})$, $x(0) = 0$ and $x$ has quadratic variation $[x]$, i.e. for each $t \in \mathbb{R}^+$ there exists a finite limit

$$ [x](t) \stackrel{\text{def}}{=} \lim_{n \to \infty} \sum_{k=1}^{r_n(t)} \left( x(t \wedge t_{n,k+1}) - x(t \wedge t_{nk}) \right)^2 $$

for some fixed sequence of discretizations $T = \{T_n\}_{n \in \mathbb{N}}$. Therefore $[x] \in D(\mathbb{R})$ and $[x](0) = 0$. It is clear by using e.g. 3.2. in [8] that $[x \circ \varrho_n] \longrightarrow [x]$ in $D(\mathbb{R})$. Moreover by Remark 2 and (6)

$$ ([x] \circ \varrho_n, [x \circ \varrho_n]) \longrightarrow ([x], [x]) \text{ in } D(\mathbb{R}^2) . $$

Using Remark 2 once more

$$(25) \qquad \sup_{t \leq q} \left| [X] \circ \rho_n(t) - [X \circ \rho_n](t) \right| \longrightarrow 0 , \quad q \in \mathbb{R}^+ .$$

The following lemma is an easy corollary of (25) .

Lemma 1. Let $X$ be a local martingale. Then

$$\sup_{t \leq q} \left| [X] \circ \rho_n(t) - [X \circ \rho_n](t) \right| \xrightarrow{P} 0 , \quad q \in \mathbb{R}^+ . \qquad \blacksquare$$

## 3.2 The Lenglart type inequality.

The following lemma follows readily from the concept of domination introduced by Lenglart [13] .

Lemma 2. Let $X$ be a process with bounded variation . Then for all $\varepsilon, \eta > 0$ and for every $\mathcal{F}$ stopping time $\tau$ :

$$P\left[ \operatorname{Var} \tilde{X}(\tau) > \varepsilon \right] \leq 4 \varepsilon^{-1} E \operatorname{Var} X(\tau) \wedge \left( \eta + \sup_{t \leq \tau} |\Delta X(t)| \right) + 2 P\left[ \operatorname{Var} X(\tau) > \eta \right] .$$

Proof. Let $X^+$ and $X^-$ be two increasing processes such that $X = X^+ - X^-$ and $\operatorname{Var} X = X^+ + X^-$ . Therefore

$$P\left[ \operatorname{Var} \tilde{X}(\tau) > \varepsilon \right] \leq P\left[ \tilde{X^+}(\tau) > \frac{\varepsilon}{2} \right] + P\left[ \tilde{X^-}(\tau) > \frac{\varepsilon}{2} \right] .$$

Using the inequality of Rebolledo [16] to the first component on the right-hand side in the above inequality we obtain :

$$P\left[ \tilde{X^+}(\tau) > \frac{\varepsilon}{2} \right] \leq 2 \varepsilon^{-1} E\, X^+(\tau) \wedge \left( \eta + \sup_{t \leq \tau} |\Delta X^+(t)| \right) + P\left[ X^+(\tau) > \eta \right] .$$

The same estimation is also true in the case of the process $X^-$ Therefore the proof is complete. $\blacksquare$

Corollary 3. Let $\{X^n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{F}^n$ adapted processes with bounded variation such that $\left\{ \sup_{t \leq \tau_n} |\Delta X^n(t)| \right\}_{n \in \mathbb{N}}$ is uniformly integrable for some sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of $\mathcal{F}^n$ stopping times. If $\operatorname{Var} X^n(\tau_n) \xrightarrow{P} 0$ then

$$\operatorname{Var} \tilde{X}^n(\tau_n) \xrightarrow{P} 0 . \qquad \blacksquare$$

## 3.3 The predictable characteristics of semimartingales and processes tangent to PII .

Let $X$ be a semimartingale. Let $h$ be a continuous function $h : \mathbb{R} \longrightarrow [-1,1]$ such that $h(x) = x$ for $|x| \leq 1/2$ and $h(x) = 0$ for $|x| > 1$ . By $X^h$ we denote the process given by the formula

$$X^h(t) \stackrel{df}{=} X(t) - \sum_{s \leq t} \left( \Delta X(s) - h(\Delta X(s)) \right) \quad , \quad t \in \mathbb{R}^+ . \tag{26}$$

The process $X^h$ is a semimartingale with bounded jumps , $\sup_t |\Delta X^h(t)| \leq 1$ , hence it is also a special semimatingale and can be uniquely decomposed into the sum :

$$X^h(t) = B^h(t) + M^h(t) \quad , \quad t \in \mathbb{R}^+ . \tag{27}$$

Where $B^h$ is a predictable process with bounded variation , $B^h(0) = 0$ , $\sup_t |\Delta B^h(t)| \leq 1$ and $M^h$ is a local martingale , $M^h(0) = 0$ , $\sup_t |\Delta M^h(t)| \leq 2$ .

Let $X^c$ be the unique continuous martingale part of the semimartingale $X$ . We define

$$\sigma^2(t) \stackrel{df}{=} \langle X^c \rangle (t) \quad , \quad t \in \mathbb{R}^+ , \tag{28}$$

where $\langle X^c \rangle = [X^c]$ is the quadratic variation process of $X^c$ .

Let $\nu = \nu(dt*dx)$ be the dual predictable projection of the jump-measure $N(dt*dx)$ of the process $X$

$$N((0,t] \times A) \stackrel{df}{=} \sum_{s \leq t} I\left( \Delta X(s) \in A \ , \ \Delta X(s) \neq 0 \right) \quad t \in \mathbb{R}^+ , \ A \in \mathcal{B}(\mathbb{R}). \tag{29}$$

The triple $(B^h, \sigma^2, \nu)$ is called a system of local predictable characteristics of the semimartingale $X$ . It can be observed that this system satisfies the following properties :

(30) $B^h$ is a predictable process with bounded variation , $B^h(0) = 0$ , $\sup_t |\Delta B^h(t)| \leq 1$ ,

(31) $\sigma^2$ is a process with continuous and nondecreasing trajectories , $\sigma^2(0) = 0$ ,

(32) $\nu$ is a random measure on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ such that : $\nu(\{0\} \times \mathbb{R}) = 0$ , $\nu(\mathbb{R}^+ \times \{0\}) = 0$ , $\int_{\mathbb{R}} x^2 \wedge 1 \, \nu((0,t] \times dx) < +\infty$ , $t \in \mathbb{R}^+$ .

It is clear that in general, it is not true that $B^h$ belongs

to $B_{loc}(T,D)$ . However comparing (30) - (32) with the properties of predictable characteristics of processes tangent to PII we can conclude that the system $(B^h, G^2, \nu)$ fulfills the conditions (22) - (24) , too .

### 3.4 The processes with independent increments.

Let $X$ be a process with independent increments. As it is proved in Jacod [8] and in Grigelionis [5] there exists a non-random system of characteristics $(B^h_g, G^2_g, \nu_g)$ satisfying (22) - (24) . Moreover if we denote $D_0 = \{ t \in \mathbb{R}^+ : \nu_g(\{t\}\times\mathbb{R}) = 0 \}$ then for every $s,t \in \mathbb{R}^+$ , $s \leq t$

$$E \exp i\theta(X(t) - X(s)) = \prod_{s<r\leq t}\{[1 + \int_{\mathbb{R}}(e^{i\theta x} - 1)\nu_g(\{r\}\times dx)]\, e^{-i\theta\Delta B^h_g(r)}\}$$

$$\exp\{i\theta(B^h_g(t) - B^h_g(s)) - \tfrac{1}{2}\theta^2(G^2_g(t) - G^2_g(s)) \tag{33}$$

$$+ \int_s^t\int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta h(x))I(r\notin D_0)\,\nu_g(dr\times dx)\} .$$

Conversely if $(B^h_g, G^2_g, \nu_g)$ is a nonrandom system of characteristics with properties (22) - (24) then there exists a process with independent increments $X$ for which the condition (33) holds. Therefore the law of $X$ , $\mathcal{L}(X)$ is uniquely determined by the triple $(B^h_g, G^2_g, \nu_g)$ . In the sequel we will use the notation $\mathcal{L}(B^h_g, G^2_g, \nu_g) \stackrel{df}{=} \mathcal{L}(X)$ .

In [8] Jacod has gived also necessary and sufficient conditions for the weak convergence of sequence of processes with independent increments. Let $\{X^n\}_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of processes with independent increments with the sequence of their characteristics $\{(B^{h,n}_g, G^{2,n}_g, \nu^n_g)\}_{n\in\mathbb{N}\cup\{\infty\}}$ .

**Theorem J4** ([8]). $\mathcal{L}(X^n) \longrightarrow \mathcal{L}(X^\infty)$ *in* $\mathcal{P}(D(\mathbb{R}))$ *iff the following conditions are satisfied :*

$$B^{h,n}_g \longrightarrow B^{h,\infty}_g \quad \text{in} \quad D(\mathbb{R}) , \tag{34}$$

$$C^{h,n}_g \longrightarrow C^{h,\infty}_g \quad \text{in} \quad D(\mathbb{R}) , \tag{35}$$

$$\int_{\mathbb{R}} f(x)\,\nu^n_g(dx) \longrightarrow \int_{\mathbb{R}} f(x)\,\nu^\infty_g(dx) \quad \text{in} \quad D(\mathbb{R}) , \ f \in C_{\nu(Q)} \tag{36}$$

where $C^{h,n}_g(t) \stackrel{df}{=} G^{2,n}_g(t) + \sum_{s\leq t}\int_{\mathbb{R}}(h(x) - \Delta B^{h,n}_g(s))^2\,\nu^n_g(\{s\}\times dx) +$

$$+ \sum_{s \le t} \left[1 - \nu_g^n(\{s\}\times\mathbb{R})\right] \left(\Delta B_g^{h,n}(s)\right)^2 , \qquad t \in \mathbb{R}^+, \ n \in \mathbb{N}\cup\{\infty\},$$

and $C_{v(0)}$ is a family of positive and bounded, continuous functions vanishing in some open neighbourhood of $0$.

## 4. Fundamental properties of processes tangent to PII.

### 4.1 Necessary and sufficient conditions for the processes from $S_g(T,D)$.

It is possible to characterise a process $X \in S_g(T,D)$ in terms of convergence in probability of the predictable characteristics of their discretizations $\{X\circ\rho_n\}_n$

Proposition 1. A process $X$ is $T$ tangent to PII iff the following conditions are fulfilled :

$$(37) \qquad \widetilde{(X\circ\rho_n)}^h \xrightarrow{P} B_g^h$$

$$(38) \qquad \left[\widetilde{(X\circ\rho_n)}^h - \widetilde{(X\circ\rho_n)}^h\right] \xrightarrow{P} C_g^h(\cdot) \stackrel{df}{=} \sigma_g^2(\cdot) + \sum_{s\le\cdot}\int_{\mathbb{R}} (h(x) - \Delta B_g^h(s))^2 \nu_g(\{s\}\times dx) + \sum_{s\le\cdot}\left[1 - \nu_g(\{s\}\times\mathbb{R})\right]\left(\Delta B_g^h(s)\right)^2$$

$$(39) \qquad \int_{\mathbb{R}} f(x)\, \widetilde{N\circ\rho_n}(dx) \xrightarrow{P} \int_{\mathbb{R}} f(x)\, \nu_g(dx) , \quad f \in C_{v(0)} ,$$

where the triple $(B_g^h, \sigma_g^2, \nu_g)$ posseses the properties (22) - (24) In this case

$$(40) \qquad X^h - B_g^h \text{ is a local martingale} ,$$

$$(41) \qquad \sigma_g^2 = \langle (X^h - B_g^h)^c \rangle ,$$

$$(42) \qquad \int_{\mathbb{R}} f(x) N(dx) - \int_{\mathbb{R}} f(x)\, \nu_g(dx) \text{ is a local martingale for every } f \in C_{v(0)} .$$

Proof. Let us assume that the process $X$ is $T$ tangent to PII. By a routine technique of subsequences and by Theorem J4 used for fixed $\omega \in \Omega$ we can readily see that the triplet $(B_g^h(\omega), \sigma_g^2(\omega), \nu_g(\omega))$ is well defined.

First we check the properties (22) and (40) for the process $B_g^h$. To prove the predictability of $B_g^h$ we use Theorem 88 C from [4]. It is clear by (37) that $B_g^h$ is adapted to $\mathcal{F}$.

Therefore we have to verify that $B_g^h(\sigma)$ is $\mathcal{F}(\sigma-)$ measurable for every predictable $\mathcal{F}$ stopping time $\sigma$ and $\Delta B_g^h(\sigma) = 0$ for every totally inaccessible $\mathcal{F}$ stopping time.

Let $\{\sigma^{ik}\}$ be the array of $\mathcal{F}$ stopping time defined by the equalities :

$$(43) \qquad \sigma^{i0} = 0 \ , \quad \sigma^{ik} = \inf\left[t > \sigma^{i,k-1} \ , \ |\Delta B_g^h(t)| > \varepsilon_i\right]$$

$i,k \in \mathbb{N}$ , where $\{\varepsilon_i\}_{i\in\mathbb{N}}$ is a sequence of positive constants such that $\varepsilon_i \downarrow 0$ , $P\left(|\Delta B_g^h(t)| = \varepsilon_i \ , \ t \in \mathbb{R}^+\right) = 0$ and $\inf \emptyset \overset{df}{=} +\infty$ .

Let $\{\tau_n^{ik}\}$ be defined analogously for fixed $n \in \mathbb{N}$ as the following array of predictable $\mathcal{F}\circ\rho_n$ stopping times :

$$(44) \qquad \tau_n^{i0} = 0 \ , \quad \tau_n^{ik} = \inf\left[t > \tau_n^{i,k-1} \ , \ |\widetilde{\Delta(X\circ\rho_n)^h}(t)| > \varepsilon_i\right]$$

$i,k \in \mathbb{N}$ . Let us fix $i,k \in \mathbb{N}$ . For simplicity we will write $\tau_n$, $\sigma$ instead of $\tau_n^{ik}$ , $\sigma^{ik}$ .

By elementary computations : $\tau_n I(\tau_n < +\infty) \xrightarrow{P} \sigma$ and $\widetilde{\Delta(X\circ\rho_n)^h}(\tau_n) I(\tau_n < +\infty) \xrightarrow{P} \Delta B_g^h(\sigma)$ on the set $[\sigma < +\infty]$. Let us put for every $n \in \mathbb{N}$ :

$$\delta_n \overset{df}{=} \begin{cases} \tau_n & \text{if} \quad \tau_n \leqslant \rho_n^*(\sigma) \\ +\infty & \text{if} \quad \tau_n > \rho_n^*(\sigma) \end{cases}$$

where $\rho_n^*(t) \overset{df}{=} \min\left[t_{nk} : t_{nk} \geqslant t\right]$ , $t \in \mathbb{R}^+$ ( $\rho_n^*(\sigma)$ is $\mathcal{F}\circ\rho_n$ stopping time ! ). According to (10) $\Delta(X\circ\rho_n)^h(\rho_n^*(\sigma)) \xrightarrow{P} \Delta X^h(\sigma)$ on the set $[\sigma < +\infty]$ . Therefore $\Delta(X\circ\rho_n)^h(\tau_n) I(\rho_n^*(\sigma) \neq \tau_n, \tau_n < +\infty) \xrightarrow{P} 0$ on the set $[\sigma < +\infty]$ and as a consequence :

$$(45) \qquad \delta_n I(\delta_n < +\infty) \xrightarrow{P} \sigma \qquad \text{on the set } [\sigma < +\infty],$$

$$(46) \quad \widetilde{\Delta(X\circ\rho_n)^h}(\delta_n) I(\delta_n < +\infty) \xrightarrow{P} \Delta B_g^h(\sigma) \text{ on the set } [\sigma < +\infty],$$

$$(47) \qquad \delta_n \leqslant \rho_n^*(\sigma) \qquad \text{on the set } \left[\delta_n < +\infty\right] , \ n \in \mathbb{N}.$$

Now we will show that $\sigma$ is a predictable $\mathcal{F}$ stopping time. Let $\gamma$ be a positive constant and $\{k_n\}_{n\in\mathbb{N}}$ be a subsequence $\{k_n\} \subset \{n\}$ , $k_n \uparrow +\infty$ for which $\sum_{n=1}^{\infty} P\left[\delta_{k_n} = +\infty \ , \ \sigma < +\infty\right] < \gamma$. Since by (47)

$$\left[\delta_{k_n} = +\infty \ , \ \sigma < +\infty\right]^c = \left[\delta_{k_n} \leqslant \rho_{k_n}^*(\sigma), \ \delta_{k_n} < +\infty\right] \cup \left[\rho_{k_n}^*(\sigma) = +\infty\right]$$

we have $[\delta_{k_n} = +\infty \ , \ \sigma < +\infty] \in \mathcal{F}_{\rho_{k_n}}(\rho_{k_n}^*(\sigma)-)$ .

By the following simple lemma

Lemma 3. Let $\sigma$ be a $\mathcal{F}$ stopping time. Then for every $n \in \mathbb{N}$ $\mathcal{F}_{\rho_n}(\rho_n^*(\sigma)-) \subset \mathcal{F}(\sigma-)$ and moreover

$$\mathcal{F}_{\rho_n}(\rho_n^*(\sigma)-) \uparrow \mathcal{F}(\sigma-) \quad . \blacksquare$$

we obtain that $S_\gamma \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} [\delta_{k_n} = +\infty \ , \ \sigma < +\infty] \in \mathcal{F}_{k_n}(\sigma-)$ .

Hence we can define new stopping time $\sigma_\gamma$:

$$\sigma_\gamma \overset{\text{def}}{=} \begin{cases} \sigma & \text{on the set } (S_\gamma)^c \\ +\infty & \text{on the set } S_\gamma \ . \end{cases}$$

For every $n \in \mathbb{N}$ we also define :

$$\sigma_{k_n} \overset{\text{def}}{=} \begin{cases} t_{k_n,j-1} & \text{if } \delta_{k_n} = t_{k_n,j} \ , \ j \in \mathbb{N} \\ +\infty & \text{if } \delta_{k_n} = +\infty \ . \end{cases}$$

If we put $\sigma_{k_n,\gamma} \overset{\text{def}}{=} \max_{i \le n} (\sigma_{k_i} \wedge n)$ then $\sigma_{k_n,\gamma} < \sigma_\gamma$ $n \in \mathbb{N}$ and $\sigma_{k_n,\gamma} \uparrow \sigma_\gamma$.

Therefore $\sigma_\gamma$ is a predictable $\mathcal{F}$ stopping time. Taking a sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ , $\gamma_i \downarrow 0$ we define a stationary decreasing sequence $\{\sigma_{\gamma_i}\}$ of predictable $\mathcal{F}$ stopping times $\sigma_{\gamma_i} \downarrow \sigma$ . Thus $\sigma$ is a predictable $\mathcal{F}$ stopping time , too.

As a consequence $\Delta B_g^h(\sigma) = 0$ for every totally inaccessible $\mathcal{F}$ stopping time $\sigma$ .

Finally we have to verify that $B_g^h(\sigma)$ is $\mathcal{F}(\sigma-)$ measurable for every predictable $\mathcal{F}$ stopping time $\sigma$ . This is clear if $\Delta B_g^h(\sigma) = 0$ . In this case for stopped processes

$$(48) \qquad \widetilde{(X \circ \rho_n)}^h, \rho_n^*(\sigma) \xrightarrow[P]{} B_g^{h,\sigma} \quad .$$

On the other hand let $\sigma$ be of the form $\sigma = \sigma^{ik}$ . Then by (45) and (46) we have

$$\widetilde{(X \circ \rho_n)}^h, \delta_n \xrightarrow[P]{} B_g^{h,\sigma} \quad .$$

And the property (47) together with Remark 2 implies that the conclusion (48) follows, too. Thus (48) holds for every $\mathcal{F}$ stopping time $\sigma$ . Since the left-hand side of (48) is $\mathcal{F}_{\rho_n}(\rho_n^*(\sigma)-)$

measurable so it follows by Lemma 3 that $B_g^{h,\sigma}$ and in particular $B_g^h(\sigma)$ are $\mathcal{F}(\sigma-)$ measurable. Therefore the process $B_g^h$ is predictable.

In the next step we will show that $X^h - B_g^h$ is a local martingale. Let $\sigma$ be a fixed $\mathcal{F}$ stopping time. First let us note that the property

$$(49) \qquad \operatorname{Var}\left( (X\circ\rho_n)^h, \rho_n^*(\sigma) - (X^{\sigma}\circ\rho_n)^h \right)(q) \xrightarrow{P} 0 \quad , \quad q\in\mathbb{R}^+$$

together with Corollary 3 implies the convergence

$$(50) \qquad \sup_{t\le q} \left| \widetilde{(X\circ\rho_n)^h}, \rho_n^*(\sigma)(t) - \widetilde{(X^{\sigma}\circ\rho_n)^h}(t) \right| \xrightarrow{P} 0 \quad , \quad q\in\mathbb{R}^+.$$

For proving (49) the following simple lemma will be used.

Lemma 4.

$$\operatorname{Var}\left( X^h\circ\rho_n - (X\circ\rho_n)^h \right)(q) \xrightarrow{P} 0 \quad , \quad q\in\mathbb{R}^+ .$$

Proof of lemma. By the definition

$$(X\circ\rho_n)^h(t) = X\circ\rho_n(t) - \sum_{i=1}^{\infty} \left( \Delta(X\circ\rho_n)(\sigma_n^i) - h\left(\Delta(X\circ\rho_n)(\sigma_n^i)\right) \right) I\left(t \ge \sigma_n^i\right)$$

$$= X\circ\rho_n(t) - J_n(t)$$

where $\sigma_n^0 = 0$, $\sigma_n^i = \inf\left[ t > \sigma_n^{i-1} , |\Delta(X\circ\rho_n)(t)| > \varepsilon \right]$ and and $0 < \varepsilon < 1/2$, $P\left( |\Delta X(t)| = \varepsilon \ , \ t\in\mathbb{R}^+ \right) = 0$.

We denote

$$J_n^1(t) = \sum_{i=1}^{\infty} \left( \Delta(X\circ\rho_n)(\rho_n^*(\sigma^i)) - h\left(\Delta(X\circ\rho_n)(\rho_n^*(\sigma^i))\right) \right) I\left(t \ge \rho_n^*(\sigma^i)\right)$$

$t\in\mathbb{R}^+$, where $\sigma^0 = 0$, $\sigma^i = \inf\left[ t > \sigma^{i-1} , |\Delta X(t)| > \varepsilon \right]$.
Since $\max\left[ i : q \ge \sigma^i \right] < +\infty$, $q\in\mathbb{R}^+$ it follows by the convergence $\lim_{n\to\infty} P\left[ \rho_n^*(\sigma^i) \ne \sigma_n^i \ , \ \sigma^i < +\infty \right] = 0$ that

$$\lim_{n\to\infty} P\left[ J_n^1(t) \ne J_n(t), \ t \le q \right] = 0 \qquad q\in\mathbb{R}^+.$$

Hence $\operatorname{Var}\left( J_n^1 - J_n \right)(q) \xrightarrow{P} 0$, $q\in\mathbb{R}^+$ and thus the estimation of $\operatorname{Var}\left( X^h\circ\rho_n - (X\circ\rho_n - J_n^1) \right)(q)$, $q\in\mathbb{R}^+$ finishes the proof.

Let us observe that

$$\operatorname{Var}\left( X^h\circ\rho_n - (X\circ\rho_n - J_n^1) \right)(q) =$$

$$= \sum_{k=1}^{m_n(q)} \left| \Delta_k^n(X^h - X) + \sum_{i=1}^{\infty} \left( \Delta(X\circ\rho_n)(\rho_n^*(\sigma^i)) - h\left(\Delta(X\circ\rho_n)(\rho_n^*(\sigma^i))\right) \right) I\left(t_{nk} = \rho_n^*(\sigma^i)\right) \right|$$

$$= \sum_{k=1}^{r_n(q)} \Big| \sum_{i=1}^{\infty} \big[ \big( \Delta X(\sigma^i) - h(\Delta X(\sigma^i)) \big) - \big( \Delta X \circ \varrho_n(\varrho_n^*(\sigma^i)) - h(\Delta X \circ \varrho_n(\varrho_n^*(\sigma^i))) \big) \big] \Big| \cdot I(t_{nk} = \varrho_n^*(\sigma^i))$$

$$\leq \sum_{k=1}^{r_n(q)} \sum_{i=1}^{\infty} \Big| \big( \Delta X(\sigma^i) - h(\Delta X(\sigma^i)) \big) - \big( \Delta X \circ \varrho_n(\varrho_n^*(\sigma^i)) - h(\Delta X \circ \varrho_n(\varrho_n^*(\sigma^i))) \big) \Big| \cdot I(t_{nk} = \varrho_n^*(\sigma^i))$$

$$= \sum_{i=1}^{\infty} \Big| \big( \Delta X(\sigma^i) - h(\Delta X(\sigma^i)) \big) - \big( \Delta X \circ \varrho_n(\varrho_n^*(\sigma^i)) - h(\Delta X \circ \varrho_n(\varrho_n^*(\sigma^i))) \big) \Big| I(q \geq \varrho_n^*(\sigma^i)).$$

Then (10) implies that the last sum converges almost surely to 0 for every $q \in \mathbb{R}^+$. ■

By Lemma 4 the estimation of (49) reduces to convergence

$$\mathrm{Var}\big( (X^h \circ \varrho_n)\varrho_n^*(\sigma) - X^{h,\sigma} \circ \varrho_n \big)(q) \xrightarrow{P} 0 \quad , \quad q \in \mathbb{R}^+ .$$

But the equality

$$\big( (X^h \circ \varrho_n)\varrho_n^*(\sigma) - X^{h,\sigma} \circ \varrho_n \big)(t) = \big( X^h(\varrho_n^*(\sigma)) - X^h(\sigma) \big) I\big( t \geq \varrho_n^*(\sigma) \big)$$

assures that convergence due to the right continuity of $X^h$ .

Comparing (50) and (48) we obtain :

$$(51) \qquad \widetilde{(X^\sigma \circ \varrho_n)^h} \xrightarrow{P} B_g^{h,\sigma} \quad .$$

Let us denote $M_g^h = X^h - B_g^h$ . Since the process $M_g^h$ has bounded jumps $\sup_t |\Delta M_g^h(t)| \leq 2$ we can choose a localizing sequence $\{\sigma_k\}$ of $\mathcal{F}$ stopping times such that $\sigma_k \uparrow +\infty$ a.s. and $\sup_{t \leq \sigma_k} |M_g^h(t)| \leq k$ , $k \in \mathbb{N}$ . Let us fix $t,s \in \mathbb{R}^+$ , $t \geq s$ and $t,s \in \mathrm{Cont}\, M_g^h = \{ t \in \mathbb{R}^+ : \Delta M_g^h(t) = 0 \}$ . For fixed $k \in \mathbb{N}$ there exists a sequence $\{\tau_n\}$ of $\mathcal{F} \circ \varrho_n$ stopping times such that

$$(52) \qquad M_{n,k}^{h,\tau_n}(t) \overset{\mathrm{df}}{=} (X^{\sigma_k} \circ \varrho_n)^{h,\tau_n}(t) - \widetilde{(X^{\sigma_k} \circ \varrho_n)^{h,\tau_n}}(t) \xrightarrow{P} M_g^{h,\sigma_k}(t)$$

(53) there exists a sequence $\{s_n\}$ of positive numbers for which $\varrho_n(s_n) \geq s$ , $n \in \mathbb{N}$ , $\varrho_n(s_n) \downarrow s$ and

$$M_{n,k}^{h,\tau_n}(s_n) \xrightarrow{P} M_g^{h,\sigma_k}(s)$$

$$(54) \qquad \sup_t \big| M_{n,k}^{h,\tau_n}(t) \big| \leq k + 1 \quad , \quad n \in \mathbb{N} .$$

It can be easily verified by Tschebyschev inequality and (52), (53)

that $E\left(M_g^{h,\sigma_k(t)} - M_g^{h,\sigma_k(s)} \mid \mathcal{F}_{\rho_n}(s_n)\right) \xrightarrow{P} 0$ . On the other hand by standard arguments and the property $\mathcal{F}_{\rho_n}(s_n) \downarrow \mathcal{F}(s)$

$$E\left(M_g^{h,\sigma_k(t)} - M_g^{h,\sigma_k(s)} \mid \mathcal{F}_{\rho_n}(s_n)\right) \xrightarrow{P} E\left(M_g^{h,\sigma_k(t)} - M_g^{h,\sigma_k(s)} \mid \mathcal{F}(s)\right).$$

Therefore $E\left(M_g^{h,\sigma_k(t)} \mid \mathcal{F}(s)\right) = M_g^{h,\sigma_k(s)}$ for every $t,s \in \mathrm{Cont}\, M_g^h$ $t \geq s$ . Hence $M_g^{h,\sigma_k}$ is a uniformly integrable martingale and the proof of (40) is complete.

It is interesting that instead of (37) we can consider a more stringent condition

$$(55) \qquad \left((X\circ\rho_n)^h, \widetilde{(X\circ\rho_n)^h}\right) \xrightarrow{P} \left(X^h, B_g^h\right) \quad \text{in } D(\mathbb{R}^2) .$$

The above property is a consequence of Remark 2 and the argument given below. Let $\sigma$ be of the form $\sigma = \sigma^{ik}$ defined by (43) . Then there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of predictable $\mathcal{F}_{\rho_n}$ stopping times such that

$$(56) \qquad \lim_{n\to\infty} P\left[\rho_n^*(\sigma) \neq \delta_n ,\ \sigma < +\infty\right] = 0 .$$

To prove (56) let us take a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ satisfying (45) - (47) . Therefore by (46) we have

$$E\left(\Delta(X\circ\rho_n)^h(\delta_n) I(\rho_n^*(\sigma) = \delta_n) \mid \mathcal{F}_{\rho_n}(\delta_n-)\right) I(\delta_n < +\infty) \xrightarrow{P} \Delta B_g^h(\sigma)$$

on the set $[\sigma < +\infty]$ . Using (47) one can see that

$$E\left(\Delta(X\circ\rho_n)^h(\delta_n) I(\rho_n^*(\sigma) = \delta_n) \mid \mathcal{F}_{\rho_n}(\delta_n-)\right) I(\delta_n < +\infty)$$

$$(57) \quad = E\left(E\left(\Delta(X\circ\rho_n)^h \rho_n^*(\sigma) \mid \mathcal{F}_{\rho_n}(\rho_n^*(\sigma)-)\right) I(\rho_n^*(\sigma) = \delta_n) \mid \mathcal{F}_{\rho_n}(\delta_n-)\right) I(\delta_n < +\infty).$$

Since by Lemma 3

$$(58) \qquad E\left(\Delta(X\circ\rho_n)^h(\rho_n^*(\sigma)) \mid \mathcal{F}_{\rho_n}(\rho_n^*(\sigma)-)\right) \xrightarrow{P} E\left(\Delta X^h(\sigma) \mid \mathcal{F}(\sigma-)\right) = \Delta B_g^h(\sigma)$$

on the set $[\sigma < +\infty]$ . so (57) and the convergence $\delta_n \xrightarrow{P} \sigma$ imply

$$E\left(\Delta B_g^h(\sigma) I(\rho_n^*(\sigma) = \delta_n ,\ \sigma < +\infty) \mid \mathcal{F}_{\rho_n}(\delta_n-)\right) \xrightarrow{P} \Delta B_g^h(\sigma) I(\sigma < +\infty) .$$

Hence

$$E\left|\Delta B_g^h(\sigma)\right| I(\sigma < +\infty) = \lim_{n\to\infty} E\left|E\left(\Delta B_g^h(\sigma) I(\rho_n^*(\sigma) = \delta_n, \sigma < +\infty) \mid \mathcal{F}_{\rho_n}(\delta_n-)\right)\right|$$

$$\leq \overline{\lim_{n\to\infty}} E\left|\Delta B_g^h(\sigma) I(\rho_n^*(\sigma) = \delta_n ,\ \sigma < +\infty)\right|$$

$$\leq E\left|\Delta B_g^h(\sigma)\right| I(\sigma < +\infty)$$

and we have $\lim_{n\to\infty} E\left|\Delta B_g^h(\sigma)\, I(\rho_n^*(\sigma) \neq \delta_n\,, \sigma < +\infty)\right| = 0$ . Finally to end the proof of (56) it remains to observe that $|\Delta B_g^h(\sigma)| > \varepsilon_i$ on the set $[\sigma < +\infty]$ .

Now let us note that Remark 2 and (56) guarantee more stringent convergence in (37) . It is clear that in fact we have the following convergence

$$(59) \qquad \sup_{t\leq q} \left| B_g^h(\rho_n(t)) - \widetilde{(X\circ\rho_n)}^h(t)\right| \xrightarrow{\mathcal{P}} 0\,, \quad q\in\mathbb{R}^+ .$$

We can prove (39) and (42) using the following Lemma 5 instead of Lemma 4 .

Lemma 5. For every $f \in C_{v(0)}$

$$\mathrm{Var}\left(\int_{\mathbb{R}} f(x)(N\circ\rho_n)(dx) - \left(\int_{\mathbb{R}} f(x)\,N(dx)\right)\circ\rho_n\right)(q) \xrightarrow{\mathcal{P}} 0\,, \quad q\in\mathbb{R}^+ .$$

The poof of (38) and (41) is essentially the same as in previous cases. In both Lemma 1 and Corollary 3 are basic and the condition (59) is very useful.

To prove the converse implication let us assume that the conditions (37) - (39) are satisfied. Using Theorem J4 for fixed $\omega\in\Omega$ once more we obtain $X \in S_g(T,D)$ and this completes the proof. ■

Using Proposition 1 we can conclude that every process $X$ T tangent to PII has triplet of predictable characteristics $(B_g^h, \sigma_g^2, \nu_g)$ or equivalently a random measure $\mu_g^X$ with values in PII such that

$$\mu_g^X(\omega) = \mathcal{L}\left(B_g^h(\omega), \sigma_g^2(\omega), \nu_g(\omega)\right)\,, \quad \omega\in\Omega .$$

Let $\tau$ be some $\mathcal{F}$ stopping time . By the stopped random measure $(\mu_g^X)^\tau$ we will mean the random measure with values in PII defined by the formulas :

$$(\mu_g^X)^{\tau(\omega)}(\omega) = \mathcal{L}\left(B_{g\,\tau(\omega)}^h, \sigma_{g\,\tau(\omega)}^2, \nu_g^{\tau(\omega)}\right)\,, \quad \omega\in\Omega .$$

By the arguments from the proof of Proposition 1 we obtain :

Corollary 5. The process $X$ belongs to $S_g(T,D)$ iff there exists a localizing sequence $\{\tau_k\}_{k\in\mathbb{N}}$ , $\tau_k \uparrow +\infty$ a.s. for which $X^{\tau_k} \in S_g(T,D)$ , $k\in\mathbb{N}$ . In this case $(\mu_g^X)^{\tau_k} = \mu_g^{X^{\tau_k}}$ , $k\in\mathbb{N}$ . ■

## 4.2 Approximation in probability for predictable compensators of special semimartingale.

The following result forms the essential part of Theorem 1 .

**Proposition 2.** *Let $X$ be a special semimartingale such that $\sup|X(t)| < c$ , $\sup \operatorname{Var} \tilde{X}(t) < c$ for some constant $c > 0$ . Then the two conditions given below are equivalent :*

(i) $\tilde{X} \circ \rho_n \xrightarrow{P} \tilde{X}$ *in* $D(\mathbb{R})$ ,

(ii) *the property* $(T)$ *is satisfied.*

*Proof.* (ii) $\Rightarrow$ (i) First let us observe that it is very convenient to have the following property $(T^*)$ instead of $(T)$ :

$(T^*)$ for every predictable $\mathcal{F}$ stopping time $\sigma$ there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of predictable $\mathcal{F}\circ\rho_n$ stopping times such that $\lim_{n\to\infty} P\left[\rho_n^*(\sigma) \neq \delta_n \ , \ A\sigma\right] = 0$ .

The equivalence $(T)\Leftrightarrow(T^*)$ is evident if the stopping time satisfies $P\left[\sigma \in T_n\right] = 0$ , $n \in \mathbb{N}$ . To obtain the general case we use the following lemma.

**Lemma 6.** *Let us suppose that the predictable $\mathcal{F}$ stopping time $\sigma$ is of the form $\sum_{k=1}^{\infty} s_k I(\sigma = s_k)$ on the set $A\sigma$ for some sequence of positive constants $\{s_k\}_{k\in\mathbb{N}}$ . Then for the stopping time $\sigma$ the conditions $(T)$ and $(T^*)$ hold.*

*Proof of lemma.* Let us note that without loss of generality we may assume that the stopping time $\sigma$ is of the form

$$\sigma = \sum_{k=1}^{\infty} s_k I(\sigma = s_k) + \{+\infty\} I(\sigma \neq s_k \ , \ k \in \mathbb{N}). \tag{60}$$

We begin with a simpler case where $\sigma$ satisfies

$$\sigma = \sum_{k=1}^{j} s_k I(\sigma = s_k) + \{+\infty\} I(\sigma \neq s_k, \ 1 \leq k \leq j), \tag{61}$$

for some fixed $j \in \mathbb{N}$ . Observe that for every $k$ , $1 \leq k \leq j$ there exists a sequence $\{s_{kn}\}_{n\in\mathbb{N}}$ , $s_{kn} \in T_n$, $s_{kn} < s_k$ , $s_{kn} \uparrow s_k$ and a sequence of positive constants $\{c_n\}_{n\in\mathbb{N}}$ , $c_n \uparrow +\infty$ for which :

$$I(A_{nk}) \overset{df}{=} I\left(E(I(\sigma = s_k) \mid \mathcal{F}(s_{kn})) > 1 - c_n^{-1}\right) \xrightarrow{P} I(\sigma = s_k)$$

for every $k$, $1 \leq k \leq j$. Finally if we define the sequence $\{\delta_n\}$ by the equalities

$$\delta_n \overset{df}{=} \begin{cases} \rho_n^*(s_1) & \text{on the set } A_{n1} \\ \rho_n^*(s_k) & \text{on the set } A_{nk} \setminus \bigcup_{i=1}^{k-1} A_{ni} \\ & \text{for every } k,\ 2_j \leq k \leq j \\ \{+\infty\} & \text{on the set } \left(\bigcup_{k=1} A_{nk}\right)^c \end{cases}$$

then the condition $(T^*)$ is fulfilled by the stopping time $\sigma$ defined by (61). If we put $\rho_n(s_k)$ instead of $\rho_n^*(s_k)$ we get a sequence $\{\sigma_n\}$ of $\mathcal{F}\circ\rho_n$ stopping times satisfying the condition $(T)$.

Now, let us suppose that $\sigma$ is of the form (60). We denote for every $j \in \mathbb{N}$ the stopping time of the form (61) by $\sigma^j$. Therefore for each $j \in \mathbb{N}$ we can define the sequence $\{\delta_n^j\}$ of predictable $\mathcal{F}\circ\rho_n$ stopping times for which $\lim_{n\to\infty} P\left[\rho_n^*(\sigma^j) \neq \delta_n^j,\ \sigma^j < +\infty\right] = 0$. Since $\lim_{j\to\infty} P\left[\sigma^j \neq \sigma\right] = 0$ we can choose a sufficiently slowly increasing sequence $\{j_n\}$, $j_n \uparrow +\infty$ such that :

$$\lim_{n\to\infty} P\left[\rho_n^*(\sigma) \neq \delta_n^{j_n},\ \sigma < +\infty\right] = 0 . \tag{62}$$

Analogously we show that the condition $(T)$ is satisfied for the stopping time $\sigma$, too. ∎

So we can assume that the condition $(T^*)$ holds.

Now, we will consider the sequence of processes $\{Y^i\}_{i\in\mathbb{N}}$ defined by $Y^i(t) = \sum_{k=1}^{\infty} \Delta^X(\sigma^{ik}) I(t \geq \sigma^{ik})$, $t \in \mathbb{R}^+$, $i \in \mathbb{N}$ where the array $\{\sigma^{ik}\}$ of predictable $\mathcal{F}$ stopping time is defined as follows :

$$\sigma^{i0} = 0 , \quad \sigma^{ik} = \inf\left[t > \sigma^{i,k-1} , |\Delta\widetilde{X}(t)| > \varepsilon_i\right] \tag{63}$$

$i,k \in \mathbb{N}$ for some sequence of positive constants $\{\varepsilon_i\}_{i\in\mathbb{N}}$, $\varepsilon_i \downarrow 0$, $P\left(|\Delta\widetilde{X}(t)| = \varepsilon_i ,\ t \in \mathbb{R}^+\right) = 0$, $i \in \mathbb{N}$.

In the next step of the proof we will show that

$$\sup_{t\leq q} \left| \widetilde{Y^i\circ\rho_n}(t) - \widetilde{Y^i}(\rho_n(t)) \right| \xrightarrow{P} 0 , \quad q \in \mathbb{R}^+ . \tag{64}$$

First let us note that by Proposition 1.49 from [8]

$$\widetilde{Y^i}(t) = \sum_{k=1}^{\infty} E\left(\Delta X(\sigma^{ik}) \mid \mathcal{F}(\sigma^{ik}-)\right) I\left(t \geq \sigma^{ik}\right)$$

$t \in \mathbb{R}^+$, $n \in \mathbb{N}$. We denote $Y^{ik}(t) \stackrel{df}{=} \Delta X(\sigma^{ik}) I\left(t \geq \sigma^{ik}\right)$ $i,k \in \mathbb{N}$. Then $\widetilde{Y^i} \circ \rho_n = \sum_{k=1}^{\infty} \widetilde{Y^{ik}} \circ \rho_n$ and $\widetilde{Y^i} = \sum_{k=1}^{\infty} \widetilde{Y^{ik}}$.

Now let us assume that the following convergence holds :

$$(65) \qquad \sup_{t \leq q} \left| \widetilde{Y^{ik}} \circ \rho_n(t) - \widetilde{Y^{ik}}(\rho_n(t)) \right| \xrightarrow{P} 0 \,, \quad q \in \mathbb{R}^+, \; i,k \in \mathbb{N}.$$

Hence for every $j \in \mathbb{N}$

$$\sup_{t \leq q} \left| \sum_{k=1}^{j} \widetilde{Y^{ik}} \circ \rho_n(t) - \sum_{k=1}^{j} \widetilde{Y^{ik}}(\rho_n(t)) \right| \xrightarrow{P} 0 \,, \quad q \in \mathbb{R}^+$$

Since $\max\left[i : \sigma^i \leq q\right] < +\infty$ we have $\limsup_{j \to \infty} \operatorname{Var}\left(\sum_{k=j}^{\infty} Y^{ik} \circ \rho_n\right)(q) = 0$ and $\lim_{j \to \infty} \operatorname{Var}\left(\sum_{k=j}^{\infty} Y^{ik}\right)(q) = 0$. Then it follows by Corollary 3 that (65) implies (64).

Therefore without loss of generality we will consider a process $Y$ of the form $Y(t) = \Delta X(\sigma) I\left(t \geq \sigma\right)$, $t \in \mathbb{R}^+$ for some predictable $\mathcal{F}$ stopping time $\sigma$. It is easy to verify that

$$Y \circ \rho_n(t) = \sum_{k=1}^{r_n(t)} \Delta X(\sigma) I\left(\rho_n^*(\sigma) = t_{nk}\right)$$

$$\widetilde{Y \circ \rho_n}(t) = \sum_{k=1}^{r_n(t)} E\left(\Delta X(\sigma) I\left(\rho_n^*(\sigma) = t_{nk}\right) \mid \mathcal{F}(t_{n,k-1})\right)$$

$t \in \mathbb{R}^+$, $n \in \mathbb{N}$.

In the next considerations the notations from the proof of Proposition 1 are used.

Let us fix $\gamma > 0$ and a subsequence $\{k_n\}_{n \in \mathbb{N}}$. We denote $Y_\gamma(t) \stackrel{df}{=} \Delta X(\sigma_\gamma) I\left(t \geq \sigma_\gamma\right)$, $t \in \mathbb{R}^+$. Since for every $n \in \mathbb{N}$ $\left[\rho_n^*(\sigma_\gamma) \neq \delta_n \,,\, \sigma_\gamma < +\infty\right] = \left[\rho_n^*(\sigma) \neq \delta_n \,,\, \sigma_\gamma < +\infty\right] \subset \left[\rho_n^*(\sigma) \neq \delta_n \,,\, \sigma < +\infty\right]$ by Corollary 3 and $(T^*)$ we have :

$$\sup_{t \leq q} \left| \widetilde{Y_\gamma \circ \rho_n}(t) - E\left(\Delta Y_\gamma \circ \rho_n(\delta_n) \mid \mathcal{F}_{\rho_n}(\delta_n-)\right) I\left(t \geq \delta_n\right) \right| \xrightarrow{P} 0$$

$q \in \mathbb{R}^+$. It is clear that $E\left(\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) \mid \mathcal{F}_{\rho_{k_n}}(\delta_{k_n}-)\right) = E\left(\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) \mid \mathcal{F}(\sigma_{k_n})\right)$. Now, let us observe that by the implication $\delta_{k_n} < +\infty \Rightarrow \delta_{k_n} \leq \rho_{k_n}^*(\sigma)$ and the definition of $\sigma_\gamma$

$$\lim_{n \to \infty} P\left[\sigma_{k_n} \neq \sigma_{k_n,\gamma} \,,\, \sigma_\gamma < \infty\right] = 0 \,.$$

Hence the convergences : $\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) \xrightarrow{P} \Delta X(\sigma_\gamma)$ on the set

$[\sigma_\gamma < +\infty]$ , $\lim_{n\to\infty} P[\delta_{k_n} \leq q \ , \ \sigma_\gamma = +\infty] = 0$ $q \in \mathbb{R}^+$ (we can assume the convergence $\delta_{k_n} \xrightarrow{P} \sigma_\gamma$ ) imply that

$$\sup_{t\leq q} \left| E\left(\Delta^{Y_{\gamma'}\circ\rho_{k_n}}(\delta_{k_n}) \middle| \mathcal{F}(\sigma_{k_n})\right) I(t \geq \delta_{k_n}) - E\left(\Delta^{X(\sigma_\gamma)} I(\sigma_\gamma < +\infty) \middle| \mathcal{F}_{\sigma_{k_n}, \gamma'}\right) I(t \geq \delta_{k_n}) \right|$$

$$\xrightarrow{P} 0 \ , \quad q \in \mathbb{R}^+.$$

Since $\sigma_{k_n}, \gamma' \uparrow \sigma_\gamma$ , $E\left(\Delta^{X(\sigma_\gamma)} I(\sigma_\gamma < +\infty) \middle| \mathcal{F}(\sigma_{k_n}, \gamma')\right) \xrightarrow{P} E\left(\Delta^{X(\sigma_\gamma)} I(\sigma_\gamma < +\infty) \middle| \mathcal{F}(\sigma_\gamma -)\right)$ . Hence

$$\sup_{t\leq q} \left| \widetilde{Y_{\gamma'}\circ\rho_{k_n}}(t) - E\left(\Delta^{X(\sigma_\gamma)} \middle| \mathcal{F}(\sigma_\gamma -)\right) I\left(t \geq \rho^*_{k_n}(\sigma_\gamma)\right) \right| \xrightarrow{P} 0$$

$q \in \mathbb{R}^+$. Therefore there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ , $\gamma_n \downarrow 0$ and one subsequence $\{\ell_{k_n}\}$ , $\{\ell_{k_n}\} \subset \{k_n\}$ such that

$$\sup_{t\leq q} \left| \widetilde{Y^{\gamma_n}\circ\rho_{\ell_{k_n}}}(t) - \widetilde{Y}\left(\rho_{\ell_{k_n}}(t)\right) \right| \xrightarrow{P} 0 \ , \quad q \in \mathbb{R}^+.$$

Since $\mathrm{Var}\left(Y^{\gamma_n}\circ\rho_{\ell_{k_n}} - Y\circ\rho_{\ell_{k_n}}\right)(q) \xrightarrow{P} 0$ , $q \in \mathbb{R}^+$ it follows by Corollary 3 that

$$\sup_{t\leq q} \left| \widetilde{Y\circ\rho_{\ell_{k_n}}}(t) - \widetilde{Y}\left(\rho_{\ell_{k_n}}(t)\right) \right| \xrightarrow{P} 0 \ , \quad q \in \mathbb{R}^+ .$$

Moreover, we could prove that for every subsequence $\{m_n\} \subset \{n\}$ there exists a further subsequence $\{\ell_{k_m}\} \subset \{m_n\}$ for which the above convergence holds. Therefore the proof of (64) is complete.

Let $\{Z^i\}_{i\in\mathbb{N}}$ be a new sequence of processes given by the equalities $Z^i(t) = X(t) - Y^i(t)$ , $t \in \mathbb{R}^+$, $i \in \mathbb{N}$ . By using the arguments of Meyer [15] :

$$E \sup_{t\leq q} \left| \widetilde{Z^i\circ\rho_n}(t) - \widetilde{Z^i}\left(\rho_n(t)\right) \right|^2 \leq 4\, E \sum_{k=1}^{r_n(q)} \left(E^n_{k-1}\Delta^n_k \widetilde{Z^i} - \Delta^n_k \widetilde{Z^i}\right)^2$$

$$\leq 4\, E \sum_{k=1}^{r_n(q)} \left(\Delta^n_k \widetilde{Z^i}\right)^2 \leq 4\, E \max_{k\leq r_n(q)} \left|\Delta^n_k \widetilde{Z^i}\right| \mathrm{Var}\, \widetilde{Z^i}(q)$$

$$\leq 4c \left\{ E \max_{k\leq r_n(q)} \left|\Delta^n_k Z^i\right|^2 \right\}^{\frac{1}{2}} .$$

Since $\lim_{i\to\infty} \overline{\lim_{n\to\infty}}\ E \max_{k\leq r_n(q)} \left|\Delta^n_k Z^i\right|^2 = 0$ we have

$$\lim_{i\to\infty} \overline{\lim_{n\to\infty}}\ P\left[\sup_{t\leq q} \left| \widetilde{Z^i\circ\rho_n}(t) - \widetilde{Z^i}\left(\rho_n(t)\right) \right| \geq \varepsilon\right] = 0 \tag{66}$$

for every $\varepsilon > 0$ and every $q \in \mathbb{R}^+$. It is easy to show that (66) and (64) imply

$$\sup_{t\leq q} \left| \widetilde{X\circ\rho_n}(t) - \widetilde{X}\left(\rho_n(t)\right) \right| \xrightarrow{P} 0 \ , \quad q \in \mathbb{R}^+ .$$

Finally (10) gives (i) .

(i) $\Longrightarrow$ (ii) First assume that $\sigma = \sigma^{ik}$ i.e $\sigma$ is of the form given by (63) . Then by the arguments from the proof of Proposition 1 the condition $(T^*)$ is fulfilled . Now, let remark that we can assume that

$$\sigma = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \sigma^{ik} I\left(\sigma = \sigma^{ik}\right) + \{+\infty\} I\left(\sigma = \sigma^{ik} \quad i,k \in \mathbb{N}\right).$$

We begin with the simpler case where $\sigma$ satisfies

$$\sigma = \sum_{l=1}^{j} \sigma^l I\left(\sigma = \sigma^l , \ \sigma^l \neq \sigma^k \quad 1 \leq k \leq l-1\right) + \{+\infty\} I\left(\sigma \neq \sigma^l \quad 1 \leq l \leq j\right) \tag{67}$$

and each $\sigma^l$ is of the form $\sigma^l = \sigma^{ik}$ . Observe that by Lemma 3 there exists a sequence $\{c_n\}_{n \in \mathbb{N}}$ , $c_n \uparrow +\infty$ for which

$$I\left(E\left(I(A_l) \mid \mathcal{F}_{\rho_n(\rho_n^*(\sigma^l)-)}\right) > 1 - c_n^{-1}\right) \xrightarrow{P} I(A_l)$$

where $A_l \overset{df}{=} [\sigma = \sigma^l , \ \sigma^l \neq \sigma^k \quad 1 \leq k \leq l-1]$ for every $l$ $1 \leq l \leq j$ . Since for each $\sigma^l$ the condition $(T^*)$ is fulfilled there exists a sequence $\{\delta_n^l\}_{n \in \mathbb{N}}$ of predictable $\mathcal{F}_{\rho_n}$ stopping times satisfying (57) . As a consequence

$$I(A_{nl}) \overset{df}{=} I\left(E\left(I(A_l) \mid \mathcal{F}_{\rho_n(\delta_n^l-)}\right) > 1 - c_n^{-1}\right) \xrightarrow{P} I(A_l) . \tag{68}$$

Therefore if we take $\delta_n^{l,*} \overset{df}{=} \delta_n^l I(A_{nl}) + \{+\infty\} I(A_{nl}^c)$ , $1 \leq l \leq j$ and $\delta_n = \min_{1 \leq l \leq j} \delta_n^{l,*}$ then it is easy to see that the condition $(T^*)$ is satisfied for the stopping time $\sigma$ of the form (67) .

Finally let us observe that we can extend this fact to every predictable $\mathcal{F}$ stopping time $\sigma$ (just as in Lemma 6) . Since $(T) \Longleftrightarrow (T^*)$ the proof is complete . ■

Let us note that in general i.e. if we do not assume that the property $(T)$ is satisfied then (i) is not true . Using " the method of Laplacians " we can obtain only that $\widetilde{X \circ \rho_n}(t) \longrightarrow \widetilde{X}(t)$ , weakly in $\mathbb{L}^1$ , $t \in \operatorname{Cont} \widetilde{X}$ .

## 4.3 Necessity of the condition $(T)$ .

Theorem 1 says that the semimartingale $X$ belongs to $S_g(T,D)$ iff $X$ satisfies the condition $(T)$ . The above result seems

to be not true in the general case. But we have .

Proposition 3. Let $X$ be a process $T$ tangent to PII . Then the condition $(T)$ is fulfilled.

Proof. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence of constants, $\varepsilon_i \downarrow 0$ such that $\nu_g(\mathbb{R}^+\times(|x|=\varepsilon_i))=0$ , $i\in\mathbb{N}$ . The family $\{\nu_g((0,t]\times(|x|\geq\varepsilon_i))\}_{t\in\mathbb{R}^+}$ is a predictable process for which by (39)

$$(69)\qquad \sum_{k=1}^{r_n(\cdot)} E^n_{k-1}\, I\left(\varepsilon_i \leq |\Delta^n_k X|\right) \xrightarrow{\mathcal{P}} \nu_g((0,\cdot]\times(|x|\geq\varepsilon_i)) \quad , \quad i\in\mathbb{N}.$$

Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence of positive constants , $\gamma_k \downarrow 0$ , $P\left(\nu_g(\{t\}\times(|x|\geq\varepsilon_i))=\gamma_k \ , \ t\in\mathbb{R}^+\right)=0 \qquad i,k\in\mathbb{N}$ . If we denote

$$\sigma^{i0}=0 \quad , \quad \sigma^{ik}=\inf\left[t>\sigma^{i,k-1}\ , \nu_g(\{t\}\times(|x|\geq\varepsilon_i))>\gamma_k\right]$$

then repeating the arguments from the proof of Proposition 2 we obtain that the property $(T)$ holds for every predictable $\mathcal{F}$ stopping time of the form

$$\sigma=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\sigma^{ik}\, I\left(\sigma=\sigma^{ik}\right)+\{+\infty\}\, I\left(\sigma\neq\sigma^{ik}\ \ i,k\in\mathbb{N}\right).$$

And as a consequence the property $(T)$ is fulfilled also in the general case. ■

## 4.4 The class $B_{loc}(T,D)$ .

Exactly in the same way as in the proof of Proposition 1 we obtain that the bounded process $B$ belongs to $B(T,D)$ iff one of the following two conditions is satisfied

$$(70)\qquad \left(B\circ\rho_n,\ \widetilde{B\circ\rho_n}\right)\xrightarrow{\mathcal{P}}(B\ ,\ B) \qquad \text{in } D(\mathbb{R}^2) \quad ,$$

$$(71)\qquad \widetilde{B\circ\rho_n}\xrightarrow{\mathcal{P}} B \qquad \text{in } D(\mathbb{R}) \ .$$

Now we collect fundamental properties of the class $B_{loc}(T,D)$ .

Proposition 4. (i) $B_{loc}(T,D)$ is a vector space.

(ii) If $B\in B_{loc}(T,D)$ and $B$ is a local martingale then $B=0$ .

(iii) If $B\in B_{loc}(T,D)$ then $B\in S_g(T,D)$ and has a triplet of characteristics $(B^h_g,\sigma^2_g,\nu_g)$ for which : $B^h_g=B^h$ ,

$\sigma_g^2 = 0$ , and $\nu_g$ is equal to the jump-measure $N$ associated to the process $B$ .

Proof. It is clear that in the proof of (i) , (ii) and also (iii) (by Corollary 4 ) it suffices to consider $B(T,D)$ instead of $B_{loc}(T,D)$. In this case (i) and (ii) are evident .

Therefore we give a proof of (iii) only . Let $B \in B(T,D)$ . We will show that the conditions (37) - (39) in Proposition 1 are satisfied. Since the process $B$ satisfies the condition (T) it is obvious that $B - B^h$ fulfills the condition (T) too.

By Proposition 2

$$\widetilde{B \circ f_n} - \widetilde{B^h \circ f_n} = \widetilde{(B - B^h) \circ f_n} \xrightarrow{P} B - B^h \ .$$

By Lemma 4 and Corollary 3

$$(72)\quad \widetilde{(B \circ f_n)^h} = \widetilde{(B \circ f_n)^h - B \circ f_n} + \widetilde{B \circ f_n} \xrightarrow{P} (B^h - B) + B = B^h$$

i.e. the condition (37) is satisfied with $B_g^h = B^h$ .

By the arguments used previously , (72) and Davis-Burkholder-Gundy inequality imply that $[(B \circ f_n)^h - \widetilde{(B \circ f_n)^h}](q) \xrightarrow{P} 0$ , $q \in \mathbb{R}^+$.

Finally by Corollary 3

$$\widetilde{[(B \circ f_n)^h - (\widetilde{B \circ f_n})^h]}(q) \xrightarrow{P} 0 \ , \quad q \in \mathbb{R}^+ \ ,$$

i.e. the condition (38) follows with $\sigma_g^2 = 0$ .

Similarly by Proposition 2 , Lemma 5 and Corollary 3

$$\widetilde{\int_{\mathbb{R}} f(x)\,(N \circ f_n)\,(dx)} \xrightarrow{P} \int_{\mathbb{R}} f(x)\,N(dx) \qquad , f \in C_v(0)$$

where $N$ is the jump-measure associated to the process $B$ .

Therefore the condition (39) is satisfied too . Hence $B \in S_g(T,D)$. ■

## 5. Proofs of theorems.

### 5.1 Proof of Theorem 1.

Let us suppose that $X$ is a semimartingale for which the condition (T) holds. By Proposition 1 it is sufficient to check that the set of conditions (37) - (39) is fulfilled. The following proposition is very useful in the proof of (37) - (39) .

Proposition 5. Let $X$ be an $\mathcal{F}$ adapted process and let $c$ be some constant $c > 0$ . The following implications are true :

(i) if $\sup_t |X^h(t)| < c$ then

$$\sup_{t \le q} \left| \widetilde{X^h \circ \rho_n}(t) - \widetilde{(X \circ \rho_n)}^h(t) \right| \xrightarrow{\mathcal{P}} 0 \quad , q \in \mathbb{R}^+ ,$$

(ii) if (37) holds and $\sup_t |X^h(t)| < c$ , $\sup_t [M_g^h](t) < c$ (where $M_g^h \overset{df}{=} X^h - B_g^h$) then

$$\sup_{t \le q} \left| \widetilde{[M_g^h] \circ \rho_n}(t) - \left[ \widetilde{(X \circ \rho_n)}^h - \widetilde{(X \circ \rho_n)^h} \right](t) \right| \xrightarrow{\mathcal{P}} 0 \quad , q \in \mathbb{R}^+,$$

(iii) if $f \in C_{v(0)}$ and $\sum_t f(\Delta X(t)) < c$ then

$$\sup_{t \le q} \left| \widetilde{\int_{\mathbb{R}} f(x) N(dx) \circ \rho_n}(t) - \widetilde{\int_{\mathbb{R}} f(x) (N \circ \rho_n)(dx)}(t) \right| \xrightarrow{\mathcal{P}} 0 , \ q \in \mathbb{R}^+.$$

Proof. The conditions (i) and (iii) are easy consequences of Corollary 3 and, Lemma 4 and 5 respectively. In order to prove (ii) first let us observe that

$$\widetilde{[M_g^h] \circ \rho_n} = \widetilde{[M_g^h \circ \rho_n]} = \widetilde{[(X^h - B_g^h) \circ \rho_n]} .$$

On other hand we have the following estimation :

$$\begin{aligned} &\mathrm{Var}\left( \left[ X^h \circ \rho_n - \widetilde{X^h \circ \rho_n} \right] - \left[ \widetilde{(X \circ \rho_n)}^h - \widetilde{(X \circ \rho_n)^h} \right] \right)(q) \\ &\le 8c \sum_{k=1}^{r_n(q)} \left( |\Delta_k^n X^h - h(\Delta_k^n X)| + |E_{k-1}^n \Delta_k^n X^h - E_{k-1}^n h(\Delta_k^n X)| \right) \\ &= 8c \mathrm{Var}\left( X^h \circ \rho_n - \widetilde{(X \circ \rho_n)}^h \right)(q) + 8c \mathrm{Var}\left( \widetilde{X^h \circ \rho_n} - \widetilde{(X \circ \rho_n)}^h \right)(q) . \end{aligned}$$

Thus twofold application of Corollary 3 enables us to test (ii) by simply examining if

$$(73) \quad \sup_{t \le q} \left| \left[ \widetilde{(X^h - B_g^h) \circ \rho_n} \right](t) - \left[ \widetilde{X^h \circ \rho_n} - \widetilde{X^h \circ \rho_n} \right](t) \right| \xrightarrow{\mathcal{P}} 0 , \ q \in \mathbb{R}^+.$$

It is clear that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$

$$\begin{aligned} \left[ \widetilde{(X^h - B_g^h) \circ \rho_n} \right](t) - \left[ \widetilde{X^h \circ \rho_n} - \widetilde{X^h \circ \rho_n} \right](t) &= \sum_{k=1}^{r_n(t)} E_{k-1}^n \left( \Delta_k^n B_g^h - E_{k-1}^n \Delta_k^n B_g^h \right)^2 \\ &\quad - 2 \sum_{k=1}^{r_n(t)} E_{k-1}^n \left( \Delta_k^n X^h - E_{k-1}^n \Delta_k^n X^h \right) \left( \Delta_k^n B_g^h - E_{k-1}^n \Delta_k^n B_g^h \right) . \end{aligned}$$

Since $B_g^h \in B_{loc}(T,D)$ the first term converges to 0 in probability. Now, let us note that second sum is of the form $\widetilde{[M^n, N^n]}(t)$, where $M^n$ , $N^n$ are two local martingales given by the formulas $N^n \overset{df}{=} \widetilde{B_g^h \circ \rho_n} - \widetilde{B_g^h \circ \rho_n}$ , $M^n \overset{df}{=} \widetilde{X^h \circ \rho_n} - \widetilde{X^h \circ \rho_n}$ . By the Kunita -Wata-

nabe inequality

$$\mathrm{Var}\,[M^n,N^n](q) \leq \{[M^n](q)[N^n](q)\}^{\frac{1}{2}} \qquad , q \in \mathbb{R}^+ .$$

Since by the arguments used previously $[N^n](q) \xrightarrow{\mathcal{P}} 0$ and $[M^n] \xrightarrow{\mathcal{P}} [M_g^h]$ in $D(\mathbb{R})$ where $\sup_t [M_g^h](t) < c$ it follows by Corollary 3 that (73) and (ii) are satisfied. ■

Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a localizing sequence $\tau_k \uparrow +\infty$ for which $\sup_{t\leq\tau_k} |X^h(t)| \leq k$ , $k \in \mathbb{N}$ . By Proposition 2

$$(74) \qquad \widetilde{X^{\tau_k,h}\circ\rho_n} \xrightarrow{\mathcal{P}} \widetilde{X^{\tau_k,h}} = B^{h,\tau_k} \qquad , k\in\mathbb{N} .$$

Therefore by (i) we have

$$\widetilde{(X^{\tau_k}\circ\rho_n)^h} \xrightarrow{\mathcal{P}} B^{h,\tau_k} \qquad , k\in\mathbb{N} .$$

Hence there exists a sufficiently slowly increasing sequence $\{k_n\}$ $k_n \uparrow +\infty$ such that

$$\widetilde{(X^{\tau_{k_n}}\circ\rho_n)^h} \xrightarrow{\mathcal{P}} B^h .$$

Finally by Corollary 3 the condition (37) is fulfilled. By exactly the same arguments the conditions (38), (39) are satisfied, too . To obtain the converse implication we use Proposition 3. ■

Proof of Corollary 1. First let us note that if a predictable $\mathcal{F}$ stopping time $\sigma$ is of the form (18) then the condition (T) follows by Lemma 6 . Next let $\sigma$ be of the form (19). Then without loss of generality we may assume that $\sigma \leq q$ for some constant $q > 0$ . Let us put $\varepsilon_n \overset{df}{=} \max_{k\leq r_n(q)} (t_{n,k+1} - t_{nk})$ , $n \in \mathbb{N}$ . Since for $\varepsilon_n \leq c$ $\rho_n(\tau + c)$ is an $\mathcal{F}^{\rho_n}$ stopping time the convergence $\varepsilon_n \downarrow 0$ implies the condition (T) . ■

## 5.2 Proof of Theorem 2.

We start with the proof of property (iii) . Let us assume that $X \in S_g(T,D)$ . Therefore by Proposition 1 the condition (37) is fulfilled. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a localizing sequence for which $\sup_{t\leq\tau_k} |B_g^h(t)| \leq k$ , $k \in \mathbb{N}$ . By (71) $B_g^{h,\tau_k} \in B(T,D)$ . Now, let us consider the process $X - B_g^h$ . Repeating the arguments from Jacod [8] we can prove that $X - B_g^h$ is a semimartingale with the triple of predictable characteristics $(B^h, C^2, \nu^h)$.

By Proposition 3 and Proposition 4 the processes $X$, $B_g^h$ satisfy the condition $(T)$. As a consequence the semimartingale $X - B_g^h$ fulfills the condition $(T)$, too. Hence Theorem 1 implies that $X - B_g^h \in S_g(T,D)$.

Let us suppose that $X$ is a semimartingale $T$ tangent to PII and the process $B$ belongs to $B_{loc}(T,D)$. We show that $X + B \in S_g(T,D)$. Let $\{\sigma^k\}_{k\in\mathbb{N}}$ be a sequence of $\mathcal{F}$ stopping times such that :

$$\sigma^0 = 0 \quad , \quad \sigma^k = \inf\left[t > \sigma^{k-1} \, , \, \max(|\Delta X(t)|, |\Delta B(t)|) > 4^{-1}\right].$$

We will consider new processes defined as follows :

$$X_1(\cdot) = \sum_{\sigma^k \le \cdot} \Delta X(\sigma^k) \quad , \quad X_2 = X - X_1 \quad , \quad B_1(\cdot) = \sum_{\sigma^k \le \cdot} \Delta B(\sigma^k) \quad , \quad B_2 = B - B_1 \, .$$

Let us observe that we have the following equality

$$(75) \qquad \begin{aligned} (X + B)^h &= (X_2 + B_2)^h + (X_1 + B_1)^h \\ &= X_2 + B_2 + (X_1 + B_1)^h \, . \end{aligned}$$

Since the processes $X_1^h$, $B_1^h$, $(X_1 + B_1)^h$ have locally integrable variation and satisfy the condition $(T)$ by Proposition 2 and Proposition 5 (i) : $(X_1 \circ \rho_n)^h \xrightarrow{\mathcal{P}} X_1^h$, $(B_1 \circ \rho_n)^h \xrightarrow{\mathcal{P}} B_1^h$, $((X_1 + B_1) \circ \rho_n)^h \xrightarrow{\mathcal{P}} (X_1 + B_1)^h$. It is easy to see that : $(X_2 \circ \rho_n)^h \xrightarrow{\mathcal{P}} X_2$ and $(B_2 \circ \rho_n)^h \xrightarrow{\mathcal{P}} B_2$. Therefore by (75) and Proposition 5 (i)

$$((X + B) \circ \rho_n)^h \xrightarrow{\mathcal{P}} X_2 + B_2 + (X_1 + B_1)^h$$

and the condition $(37)$ is fulfilled. The remaining conditions $(38)$ and $(39)$ are also corollaries from Proposition 2, 3 and 5 (ii), (iii). Hence the proof of (iii) and (ii) is complete.

The property (i) is an easy consequence of (ii), Proposition 4 and the simple remark that the set of semimartingales $T$ tangent to PII forms a vector space. Let us also observe that the property (iv) is clear by Proposition 2 and (10). ■

Proof of Corollary 2. Let us suppose that $X$ is a process with conditionally independent increments given $\sigma$ algebra $G$. By the arguments from Jacod [8] there exists a system of $G$ measurable characteristics $(B_g^h, \sigma_g^2, \nu_g)$ satisfying the properties (22) - (24) for which $X - B_g^h$ is a semimartingale. Since $G \subset \mathcal{F}(0)$

and the predictable stopping times $\{\sigma^k\}_{k\in\mathbb{N}}$ exhausting the predictable jumps of $X$ are $G$ measurable so for all $k,n \in \mathbb{N}$ $\rho_n(\sigma^k)$ is $\mathcal{F}_{\rho_n}$ stopping time. Therefore by Theorem 1 $X - B^h_g \in S_g(T,D)$. Similarly by Theorem 2 (iv) $B^h_g \in B_{loc}(T,D)$. Using Theorem 2 (i) the proof is complete. ■

## 5.3 Proof of Theorem 3.

Let $X$ be a process $T$ tangent to PII with random measure $\mu^X_g$. First we define the family of characteristic functions of $\Phi^X_g$. We take

$$\Phi^X_g(\theta,t) \stackrel{df}{=} \int_{\mathbb{R}} \exp i\theta x \, \mu^X_g(t,dx) \qquad \theta\in\mathbb{R}, t\in\mathbb{R}^+.$$

**Proposition 6.** *Let $X \in S_g(T,D)$. Then for each $\theta\in\mathbb{R}$ $\Phi^X_g$ is a predictable process such that the process $Y_\theta$ defined by formula :*

$$Y_\theta(t) \stackrel{df}{=} \exp i\theta X(t) / \Phi^X_g(\theta,t) \qquad t\in\mathbb{R}^+.$$

*is a local martingale on the stochastic interval $[\![ 0, R_\theta [\![$ where $R_\theta = \inf [t : |\Phi^X_g(\theta,t)| = 0]$.*

**Proof.** Let $Z = X - B^h_g$. Then

$$Y_\theta(t) = \left(\exp i\theta Z(t) \,/\, \Phi^X_g(\theta,t)\right) \exp(-i\theta B^h_g(t))$$

and a simple computation based on Theorem 2 (iii) shows that

$$\Phi^X_g(\theta,t)\exp(-i\theta B^h_g(t)) = \Phi^Z_g(\theta,t) \quad .$$

Since the local martingale property for $\{\exp i\theta Z(t) / \Phi^Z_g(\theta,t)\}_{t\in\mathbb{R}^+}$ is well known the proof is finished. ■

**Acknowledgements.** I would like to thank Professor Jean Jacod for careful reading of the manuscript which enabled to avoid several mistakes and misprints.

Institute of Mathematics
Nicholas Copernicus University
ul. Chopina 12/18
87-100 Toruń , Poland

References.

1. Aldous, D.J.: A concept of weak convergence for stochastic processes viewed in the Strasbourg manner. Preprint, Statist.Laboratory Univ.Cambridge 1979

2. Billingsley,P.: Weak Convergence of Probability Measures. New York : Wiley 1968

3. Dellacherie,C, Meyer,P.A. : Probabilities and Potential. North.-Holland 1978

4. Dellacherie,C,Doleans-Dade,C : Un contre-example au probleme des Laplaciens approches. Lecture Notes in Math. 191 , 128-140 , Springer-Verlag 1970

5. Grigelionis,B. : On the martingale characterisation of stochastic processes with independent increments. Lietuvos Mat. Rinkynys XVII, 1, 75-86, 1977

6. Grigelionis,B,Mikulevicius,R : On weak convergence of semimartingales. Lietuvos Mat. Rinkynys XXI, 3, 9-24, 1981

7. Jacod,J. : Calcul stochastique et problemes de martingales. Lecture Notes in Math. 714 , 1979

8. Jacod,J. : Processus a accroissements independants : une condition necessaire et suffisant de convergence. Z.Wahr.Verw. Gebiete 63, 109-136, 1983

9. Jacod,J. : Une generalization des semimartingales : les processus admettant un processus a accroissements independants tangent . Lecture Notes in Math. 1059 , 91-118, 1984

10. Jacod,J.,Kłopotowski,A.,Memin,J. : Theoreme de la limite centrale et convergence fonctionnelle vers un processus a accroissements independants, la methode des martingales. Ann. IHP B XVIII , 1-45, 1982

11. Jakubowski,A.,Słomiński,L. : Extended convergence to continuous in probability processes with independent increments. Probab. Th. Rel. Fields 72, 55-82, 1986

12. Kubilius,K. : On necessary and sufficient conditions for the convergence to quasicontinuous semimartingale. Preprint , 1985

13. Lenglart,E. : Relation de domination entre deux processes. Ann. IHP B XIII , 171-179, 1977

14. Lindvall,T. : Weak convergence of probability measures and random functions in the function space $D[0,\infty)$ . J.Appl. Probab. 10, 109-121, 1973

15. Meyer,P.: Integrales Stochastiques. Lecture Notes in Math. 39 , 72-94, 1967

16. Rebolledo,R. : Central limit theorems for local martingales. Z.Wahr.Verw. Gebiete 51, 262-286, 1982

17. Słomiński,L. : Necessary and sufficient conditions for extended convergence of semimartingales. Probab. and Math. Statistics 7, Fasc.1, 77-93, 1986