SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

P. MC GILL

Integral representation of martingales in the brownian excursion filtration

Séminaire de probabilités (Strasbourg), tome 20 (1986), p. 465-502

<http://www.numdam.org/item?id=SPS_1986__20__465_0>

Integral representation of martingales in the Brownian excursion filtration

Paul McGill

Department of Mathematics, Maynooth College, Co. Kildare, Ireland

If B_t is a Brownian motion, whose natural filtration we write as $\mathcal{B}_t$, then it is a well-known theorem of Ito that every $\mathcal{B}_t$ martingale M_t can be represented as a stochastic integral

$$M_t = M_0 + \int_0^t u_s dB_s$$

where u_t is a $\mathcal{B}_t$ predictable process. This means that every $\mathcal{B}_t$ martingale is continuous and that, in a certain sense, $\mathcal{B}_t$ is a one-dimensional filtration. In [14] Williams has introduced the notion of the excursion σ-field $\mathcal{E}^x$ below x, defined as the σ-field generated by the time change of B_t which deletes the excursions above x. Walsh [12] takes the theory a stage further by considering the associated filtration $(\mathcal{E}^x, x \in R)$. He points out that the Ray-Knight results (see also [6]) on the Markovian properties of the local time in the space variable enable us to generate an infinite number of orthogonal $\mathcal{E}^x$ martingales, so that this filtration has infinite dimension [2]. He also poses the following questions.

(a) Is every $\mathcal{E}^x$ martingale continuous?

(b) Can every $\mathcal{E}^x$ martingale be represented as a stochastic integral?

Williams [15] has proved that the answer to (a) is yes. See [7] for more details on the recurrent case. The precise formulation of (b) is more difficult because there is no obvious choice of integrating process. However Walsh has suggested that one look for an integral with respect to the local time $L(x,t)$, using the family of $\mathcal{E}^x$ martingales associated with the Ray-Knight theorems. On the other hand the conditional excursion formulae of Williams [15] provide us with a dense family $\mathcal{E}^x$

martingales. These have the added advantage that they are defined for all levels, in contrast to the Ray-Knight martingales which are only defined on a semi-infinite interval. In [13] Walsh defines a double integral, working directly with $L(x,t)$, and he then proves that the conditional excursion formulae of Williams have the required integral representation. To do this he needs to derive certain complicated analogues of Green's formula.

Our idea is to modify Walsh's procedure by integrating with respect to a different process. In order to explain more fully let us write Tanaka's formula

$$B_t \wedge x = B_0 \wedge x + \int_0^t 1_{(B_s<x)}dB_s - \tfrac{1}{2}L(x,t)$$

Time changing (see section one below) so that we only measure time spent below x gives

$$\tilde{B}(x,t) = B_0 \wedge x + \tilde{\beta}(x,t) - \tilde{L}(x,t)$$

where $\tilde{B}(x,t)$ is now a reflecting Brownian motion on $(-\infty,x]$, $\tilde{\beta}(x,t)$ is a Brownian motion, and $\tilde{L}(x,t)$ is the reflecting Brownian local time at x. It is easy to see, from the Ray-Knight theorem, that $(\tilde{L}(x,t), x \in R)$ is an $\mathcal{E}^x$ semimartingale so that we can use it to define stochastic integrals in the variable x. If we write $R^x_\lambda f(y)$ to denote the resolvent of Brownian motion killed when it hits $(-\infty,x]$ then the first order conditional excursion formula of [7] can be written as

$$\mathbf{E}[\int_0^\infty e^{-\lambda t}f(B_s)ds|\mathcal{E}^x] \quad = \quad R^x_\lambda f(0) +$$

$$\exp\{-\sqrt{2\lambda}x^-\}[R^x_\lambda f'(x+)\int_0^\infty \exp\{-\lambda t - \sqrt{2\lambda}\tilde{L}(x,t)\}d_t\tilde{L}(x,t)+$$

$$\int_0^\infty \exp\{-\lambda t - \sqrt{2\lambda}\tilde{L}(x,t)\}f(\tilde{B}(x,t))dt]$$

where we have assumed that $B_0 = 0$. Here, and throughout the rest of this paper, $R^x_\lambda f'(x+)$ is the derivative in y of $R^x_\lambda f(y)$ evaluated at $y = x+$. One is now led to guess that this $\mathcal{E}^x$ martingale can be written as some sort of stochastic integral in the space variable involving $\tilde{L}(x,t)$, the essential contrast with [13] being that we work in an intrinsic time scale. The integrals we use are parameterised stochastic integrals rather than the more sophisticated double integrals of Walsh and our approach has two important advantages. First of all we obtain all the $\mathcal{E}^x$ martingales from a single infinite-dimensional Markovian process. And also we identify a special dense subclass of martingales for which the representation takes a particularly simple form. It is this, rather than the representation itself, which we hope will prove to be most useful for subsequent calculation.

The article is organised as follows. In the first section we look at the basic continuity properties of the processes $\tilde{B}$, $\tilde{\beta}$ and $\tilde{L}$. We show here that, for each fixed t, $L(x,t)$ is a semimartingale in the filtration $\mathcal{E}^x$. It is perhaps surprising, in view of the complexity of some if the subsequent calculations, that this is an immediate consequence of the Ray-Knight theorem once we note an obvious remark. In the second section we prove that the excursion filtration is right continuous. For this we use the dense set of martingales found by Williams [15]. And from this we are able to see that $\tilde{L}$ and $\tilde{B}$ are strongly Markovian in the space variable. Such results have already been looked at by Walsh [12], but our reasoning is different.

The next section states the conditional excursion theorem (for a proof see [8]) and uses it to obtain information concerning the conditional joint law of $\tilde{L}(x,t)$ and $\tilde{B}(x,t)$ given the σ-field $\mathcal{E}^a$. The calculations are in principle fairly straightforward, though they should perhaps be omitted on a first reading. From this we are able to find the Doob-Meyer decomposition of $\tilde{L}_x$. This is the main result of the paper and yields the extra information that the process $\int^x \tilde{L}^2(y,t)dy$ has a density in the time variable. Because $\tilde{L}(x,t)$ is a semimartingale for each value of t, we can define the stochastic integral $\int^x K(y,t)d_y\tilde{L}(y,t)$ for suitable processes K. From this we are able to construct a dense family of continuous $\mathcal{E}^x$ martingales by using processes of the form $\int_0^\infty dt \int_{-\infty}^x K(y,t)d_y\tilde{L}(y,t)$. The hard part is to show that the conditional excursion martingales of Williams can be written in the appropriate form using these parameterised stochastic integrals. Once this is achieved, the general representation result is immediate. The results given here have already been announced in the note [9].

N.B. Throughout the paper C shall be a constant, whose value may differ from one equation to the next. Also it is often convenient not to specify the lower limit on certain integrals in the space variable.

Acknowledgement I wish to thank IHES (Bûres-sur-Yvette) for support during the intial stages of this work, most of which was carried out while I was employed at N.U.U. in Coleraine. Thanks go also to those who have helped by commenting on previous versions.

§1. Properties of the process at a fixed intrinsic time

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space which supports a Brownian motion B_t. We will take $B_0 = 0$. Then write

$$A(x,t) = \int_0^t 1_{(B_s < x)}\, ds$$

and denote by $\tau(x,.)$ the right continuous inverse of the increasing process $A(x,.)$. Because $A(x,t)$ is $\mathcal{B}_t$ adapted it follows that $\tau(x,t)$ is a $\mathcal{B}_t$ stopping time. Define $\tilde{B}(x,t) = B(\tau(x,t))$ noting that this process, whose completed right continuous filtration we represent by $\tilde{\mathcal{B}}(x,t)$, satisfies the inequality $\tilde{B}(x,t) \leq x$ for all $t \geq 0$. Then following David Williams [14], we define $\mathcal{E}^x = \tilde{\mathcal{B}}(x, \infty)$ to be the excursion σ-field of B_t below x. The increasing family $(\mathcal{E}^x, x \in R)$ is called the excursion filtration of B_t. It is known from the Ray-Knight theorems [12] that the martingale dimension of this filtration is infinite, so it is certainly not the filtration of any finite dimensional Brownian motion.

We will write $L(x,t)$ to denote the almost surely bicontinuous version of the local time of B_t, which we normalise in such a way that the occupation density formula becomes

$$\int_0^t f(B_s)ds = \int f(x)L(x,t)dx$$

Also write $\tilde{L}(x,t) = \frac{1}{2}L(x,\tau(x,t))$. In order to identify this as the local time of $\tilde{B}(x,t)$ at x we can proceed as follows. By the Tanaka formula

$$B_t \wedge x = B_0 \wedge x + \int_0^t 1_{(B_s < x)}\, dB_s - \tfrac{1}{2}L(x,t)$$

When we time change by $\tau(x,t)$ this becomes

$$\tilde{B}(x,t) = B_0 \wedge x + \tilde{\beta}(x,t) - \tilde{L}(x,t) \qquad (1.a)$$

where $\tilde{\beta}(x,t) = \int_0^{\tau(x,t)} 1_{(B_s < x)}\, dB_s$ is, by Paul Levy's martingale characterisation, a $\mathcal{B}(\tau(x,t))$ Brownian motion. Now regard $(1.a)$ as a stochastic differential equation driven by $\tilde{\beta}(x,t)$ and subject to the boundary condition that $\tilde{L}(x,t)$ increases only when $\tilde{B}(x,t) = x$.

Theorem 1.1 (see [4]) (i) $\tilde{B}(x,t)$ is a reflecting Brownian motion on $(-\infty, x]$ and is the unique strong solution of (1.a).

(ii) $\tilde{L}(x,t) = \sup\{(\tilde{\beta}(x,s) - x^+)^+ : 0 \le s \le t\}$ and hence it is $\tilde{\mathcal{B}}(x,t)$ adapted.
(iii) $\mathcal{B}(\tau(x,t))$ and $\mathcal{E}^x$ are conditionally independent given $\tilde{\mathcal{B}}(x,t)$.

Proof: (iii) Let F be bounded and $\mathcal{E}^x$ measurable. We must prove that $F_t = \mathbf{E}[F|\tilde{\mathcal{B}}(x,t)] = \mathbf{E}[F|\mathcal{B}(\tau(x,t))]$. But this follows if every $\tilde{\mathcal{B}}(x,t)$ martingale is a $\mathcal{B}(\tau(x,t))$ martingale. Which is true by (i) and the Ito representation theorem for $\tilde{\mathcal{B}}(x,t)$.

Let θ_t be the shift operator of B_t, and suppose that S is a $\mathcal{B}_t$ stopping time. Define $\mathcal{E}^x \circ \theta_S$ to be the σ-field generated by the process $\{B(\tau(x,t)) \circ \theta_S, t \ge 0\}$, and remark that it starts at the position $B_S \wedge x$. This need not be $\mathcal{E}^x$ measurable. The next result therefore generalises [6] Lemma 4.3.

Lemma 1.2 Let S be any $\mathcal{B}_t$ stopping time. If f is any $\mathcal{B}_S$ measurable function then for each bounded measurable g we have

$$\mathbf{E}[fg \circ \theta_S|\mathcal{E}^x] = \mathbf{E}[f\mathbf{E}_{B(S)}[g|\mathcal{E}^x]|\mathcal{E}^x]$$

Hence, if B_S is $\mathcal{E}^x$ measurable we have conditional independence of $\mathcal{B}_S$ and $\sigma\{B_t \circ \theta_S : t \ge 0\}$ given $\mathcal{E}^x$.

Proof: Note first that $\sigma(\mathcal{B}_S, \mathcal{E}^x) = \sigma(\mathcal{B}_S, \mathcal{E}^x \circ \theta_S)$ so by the strong Markov property applied at the time S

$$\mathbf{E}[fh(gH \circ \theta_S)] = \mathbf{E}[fh\mathbf{E}_{B(S)}[gH]]$$

where h is $\mathcal{B}_S$ measurable, H is $\mathcal{E}^x$ measurable, and both functions are bounded. Thus we see that $\mathbf{E}[f(gH \circ \theta_S)|\mathcal{B}_S] = f\mathbf{E}_{B(S)}[gH]$ which implies that $\mathbf{E}[fg \circ \theta_S|\sigma(\mathcal{B}_S, \mathcal{E}^x \circ \theta_S)] = f\mathbf{E}_{B(S)}[g|\mathcal{E}^x]$. The proof is finished by doing the projection onto $\mathcal{E}^x$.

The next remark will prove extremely useful. Although it is almost obvious it plays a key role in what follows.

Remark 1.3 Let $\rho : \Omega \mapsto R^+$ be measurable. Then $\tau(x,\rho) \ge \tau(y,\rho)$ whenever $x \le y$.

We shall need the following version of the Ray-Knight theorem. This formulation is due essentially to Williams [14] and a proof can be found in either [6] or [12]. First let us note that the stochastic differential equation

$$Z(a,x) = Z(0,x) + 2\int_0^a \sqrt{Z(b,x)}d\beta_b + 2(a \wedge x)$$

where $(\beta_a, a \geq 0)$ is a Brownian motion, has a unique continuous strong solution (see [6]) which is absorbed when the process reaches zero. And the hitting time of zero is finite almost surely.

Theorem 1.4 Let $\xi \geq 0$ be any $\mathcal{E}^x$ measurable random variable. Then given $\mathcal{E}^x$, the process $\{L(x+a, \tau(x,\xi)), a \geq 0\}$ has the same law as the process $\{Z(a, x^-), a \geq 0\}$ started at $2\tilde{L}(x,\xi)$.

We point out that by Theorem 1.1 $\tilde{L}(x,t)$ is $\mathcal{E}^x$ measurable. Also notice that if $y > x$ then

$$\tau(x,\xi) = \tau(y, A(y, \tau(x,\xi))) \tag{1.b}$$

and $A(y,\tau(x,\xi))$ is $\mathcal{E}^y$ measurable. Therefore we see that $L(y,\tau(x,\xi)) = 2\tilde{L}(y, A(y,\tau(x,\xi)))$ is $\mathcal{E}^y$ measurable. Also starting from (1.b) we can show that the analogues of the identifiable rectangles, defined by Walsh in [13], form a pre-ring. However we do not need this here. What we shall need though is the following formula due to Williams [14]. His sketched proof is given in more detail in [6] or [12].

Corollary 1.5 If $x \leq a$ then

$$\mathbf{E}[\exp\{-\lambda\tau(x,\xi)\}|\mathcal{E}^a] = \exp\{-\sqrt{2\lambda}a^- - \lambda\varsigma - \sqrt{2\lambda}\tilde{L}(a,\varsigma)\}$$

where $\varsigma = A(a, \tau(x,\xi))$.

The next lemma should be compared with some of the results of [1] although it is much easier. However we also use the Ray-Knight theorem in the proof.

Lemma 1.6 If K is any compact subset of the real line then for every $t \geq 0$ and $x \leq \inf K$ then we have

$$\mathbf{E}[\sup_{a\in K} L^p(a, \tau(x,t))|\mathcal{E}^x] \leq C[1 + \tilde{L}(x,t)]^p$$

for all $p \geq 0$.

Proof: By the previous theorem the process $\{L(a, \tau(x,t)), a \geq x\}$ is equivalent in law to a process which is majorised on every sample path (see [17]) by a $BES^2(2)$ process $\{Z(a-x,\infty), a \geq x\}$. But by writing the latter as the sum of the squares of two independent Brownian motions we find

$$\mathbf{E}[\sup_{a\in K} L^p(a,\tau(x,t))|\mathcal{E}^x] \leq \mathbf{E}[\sup_{a\in K} Z^p(a-x,\infty)|\mathcal{E}^x] \leq$$

$$\mathbf{E}(\sup_{a\in K} \beta_a^2 + \sup_{a\in K} \bar{\beta}_a^2)^p] \leq C(1+\tilde{L}(x,t))^p$$

where for the last part we use independence and the inequality (see [3]) of Burkholder-Davis-Gundy.

Notice that $\sup_a L(a,\tau(x,t))$ is not in L_p, for this would imply uniform integrability of the $BES^2(0)$ process.

Lemma 1.7 For every $p \geq 1$

$$E[|\tilde{\beta}(x,t) - \tilde{\beta}(y,s)|^{2p}] \leq C||(x,t)-(y,s)||^p$$

where C is some constant and $||.||$ is the usual norm on R^2.

Proof: Let us write, for $y < x$

$$M(x,y;s,t) = \int_0^{\tau(x,t)} 1_{(B_u<x)} dB_u - \int_0^{\tau(y,s)} 1_{(B_u<y)} dB_u$$

Working on a fixed compact subset of R^2 and considering $M(x,y;s,t)$ as a stopped $\mathcal{B}_t$ martingale, we use the bilinearity of the mutual quadratic variation to compute (and forgive the abuse of notation)

$$< M(x,y,;s,t) >= t + s - 2A(y, \tau(x,t) \wedge \tau(y,s))$$

But, by Remark 1.3, notice that $A(y,\tau(x,s\wedge t)) \leq A(y,\tau(x,t)\wedge\tau(y,s)) \leq A(y,\tau(y,t\wedge s))$ which can be written as

$$t\wedge s - \int_y^x L(a,\tau(x,t\wedge s))da \leq A(y,\tau(x,t)\wedge\tau(y,s)) \leq t\wedge s$$

And so we obtain the estimate

$$< M(x,y;s,t) > \leq |t-s| + 2\int_y^x L(a,\tau(x,t\wedge s))da$$

And by the inequality of Burkholder-Davis-Gundy there is a universal constant C_p such that

$$\mathbf{E}[M^{2p}(x,y;s,t)] \leq C_p\mathbf{E}[< M(x,y;s,t) >^p]$$

If however $n \geq 1$ then by Jensen's inequality and the fact that we know the law of the local time

$$\mathbf{E}\left[\left(\int_y^x L(a,\tau(x,t\wedge s))da\right)^n\right] \leq \mathbf{E}\left[(x-y)^n\left(\int_y^x L(a,\tau(x,t\wedge s))\frac{da}{x-y}\right)^n\right]$$

$$(x-y)^{(n-1)}\mathbf{E}[\int_y^x L^n(y,\tau(x,t\wedge s))da] \leq C(x-y)^n$$

Thus for integer values of p we can use the binomial theorem to expand the estimate for $< M(x,y;s,t) >^p$, which when we use the above inequality gives us (once we put everything together again)

$$\mathbf{E}[M^{2p}(x,y;s,t)] \leq C(|s-t|+|x-y|)^p$$

as required.

We now recall the following well-known result. For the proof we can proceed as in the appendix to [10].

Kolmogoroff Criterion: Suppose that X_t is a process indexed by R^d which satisfies

$$\mathbf{E}[|X_t - X_s|^p] \leq C||t-s||^{d+\gamma} \qquad (p>0, \gamma>0)$$

Then X_t has a version whose paths are almost surely Hölder continuous of order δ for every $\delta < (\gamma/p)$. More precisely for each bounded region of R^2 there exists a constant C, independent of ω, such that

$$|\tilde{\beta}(x,t) - \tilde{\beta}(y,s)| < C||(x,t)-(y,s)||^\alpha$$

whenever $||(x,t)-(y,s)|| < \delta(\omega)$.

Corollary 1.8 $\tilde{\beta}(x,t)$ has a version satisfying a local Hölder inequality of order α for each $\alpha < \frac{1}{2}$.

Proof: This follows from the Kolmogoroff criterion since the statement of the above lemma holds for arbitrarily large values of p.

From Theorem 1.1 (ii) we can immediately deduce the following.

Corollary 1.9 The process $(\tilde{L}(x,t),(x,t) \in R \times R^+)$ has a version satisfying a local Holder condition of order α for every $\alpha < \frac{1}{2}$.

Proof: We have

$$\begin{aligned} |\tilde{L}(x,t) - \tilde{L}(y,s)| &\leq C(|t-s|^\alpha + |x-y|^\alpha) \\ &\leq 2C(|t-s| + |x-y|)^\alpha \end{aligned}$$

Which gives the result.

Theorem 1.10 (i) The random variables $(\tilde{L}(x,t), x \leq 0)$ are identical in law.
(ii) $(\tilde{L}(x,t), x \geq 0)$ is an $\mathcal{E}^x$ potential i.e. it is a supermartingale which vanishes at infinity.
(iii) The process $(\tilde{L}(x,t), x \in R)$ is a continuous uniformly integrable semimartingale.

Proof: (i) This follows by using the strong Markov property at the hitting time of $(-\infty, x]$.
(ii) For the supermartingale property let $0 < x < y$ and, using Remark 1.3, we get

$$\mathbf{E}[2\tilde{L}(y,t)|\mathcal{E}^x] = \mathbf{E}[L(y,r(y,t)) - L(y,r(x,t))|\mathcal{E}^x] + \mathbf{E}[L(y,r(x,t))|\mathcal{E}^x] \leq 2\tilde{L}(x,t)$$

by Theorem 1.4. To see that it is a potential let T_y be the first hitting time of y so that $\lim_n T_n = +\infty$ almost surely. But $\tilde{L}(x,t) = 0$ on the set $\{T_y > t\}$.
(iii) Continuity is proved above. Uniform integrability follows because for all $p \geq 0$

$$\mathbf{E}[\tilde{L}^p(x,t)] \leq \mathbf{E}[\tilde{L}^p(0,t)]$$

since $\tilde{L}(0,t)$ has the same law as the maximum of a Brownian motion. We now take $0 > y > x$ and find that

$$\mathbf{E}[2\tilde{L}(y,t)|\mathcal{E}^x] = \mathbf{E}[L(y,\tau(y,t)) - L(y,\tau(x,t))|\mathcal{E}^x]+$$

$$\mathbf{E}[L(y,\tau(x,t))|\mathcal{E}^x] \leq 2\tilde{L}(x,t) + 2(y-x).$$

again using Theorem 1.4 and Remark 1.3.

This proves a little more than we claimed, namely that $\tilde{L}(x,t) + x^-$ is an $\mathcal{E}^x$ supermartingale.

§2. Right continuity of $\mathcal{E}^x$

Most of the standard results in the general theory of processes [3] require that the filtration be right continuous. In the present context this is quite a difficult problem. The excursion filtration, as we have defined it, is assumed to be complete and traditionally this suffices if the underlying Markov process is a 'good' one. However we have not as yet been able to exhibit the process. So we try to show directly that $\mathcal{E}^{x+}$ differs from $\mathcal{E}^x$ only by null sets of the measure $\mathbf{P}$. The main difficulty is in applying the dominated convergence theorem to a suitably large class of projections.
In [12] the proof is carried out by using what is called the 'strong Markov property' of the excursion process. However our method turns on the use of the CMO martingales of Williams [15]. We have already shown in [7] how these may be calculated without the use of excursion theory. So first we write

$$K_t(n,\lambda,\mathbf{f}) = K_t(\lambda_1,\lambda_2,\ldots,\lambda_n;f_1,f_2,\cdots f_n)$$

$$= \int_0^t dt_n e^{-\lambda_n t_n} f_n(B_{t_n}) \int_0^{t_n} \cdots \int_0^{t_2} dt_1 e^{-\lambda_1 t_1} f_1(B_{t_1})$$

where the functions $\{f_n\}$ are always assumed to be continuous and to have compact support. Also it will be convenient to let $K_t(0,\lambda,\mathbf{f}) \equiv 1$.

Theorem 2.1 The filtration $\{\mathcal{E}^x, x \geq 0\}$ satisfies the usual conditions.

Proof: Since each $\mathcal{E}^x$ is defined to be complete it suffices to prove right continuity. Namely that for F any bounded measurable function on Ω we have

$$\lim_{\epsilon\downarrow 0} \mathbf{E}[F|\mathcal{E}^{x+\epsilon}] = \mathbf{E}[F|\mathcal{E}^x]$$

almost surely. This proves that $\mathcal{E}^{x+}$ and $\mathcal{E}^x$ differ by no more than $\mathbf{P}$ null sets. Moreover it suffices to project only those functionals of the Brownian path which are supported above the level x. And since it is enough to give the proof for a dense set of such F we can restrict ourselves to those functionals of the form $K_\infty(n, \lambda, \mathbf{f})$ where the functions $\{f_n\}$ are all supported on a compact subset of $(x, +\infty)$. To begin with we have the evaluation from [7] Theorem 2.2 of the first order CMO formula

$$\mathbf{E}[\int_0^\infty e^{-\lambda t} f(B_t)dt|\mathcal{E}^a] =$$

$$R_\lambda^a f(0) + \exp(-\sqrt{2\lambda}a^-)R_\lambda^a f'(a+)\int_0^\infty \exp\{-\lambda t - \sqrt{2\lambda}\tilde{L}(a,t)\}d_t\tilde{L}(a,t)$$

where we recall from the introduction that $R_\lambda^a f(x)$ is the resolvent of Brownian motion killed at a. We can now examine each term in turn. The first one $R_\lambda^a f(0)$ is continuous in a as can be seen from the explicit formula

$$R_\lambda^a f(x) = \frac{1}{\sqrt{2\lambda}}\int_a^\infty \left(e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}|2a-x-y|}\right) f(y)dy$$

The second term is continuous in a by the bicontinuity of $\tilde{L}$, and the explicit formula

$$R_\lambda^a f'(a+) = 2\int_a^\infty e^{-\sqrt{2\lambda}|a-y|} f(y)dy$$

We now consider the higher order formulae

$$\mathbf{E}[\int_0^\infty e^{-\lambda t} f(B_t)K_t(n,\lambda,\mathbf{f})dt|\mathcal{E}^a] =$$

$$\mathbf{E}[\int_0^{T_a} e^{-\lambda t} f(B_t)K_t(n,\lambda,\mathbf{f})dt] + \mathbf{E}[\int_{T_a}^\infty e^{-\lambda t} f(B_t)K_t(n,\lambda,\mathbf{f})dt|\mathcal{E}^a]$$

Examining each of these terms in turn we see that the first calculates by using the expression

$$\mathbf{E}_y[K_{T_a}(n,\lambda,\mathbf{f})] = R_{\mu_1}^a[f_1 R_{\mu_2}^a[f_2 \ldots [R_{\mu_n}^a[f_n]\ldots](y)$$

where $\mu_i = \lambda_i + \ldots + \lambda_n$. The second term is reduced, via the facilities of [7] Lemma 3.1 and 3.2, to the evaluation of a first order formula. The fact that the limit gives what we want follows by the same argument as in the first order case applied inductively, since at each stage the projection is continuous in the variable a.

One can certainly interpret the above calculations as confirming the Markovian properties of certain infinite dimensional processes in the space variable. We can define the process $(\tilde{L}_x, x \in R)$ with state space $C([0,\infty))$ by $\tilde{L}_x(t) = \tilde{L}(x,t)$. The state space is Polish but is not locally compact, so one cannot directly apply standard Markov process theory. The processes $\tilde{\beta}_x$ and $\tilde{B}_x$ are defined similarly.

Corollary 2.2 If K is a functional of the Brownian path which is supported above the $\mathcal{E}^x$ stopping time X then

$$\mathbf{E}[K|\mathcal{E}^{X+}] = \mathbf{E}[K|\tilde{L}_X]$$

Proof: It is enough to prove this for a dense set of such K. Suppose that $\{X_n\}$ is a sequence of discrete stopping times which decreases to X. Thus we may suppose that K has the form $K_\infty(n, \lambda, \mathbf{f})$ and we can allow all the functions to be supported above a fixed X_N. The result clearly being true for discrete stopping times, the general case now follows by taking limits.

The above appears in Walsh's article [12] while the following result and proof are taken from [13].

Lemma 2.3 The σ-field $\mathcal{E}^{-\infty}$ is $\mathbf{P}$ trivial.

Proof: Let $A \in \mathcal{E}^{-\infty}$. Then by the strong Markov property applied at the times T_{-n} we see that A is independent of the generating set $\bigcup_{n\geq 1} \mathcal{B}(T_{-n})$ for $\mathcal{E}^\infty$. Which proves the result.

The results of this section are very important since, once we know that the excursion filtration satisfies the (so-called) usual conditions, we can proceed to deploy the machinery of the general theory of processes.

§3. Calculations with the conditional excursion theorem

Let us fix the semi-infinite interval $(-\infty, a]$. The excursions of B_t from this set take their values in the space $\mathcal{W}^a$, the collection of all continuous paths γ starting at a and absorbed when they return to $(-\infty, a]$ again. The excursion process from $(-\infty, a]$ is then a mapping $\mathcal{E}^a : D \times R^+ \mapsto \mathcal{W}^a$ defined by

$$\begin{aligned} \mathcal{E}^a(\omega, s) &= B_{t\wedge T_a} \circ \theta_{\tau(a,s)-} && (\Delta\tau(a,s) \neq 0) \\ &= \Delta && (\Delta\tau(a,s) = 0) \end{aligned}$$

where Δ is the null excursion and T_a is the hitting time of $(-\infty, a]$ (we recognise that there may be a temporary risk of confusion between the two meanings of the symbol $\mathcal{E}$ but this should not cause difficulties later on). The initial excursion $\mathcal{E}^a(\omega, 0)$ from $B_0 > a$ to $(-\infty, a]$, when it exists, is independent of $\mathcal{E}^a$ and usually needs to be looked at separately.

On this space $\mathcal{W}^a$ we can define the so-called excursion measure $\mathcal{Q}^a$. If $p_t^a(x,y)$ denotes the transition density of $B_{t\wedge T_a}$ then the excursion measure has entrance law given by

$$Q_t^a[dy] = dy \frac{\partial}{\partial x} p_t(x,y)|_{x=a+} \qquad (a < y)$$

The terminology means that if $t > 0$ and Y is a Borel subset of $(a, +\infty)$ then

$$\mathcal{Q}^a[\gamma(t) \in Y] = \int_Y Q_t^a[dy]$$

$\mathcal{Q}^a$ is now completely specified by declaring that the $\mathcal{Q}^a$ conditional distribution of $\{\gamma(t+s) : s \geq 0\}$, given that $\gamma(t) > a$, is that of a Brownian motion started at $\gamma(t)$ and absorbed at a. The following is (see [8] for example) a variant of the general conditional excursion theorem.

Theorem 3.1 Let $\mathcal{A}^a \geq 0$ be a time-homogeneous function defined on the excursion space $\mathcal{W}^a$. Suppose that $\mathcal{Q}^a[\mathcal{A}^a] < +\infty$. Then for every bounded $\mathcal{B}(\tau(a,t))$ predictable process Y_t

$$Z_t = \sum_{0<s\leq t} Y_s \mathcal{A}^a \circ \mathcal{E}^a(\omega, s) - \mathcal{Q}^a[\mathcal{A}^a] \int_0^t Y_s d_s \tilde{L}(a,s)$$

is a $\mathcal{B}(\tau(a,t))$ martingale which is orthogonal to $\tilde{\beta}(a,t)$.

The effectiveness of this result for performing calculations depends to a large extent on the following auxiliary facts. They are certainly well known to experts but we think it useful to underline their importance.

Lemma 3.2 (i) Let $A(t) \geq 0$ be any continuous functional of the killed Brownian path $\{B_t : 0 < t \leq T_a\}$, such that $A(0) = 0$. If $\mathcal{A}^a$ is the corresponding function defined on $\mathcal{W}^a$ then we have

$$\mathcal{Q}^a[1 - \exp\{-\lambda \mathcal{A}^a\}] = H'(a+)$$

where $H(x) = \mathbf{E}_x[\exp\{-\lambda A(T_a)\}]$.
(ii) If N_t is a square integrable $\mathcal{B}(\tau(a,t))$ martingale which is orthogonal to $\beta(a,t)$ then $\mathbf{E}[N_\infty - N_0 | \mathcal{E}^a] = 0$.

Proof: (i) By our description of $\mathcal{Q}^a$ and the continuity of $A(t)$

$$\mathcal{Q}^a[1 - \exp\{-\lambda \mathcal{A}^a\}] = \lim_{t \downarrow 0} \int [1 - H(y)] Q_t^a(dy)$$

The result follows when we integrate by parts and take the weak limit, using our definition of Q_t^a.
(ii) See [6] Lemma 4.2.

The following is typical of the kind of results we can obtain by using the second part of the previous lemma as well as being a vital step in our main calculation.

Lemma 3.3 For $x > a$

$$\mathbf{E}_y[\exp\{-\frac{\mu^2}{2} \int_a^x L(b, \tau(a,t)) db - \gamma L(x, \tau(a,t))\} | \mathcal{E}^a] =$$

$$K(\gamma, \mu, y) \exp\{-\mu Z(\gamma, \mu, x) \tilde{L}(a,t)\}$$

where we write

$$Z(\gamma, \mu, x) = \frac{\mu \sinh \mu(x-a) + 2\gamma \cosh \mu(x-a)}{\mu \cosh \mu(x-a) + 2\gamma \sinh \mu(x-a)}$$

$$K(\gamma, \mu, y) = \frac{\mu \cosh \mu(x-y) + 2\gamma \sinh \mu(x-y)}{\mu \cosh \mu(x-a) + 2\gamma \sinh \mu(x-a)}$$

when $a < y \leq x$. Furthermore if $\mu = 0$ we replace

$$\mu Z(\gamma,\mu,x) \mapsto \frac{2\gamma}{1+2\gamma(x-a)} \quad ; \quad K(\gamma,\mu,x) \mapsto \frac{1+2\gamma(x-y)}{1+2\gamma(x-a)}$$

Proof: By [6] Theorem 4.5

$$\mathbf{E}_y[\exp\{-\frac{\mu^2}{2}\int_a^x L(b,\tau(a,t))db - \gamma L(x,\tau(a,t))\}|\mathcal{E}^a] = f(0)\exp\{f'(a+)\tilde{L}(a,t)\}$$

where f is the unique solution of the system

$$f'' = \mu^2 f \quad (a \leq z \leq x) \quad ; \quad f'(x+) - f'(x-) = 2\gamma f(x)$$

subject to the boundary conditions $f(a) = 1$ and $f'(z) = 0$ for $z > x$. Now we solve explicitly for f and substitute.

Next we introduce some more notation. If $W_s = \int_a^x L(b,\tau(a,s)-)db$ we define Y_s be the indicator function of the event $\{s + W_s < t\}$. Note that Y_s is $\mathcal{B}(\tau(a,s))$ predictable. This will be important later on. Also we shall use $T_a(x)$ to denote the hitting time of $(-\infty, a]$ by the process $\tilde{B}(x,t)$. Then in our proof we consider the function $\mathcal{A}^a(x)$ defined on the space $\mathcal{W}^a$ so that

$$\mathcal{A}^a(x)\circ\mathcal{E}(\omega,s) = \int_{\tau(a,s)-}^{\tau(a,s)} \exp\{-\gamma\tilde{L}(x,t) - \frac{\lambda^2}{2}t\}d_t\tilde{L}(x,t).$$

Lemma 3.4 If $x > a$,

$$\mathcal{Q}^a[\mathcal{A}^a(x)] = \lambda e^{\lambda(a-x)}[1 + Z(\gamma/2,\lambda,x)]/(\lambda+\gamma)$$

Proof: Following the prescription of Lemma 3.2 we first calculate

$$\mathbf{E}_y[\int_0^{T_a(x)} \exp\{-\gamma\tilde{L}(x,t) - \frac{\lambda^2}{2}t\}d_t\tilde{L}(x,t)].$$

By Ito's formula, since $\tilde{B}(x,t) = x$ on the support of $\tilde{L}(x,t)$, we have

$$\exp\{\lambda\tilde{B}(x,t) - \gamma\tilde{L}(x,t) - \frac{\lambda^2}{2}t\} - e^{\lambda y} +$$

$$(\lambda+\gamma)\int_0^t \exp\{\lambda x-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{L}(x,s)$$
$$=\lambda\int_0^t \exp\{\lambda\tilde{B}(x,s)-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{\beta}(x,s).$$

and this martingale is uniformly integrable. Stopping at the time $T_a(x)$ and taking the expectation Doob's theorem gives

$$e^{\lambda a}\mathbf{E}_y[\exp\{-\gamma\tilde{L}(x,T_a(x))-\frac{\lambda^2}{2}T_a(x)\}]-e^{\lambda y}=$$
$$-(\lambda+\gamma)\mathbf{E}_y[\int_0^t \exp\{\lambda x-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{L}(x,s)]$$

However Lemma 3.3 and a time change shows that

$$\mathbf{E}_y[\exp\{-\frac{\gamma}{2}L(x,T_a)-\frac{\lambda^2}{2}\int_a^x L(b,T_a)db\}]=$$
$$\mathbf{E}_y[\exp\{-\gamma\tilde{L}(x,T_a(x))-\frac{\lambda^2}{2}T_a(x)\}]=K(\gamma/2,\lambda,y)$$

Thus we find that

$$\mathbf{E}_y\int_0^{T_a(x)} \exp\{-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{L}(x,s)=$$
$$[e^{\lambda(y-x)}-e^{\lambda(a-x)}K(\gamma/2,\lambda,y)]/(\gamma+\lambda)$$

Taking the derivative of this in y, and evaluating at $y=a+$, we get the value of $\mathcal{Q}^a[\mathcal{A}^a(x)]$

Corollary 3.5 If $\lambda=0$ then

$$\mathcal{Q}^a[\mathcal{A}^a(x)]=1/[1+\gamma(x-a)]$$

With these preliminary calculations out of the way we are now in position to use the conditional excursion theorem.

Lemma 3.6

$$\mathbf{E}[\exp\{\lambda\tilde{B}(x,t)-\gamma\tilde{L}(x,t)-\frac{\lambda^2}{2}t\}|\mathcal{E}^a]=$$

$$\mathbf{E}[\exp\{\lambda\tilde{B}(x,t\wedge T_a(x))-\gamma\tilde{L}(x,t\wedge T_a(x))-\frac{\lambda^2}{2}t\wedge T_a(x)\}]+$$

$$\lambda\int_0^\infty \exp\{\lambda\tilde{B}(x,s)\}\mathbf{E}[Y_s\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))-\frac{\lambda^2}{2}(s+W_s)\}|\mathcal{E}^a]d_s\tilde{\beta}(a,s)\quad -$$

$$\lambda e^{\lambda a}[1+Z(\gamma/2,\lambda,x)]\int_0^\infty \mathbf{E}[Y_s\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))-\frac{\lambda^2}{2}(s+W_s)\}|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

Proof: By Ito's formula

$$\exp\{\lambda\tilde{B}(x,t)-\gamma\tilde{L}(x,t)-\frac{\lambda^2}{2}t\}=$$

$$\exp\{\lambda\tilde{B}(x,t\wedge T_a(x))-\gamma\tilde{L}(x,t\wedge T_a(x))-\frac{\lambda^2}{2}t\wedge T_a(x)\}+$$

$$\lambda\int_{T_a(x)}^t \exp\{\lambda\tilde{B}(x,s)-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{B}(x,s)$$

$$-\gamma\int_{T_a(x)}^t \exp\{\lambda\tilde{B}(x,s)-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{L}(x,s) \tag{3.a}$$

Working term by term we note that the martingale part of (3.a) is uniformly integrable. The contribution from the excursions above a is

$$\int_{T_a(x)}^t 1_{(\tilde{B}(x,s)>a)}\exp\{\lambda\tilde{B}(x,s)-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{\beta}(x,s)$$

so that by time change to the $\tau(a,t)$ time scale we get a square integrable $B(\tau(a,t))$ martingale which is orthogonal to $\tilde{\beta}(a,t)$. Therefore by Lemma 3.2 its projection onto $\mathcal{E}^a$ is zero. On the other hand

$$\int_{T_a(x)}^t 1_{(\tilde{B}(x,s)<a)}\exp\{\lambda\tilde{B}(x,s)-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{\beta}(x,s)$$

can be time changed to give

$$\mathbf{E}[\int_0^\infty Y_s\exp\{\lambda\tilde{B}(x,s)-\frac{\gamma}{2}L(x,\tau(a,s)-)-\frac{\lambda^2}{2}(s+W_s)\}d_s\tilde{\beta}(a,s)|\mathcal{E}^a]$$

$$=\int_0^\infty \exp\{\lambda\tilde{B}(a,s)\}\mathbf{E}[Y_s\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))-\frac{\lambda^2}{2}(s+W_s)\}|\mathcal{E}^a]d_s\tilde{\beta}(a,s)$$

Notice that we were able to replace $L(x,\tau(a,s)-)$ by $L(x,\tau(a,s))$ since the increasing process of $\tilde{\beta}(a,t)$ is continuous. Next we consider the bounded variation part of (3.a) which we can write as

$$(\lambda-\gamma)\int_{T_a(x)}^{t} \exp\{\lambda x-\gamma\tilde{L}(x,s)-\frac{\lambda^2}{2}s\}d_s\tilde{L}(x,s)$$

since $\tilde{B}(x,t)=x$ on the support of $\tilde{L}(x,t)$. But in the $\tau(a,t)$ time scale this looks as

$$(\lambda-\gamma)e^{\lambda x}\sum_{0<s\le t} Y_s\mathcal{A}^a(x)\circ\mathcal{E}^a(\omega,s)\exp\{-\frac{\gamma}{2}L(x,\tau(a,s)-)-\frac{\lambda^2}{2}(s+W_s)\}$$

Now by Theorem 3.1, because we know that $Y_s\exp\{-\frac{\gamma}{2}L(x,\tau(a,s)-)-\frac{\lambda^2}{2}(s+W_s)\}$ is $\mathcal{B}(\tau(a,t))$ predictable, this projects onto $\mathcal{E}^a$ to give

$$(\lambda-\gamma)e^{\lambda x}\mathcal{Q}^a[\mathcal{A}^a(x)]\int_0^\infty \mathbf{E}[Y_s\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))-\frac{\lambda^2}{2}(s+W_s)\}|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

where we have invoked the continuity of $\tilde{L}(x,t)$ in order to replace $L(x,\tau(a,s)-)$ by $L(x,\tau(a,s))$. The proof is completed by using the evaluation of $\mathcal{Q}^a[\mathcal{A}^a(x)]$ from Lemma 3.4.

The following result can be proved either by the same argument this time with $\exp\{\lambda\tilde{B}(x,t)-\frac{\lambda^2}{2}t\}\tilde{L}^n(x,t)$ instead of $\exp\{\lambda\tilde{B}(x,t)-\frac{\lambda^2}{2}t-\gamma\tilde{L}(x,t)\}$. However we prefer to deduce it from the above.

Corollary 3.7

$$\mathbf{E}[\exp\{\lambda\tilde{B}(x,t)-\frac{\lambda^2}{2}t\}\tilde{L}^n(x,t)|\mathcal{E}^a]=$$

$$\mathbf{E}[\tilde{L}^n(x,t\wedge T_a(x))\exp\{\lambda\tilde{B}(x,t\wedge T_a(x))-\frac{\lambda^2}{2}t\wedge T_a(x)\}+$$

$$\lambda\int_0^\infty \exp\{\lambda\tilde{B}(a,s)\}\mathbf{E}[Y_s2^{-n}L^n(x,\tau(a,s))\exp\{-\frac{\lambda^2}{2}(s+W_s)\}|\mathcal{E}^a]d_s\tilde{\beta}(a,s)$$

$$-e^{\lambda a}\int_0^\infty \mathbf{E}[Y_s(-1)^n\frac{\partial^n}{\partial\gamma^n}[(\lambda+\lambda Z(\frac{\gamma}{2},\lambda,x))\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))\}]_{\gamma=0+}$$

$$\exp\{-\frac{\lambda^2}{2}(s+W_s)\}|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

Proof: Since $\tilde{L}$ has moments of all orders the only difficulty is in showing that we can differentiate inside the integrals. However, even in the case of the stochastic integral this is quite straightforward. We simply argue in the usual way only that we take limits in the martingale H_2 norm.

We can now prove the main result of this section. Recall that, unless otherwise stated, we always have $B_0 = 0$.

Theorem 3.8 For γ, y positive and $\mu > \lambda^2/2$

$$\lim_{x\to a} \frac{1}{x-a}\int_0^\infty e^{-\mu t}(\mathbf{E}[\exp\{\lambda\tilde{B}(x,t)-\gamma\tilde{L}(x,t)\}|\mathcal{E}^a]-\exp\{\lambda\tilde{B}(a,t)-\gamma\tilde{L}(a,t)\})dt$$

$$= -e^{\lambda a}(\gamma/\mu)1_{(a<0)} - \left(\frac{\lambda}{\mu-\lambda^2/2}\right)\int_0^\infty \exp\{\lambda\tilde{B}(a,s)-\gamma\tilde{L}(a,s)-\mu s\}$$

$$\left[\gamma 1_{(a<0)} + (2\mu-\gamma^2)\tilde{L}(a,s)\right] d_s\tilde{\beta}(a,s) - \left(\frac{e^{\lambda a}}{\mu-\lambda^2/2}\right)\int_0^\infty \exp\{-\gamma\tilde{L}(a,s)-\mu s\}$$

$$[\lambda^2-\gamma^2-\gamma(\lambda+\gamma)1_{(a<0)} - (\gamma+\lambda)(2\mu-\gamma^2)\tilde{L}(a,s)]d_s\tilde{L}(a,s)$$

Proof: We use Lemma 3.6, so first we need to write

$$\exp\{\lambda\tilde{B}(a,t)-\gamma\tilde{L}(a,t)-\frac{\lambda^2}{2}t\} = \exp\{\lambda\tilde{B}(a,0)\}$$

$$+\lambda\int_0^\infty 1_{[0,t]}(s)\exp\{\lambda\tilde{B}(a,s)-\gamma\tilde{L}(a,s)-\frac{\lambda^2}{2}s\}d_s\tilde{\beta}(a,s)$$

$$-(\gamma+\lambda)\int_0^\infty 1_{[0,t]}(s)\exp\{\lambda\tilde{B}(a,s)-\gamma\tilde{L}(a,s)-\frac{\lambda^2}{2}s\}d_s\tilde{L}(a,s)$$

Subtracting this from the result of Lemma 3.6 and taking the Laplace transform in t we look first at the martingale part. Because

$$\int_0^\infty \exp\{-\mu t+\frac{\lambda^2}{2}t\}Y_s dt = \exp\{-(\mu-\frac{\lambda^2}{2})(s+W_s)\}/[\mu-\frac{\lambda^2}{2}]$$

we find, using a stochastic Fubini theorem, that this contributes

$$\left(\frac{\lambda}{\mu-\lambda^2/2}\right)\lim_{x\to a}\frac{1}{x-a}\int_0^\infty \exp\{\lambda\tilde{B}(a,s)\}$$

$$(\mathbf{E}[\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))-\mu(s+W_s)\}|\mathcal{E}^a]-\exp\{-\gamma\tilde{L}(a,s)-\mu s\})d_s\tilde{\beta}(a,s)$$

And by Lemma 3.3, arguing as in the previous proof, this is equal to

$$\left(\frac{\lambda}{\mu-\lambda^2/2}\right)\int_0^\infty \exp\{\lambda\tilde{B}(a,s)-\gamma\tilde{L}(a,s)-\mu s\}[\gamma 1_{(a<0)}+(2\mu-\gamma^2)\tilde{L}(a,s)]d_s\tilde{\beta}(a,s)$$

Doing the same with the other integral and using Lemma 3.4 we see that we must evaluate

$$-e^{\lambda a}\left(\frac{\lambda}{\mu-\lambda^2/2}\right)\lim_{x\to a}\frac{1}{x-a}\int_0^\infty \{[1+Z(\frac{\gamma}{2},\lambda,x)]$$

$$\mathbf{E}[\exp\{-\frac{\gamma}{2}L(x,\tau(a,s))-\mu(s+W_s)\}|\mathcal{E}^a]-(1+\frac{\gamma}{\lambda})\exp\{-\gamma\tilde{L}(a,s)-\mu s\}\}d_s\tilde{L}(a,s)$$

Which we can calculate from Lemma 3.3 as before. Finally we consider the contribution from a possible initial excursion. For this take $a<0$ and use Lemma 3.3 to calculate the limit

$$\lim_{x\to a}\frac{1}{x-a}(\mathbf{E}[\exp\{\lambda a-\frac{\gamma}{2}L(x,T_a)-\frac{\lambda^2}{2}T_a(x)\}]-e^{\lambda a})=-\gamma 1_{(a<0)}e^{\lambda a}$$

Thus the contribution from the initial excursion is $-\frac{\gamma}{\mu}e^{\lambda a}1_{(a<0)}$ since we have the difference

$$\mathbf{E}[\exp\{\lambda a-\frac{\gamma}{2}L(x,T_a)-\frac{\lambda^2}{2}T_a(x)\}]-$$

$$\mathbf{E}[\exp\{\lambda\tilde{B}(x,t\wedge T_a(x))-\frac{\gamma}{2}\tilde{L}(x,t\wedge T_a(x))-\frac{\lambda^2}{2}t\wedge T_a(x)\}]$$

bounded by $e^{\lambda x}\mathbf{P}_x[t<T_a(x)]$ However from Lemma 3.3

$$\lambda\int_0^\infty e^{-\lambda t}\mathbf{P}_x[t<T_a(x)]dt=1-\operatorname{Sech}\sqrt{2\lambda}(x-a)$$

So dividing by $(x-a)$ and taking the limit we get zero.

In fact we do not need the above result in such generality. But we have given it here for convenience, and as an illustration of the power of the conditional excursion theorem. We really only require the first two conditional moments of $\tilde{L}$. And rather than apply a result on double limits it is easier to check, starting from Corollary 3.7, that the argument used above gives the following results.

Corollary 3.9 For each $\mu > 0$ we have

(i) $$\lim_{x \to a} \frac{1}{x-a} \int_0^\infty e^{-\mu t} \mathbf{E}[\tilde{L}(x,t) - \tilde{L}(a,t) | \mathcal{E}^a] dt =$$

$$\frac{1}{\mu} 1_{(a<0)} - 2 \int_0^\infty e^{-\mu s} \tilde{L}(a,s) d_s \tilde{L}(a,s).$$

(ii) $$\lim_{x \to a} \frac{1}{x-a} \int_0^\infty e^{-\mu t} \mathbf{E}[\tilde{L}^2(x,t) - \tilde{L}^2(a,t) | \mathcal{E}^a] dt =$$

$$\frac{2}{\mu} \int_0^\infty e^{-\mu s} [1 + 1_{(a<0)}] d_s \tilde{L}(a,s) - 4 \int_0^\infty e^{-\mu s} \tilde{L}^2(a,s) d_s \tilde{L}(a,s).$$

§4. Canonical decomposition in the excursion filtration

By the results of the first section we already know that $\{\tilde{L}(x,t), x \in R\}$ is a semimartingale for each fixed t. Therefore we can write it as

$$\tilde{L}(x,t) = N(x,t) + D(x,t)$$

where $N(x,t)$ and $D(x,t)$ are respectively the martingale part and the bounded variation part of $\tilde{L}(x,t)$. Recall that by a result [11] of Stricker and Yor there is a jointly measurable version of this decomposition. And moreover, since $\{\tilde{L}(x,t) - \int^x 1_{(y<0)} dy, x \in R\}$ is a supermartingale, we can assume that $D(x,t) - \int^x 1_{(y<0)} dy$ is increasing so that almost surely it generates a positive measure. We fix a normalisation for D by requiring that

$$D(w,t) \equiv 0$$

w being assumed fixed for the moment.
In this section we will find a more explicit form for this canonical decomposition as well as investigating a similar result for the process $\tilde{B}_x$. First of all we explain the general approach. The main idea is really quite simple. Suppose that X_t,

with filtration $\mathcal{X}_t$, is a process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t r(s, X_s)ds$$

where for the moment we assume that the coefficients are smooth. If we know $\mathbf{E}[X_t|\mathcal{X}_s]$ and $\mathbf{E}[X_t^2|\mathcal{X}_s]$ for all pairs $s < t$ then, in principle, we can determine the coefficients σ and r from the following formulae

$$\lim_{t \to s} \frac{1}{t-s}\{\mathbf{E}[X_t|\mathcal{X}_s] - X_s\} = r(s, X_s) \tag{4.a}$$

$$\lim_{t \to s} \frac{1}{t-s}\{\mathbf{E}[X_t^2|\mathcal{X}_s] - X_s^2\} = 2X_s r(s, X_s) + \sigma^2(s, X_s) \tag{4.b}$$

In our case we should calculate

$$\lim_{x \to a} \frac{1}{x-a}\{\mathbf{E}[\tilde{L}^j(x,t)|\mathcal{E}^a] - \tilde{L}^j(a,t)\}$$

for $j = 1, 2$. However we have no reason to suppose that the coefficients are smooth. Indeed they are not. It ought to be clear from the results of the previous section that in order to do calculations we must use some sort of smoothing in the time variable.

The results obtained below were motivated by the conditional Ray-Knight theorems as ennunciated by Walsh [12]. The guiding principle is that working in the intrinsic time scale merely introduced a drift term and that infinitesimally the relation between martingale parts remains unaffected. Consequently then one expects that disjoint time intervals will produce orthogonal martingales and that the quadratic variation will be the 'usual one'. For more information on the Ray-Knight martingales see [6] Theorem 4.8.

We begin with the following technical lemma. It is generally known [3] as 'la méthode des laplaciens approchés'. To set it up we suppose that Y_t is a continuous process which is a semimartingale in the filtration $\mathcal{Y}_t$ and having canonical decomposition there given by $M_t + A_t$. M_t is the martingale part while A_t has bounded variation. Both of these processes are continuous so by localisation we can assume that A_t is integrable.

Lemma 4.1 Suppose that for some $\epsilon > 0$ the set

$$\left\{\frac{1}{h}\int_0^t (\mathbf{E}[Y_{s+h}|\mathcal{Y}_s] - Y_s)\, ds, 0 < h < \epsilon\right\}$$

is uniformly integrable and converges weakly in $\mathcal{L}_1$ to the continuous process U_t. Then U_t is a version of A_t.

Proof: It suffices to prove that for every bounded random variable Z we have $\mathbf{E}[ZU_t] = \mathbf{E}[ZA_t]$. Let $Z_t = \mathbf{E}[Z|\mathcal{Y}_t]$ define a right continuous $\mathcal{Y}_t$ martingale. Let us define $U_t^h = \frac{1}{h}\int_0^t (\mathbf{E}[Y_{s+h}|\mathcal{Y}_s] - Y_s)\,ds$ so that by Ito's formula $d(Z_tU_t^h) = U_t^h dZ_t + Z_t dU_t^h$. Now take expectations and use the definition of conditional expectation to obtain

$$\mathbf{E}[ZU_t^h] = \mathbf{E}[Z_tU_t^h] = \mathbf{E}[\int_0^t Z_s dU_s^h] = \mathbf{E}[\frac{1}{h}\int_0^t Z_s\,(A_{s+h} - A_s)\,ds]$$

But by Fubini we transform this into $\mathbf{E}[\int_0^t dA_u \frac{1}{h}\int_{(u-h)+}^u Z_s ds]$ which converges to $\mathbf{E}[\int_0^t Z_{u-}dA_u] = \mathbf{E}[\int_0^t Z_u dA_u]$ since A_t is assumed to be continuous. However another use of Ito's formula shows this to be the same as $\mathbf{E}[ZA_t]$. And the result follows since

$$\lim_{h\downarrow 0}\mathbf{E}[ZU_t^h] = \mathbf{E}[ZU_t]$$

by hypothesis.

Lemma 4.2 If $x > a$ then $\{D(x,t) - D(a,t), t \geq 0\}$ is a decreasing process with the same Laplace transform in t as the distributional measure

$$\int_a^x 1_{(y<0)}\,dy - \frac{d}{dt}\int_a^x \tilde{L}^2(y,t)dy$$

Proof: We wish to apply Lemma 4.1 to the semimartingale

$$\left\{\int_0^\infty e^{-\mu t}\tilde{L}(x,t)dt, x \in R\right\}$$

So first we check the uniform integrability of

$$\left\{\frac{1}{h}\int_a^x dy\int_0^\infty e^{-\mu t}\left(\mathbf{E}[\tilde{L}(y+h,t)|\mathcal{E}^y] - \tilde{L}(y,t)\right)dt\right\}$$

when δ and h are sufficiently small and positive, and x lies in a fixed compact interval. However from Corollary 3.7 with $\lambda = 0$ we know (using the evaluation $\lambda Z(\gamma/2,\lambda,x) \mapsto \gamma/[1+\gamma(x-a)]$ from Lemma 3.3) that

$$\mathbf{E}[\tilde{L}(x,t)|\mathcal{E}^a] - \tilde{L}(a,t) = \mathbf{E}[\tilde{L}(x,t\wedge T_a(x))] + \int_0^\infty \mathbf{E}[Y_s - 1_{[0,t]}(s)|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

where Y_s is as defined for Lemma 3.4. Taking the Laplace transform in t gives

$$\int_0^\infty e^{-\mu t}\mathbf{E}[\tilde{L}(x, t \wedge T_a(x))]dt + \frac{1}{\mu}\int_0^\infty e^{-\mu s}\mathbf{E}[e^{-\mu W_s} - 1|\mathcal{E}^a]d_s\tilde{L}(a, s)$$

and from Lemma 3.3 the conditional expectation calculates to be

$$\operatorname{sech}(\sqrt{2\mu}1_{(a<0)}(x-a)\exp\{-\sqrt{2\mu}\tanh[\sqrt{2\mu}(x-a)]\tilde{L}(a,t)\} - 1.$$

Using the inequalities

$$1 - e^{-x} \leq x \quad ; \quad 1 - \operatorname{sech}x \leq x$$

we find that this is bounded by $4\mu(x-a)\tilde{L}(a,t) + \sqrt{2\mu}(x-a)$ when x is close to a. And by the Ray-Knight theorem we obtain the bound $\mathbf{E}[\tilde{L}(a, T_a(x))] \leq (x-a)$. This then shows that

$$\int_0^\infty e^{-\mu t}\left(\mathbf{E}[\tilde{L}(x,t)|\mathcal{E}^a] - \tilde{L}(a,t)\right)dt \quad \leq$$

$$\frac{1}{\mu}(x-a) + \int_0^\infty e^{-\mu t}(x-a)[2\tilde{L}(a,t) + \sqrt{\frac{2}{\mu}}]d_t\tilde{L}(a,t)$$

which gives us the bound

$$\frac{1}{h}\int_a^x dy \int_0^\infty e^{-\mu t}\left(\mathbf{E}[\tilde{L}(y+h,t)|\mathcal{E}^y] - \tilde{L}(y,t)\right)dt \quad \leq$$

$$\frac{1}{\mu}(x-a) + \int_a^x dy \int_0^\infty e^{-\mu t}[2\tilde{L}(a,t) + \sqrt{\frac{2}{\mu}}]d_t\tilde{L}(y,t)$$

However by Lemma 1.6 the r.h.s. is integrable so indeed the set is uniformly integrable. Moreover Corollary 3.9 shows that as $h \downarrow 0$ it converges to

$$\frac{1}{\mu}\int_a^x 1_{(y<0)}dy - \int_0^\infty e^{-\mu t}\int_a^x d_t\tilde{L}^2(y,t)dy$$

Since this process is continuous it must equal, by the previous lemma, the bounded variation part of $\int_0^\infty e^{-\mu t}\tilde{L}(x,t)dt$. However by Fubini and the joint measurability of the (unique) Doob-Meyer decomposition [11] this is given by $\int_0^\infty e^{-\mu t}D(x,t)dt$. The result follows.

Note that this automatically implies the almost sure relation

$$\int_0^t (D(x,s) - D(a,s))ds = t\int_a^x 1_{(y<0)}dy - \int_a^x dy \int_0^t d_s\tilde{L}^2(y,s)$$

This is proved by an integration by parts and then inversion of the Laplace transform. Thus we can use the r.h.s. to define the version under consideration.

Lemma 4.3 As versions we have

$$< \tilde{L}(.,t) >_x - < \tilde{L}(.,t >_a \;=\; 2\int_a^x \tilde{L}(y,t)dy$$

Proof: For this we wish to apply Lemma 4.1 to the semimartingale

$$\left\{\int_0^\infty e^{-\mu t}\tilde{L}^2(x,t)dt, x \in R\right\}$$

Which involves checking the uniform integrability of

$$\frac{1}{h}\int_a^x dy \int_0^\infty e^{-\mu t}(\mathbf{E}[\tilde{L}^2(y+h,t)|\mathcal{E}^y] - \tilde{L}^2(y,t))dt \tag{4.c}$$

when δ and h are sufficiently small and positive. However from Corollary 3.7 we know that

$$\mathbf{E}[\tilde{L}^2(x,t)|\mathcal{E}^a] - \tilde{L}^2(a,t) = \mathbf{E}[\tilde{L}^2(x,t\wedge T_a(x))]$$
$$+\int_0^\infty \mathbf{E}[Y_s\{2(x-a) + L(x,\tau(a,s))\} - 1_{[0,t]}(s)2\tilde{L}(a,s)|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

Taking the Laplace transform in t gives

$$\int_0^\infty e^{-\mu t}\mathbf{E}[\tilde{L}^2(x,t\wedge T_a(x))]dt+$$

$$\frac{1}{\mu}\int_0^\infty e^{-\mu s}\mathbf{E}[e^{-\mu W_s}\{2(x-a) + L(x,\tau(a,s))\} - 2\tilde{L}(a,s)|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

The conditional expectation $\mathbf{E}[e^{-\mu W_s}L(x,\tau(a,s))|\mathcal{E}^a]$ calculates from Lemma 3.3 (when x is close to a) as

$$[1 + 1_{(a<0)}\frac{1}{\sqrt{2\mu}}\sinh\sqrt{2\mu}(x-a)]2\,\mathrm{sech}^2[\sqrt{2\mu}(x-a)]\tilde{L}(a,t)$$

$$\exp\{-\sqrt{2\mu}\tanh[\sqrt{2\mu}(x-a)]\tilde{L}(a,t)\}.$$

Now we estimate. The Laplace transform of $\mathbf{E}[\tilde{L}^2(a,t) - \tilde{L}^2(x,t)|\mathcal{E}^a]$ is bounded by

$$\int_0^\infty e^{-\mu t}\mathbf{E}[\tilde{L}^2(x,T_a(x))]dt + \frac{2}{\mu}\int_0^\infty e^{-\mu t}(x-a)d_t\tilde{L}(a,t)\}$$

$$+\frac{2}{\mu}\int_0^\infty e^{-\mu t}[\{1 - 1_{(a<0)}\frac{1}{\sqrt{2\mu}}\sinh\sqrt{2\mu}(x-a)\}\ \mathrm{sech}^2[\sqrt{2\mu}(x-a)]$$

$$\exp\{-\sqrt{2\mu}\tanh[\sqrt{2\mu}(x-a)]\tilde{L}(a,t)\} - 1]d_t\tilde{L}(a,t)$$

Now by using the Ray-Knight theorem we find that

$$\int_0^\infty e^{-\mu t}\mathbf{E}[\tilde{L}^2(x,T_a(x))]dt \le C(x-a)$$

The same sort of inequality is valid for the second integral also. For the third we see that it is bounded above by

$$\frac{C}{\mu}\int_0^\infty e^{-\mu t}\tilde{L}^2(a,t)d_t\tilde{L}(a,t)$$

when x is sufficiently close to a. And the uniform integrability of (4.c) is immediate from Lemma 1.6 by arguing as before. Moreover from Corollary 3.9 we see that as $h \downarrow 0$ it converges to

$$\frac{2}{\mu}\int_0^\infty e^{-\mu s}\int_a^x dy(1 + 1_{(y<0)})d_s\tilde{L}(y,s) - 4\int_0^\infty e^{-\mu s}\int_a^x dy\tilde{L}^2(y,s)d_s\tilde{L}(y,s)$$

And since this is adapted it follows from the previous lemma that it is the bounded variation part of the process

$$\int_0^\infty e^{-\mu t}\tilde{L}^2(x,t)dt$$

Now we know, using the Ito formula, that

$$\tilde{L}^2(x,t) = \tilde{L}^2(a,t) + 2\int_a^x \tilde{L}(y,t)d_y\tilde{L}(y,t) + <\tilde{L}(.,t)>_x - <\tilde{L}(.,t)>_a$$

and also recall how [11] shows we can assume there is a measurable version of this. The bounded variation part we seek is given by

$$\int_0^\infty e^{-\mu t}dt\left(\int_a^x 2\tilde{L}(y,t)d_yD(y,t) + <\tilde{L}(.,t)>_x - <\tilde{L}(.,t)>_a\right)$$

However by the previous lemma we can identify the measure $d_y D(y,t)dt$ with $dt1_{(y<0)}dy - d_t\tilde{L}^2(y,t)dy$. Comparing with what we have, gives

$$\int_0^\infty e^{-\mu t}(< \tilde{L}(.,t) >_x - < \tilde{L}(.,t) >_a)dt =$$

$$\frac{2}{\mu}\int_0^\infty e^{-\mu t}\int_a^x d_t\tilde{L}(y,t)$$

And the result is immediate since $\tilde{L}$ has a bicontinuous version by Corollary 1.9.

The next problem is to see how the processes corresponding to different time intervals intefere. This is much harder than it might appear. As a general principle the CMO formulae of order two and greater are extremely difficult to write down explicitly. Nevertheless we are able to do the easiest case which turns out to suffice for the next lemma. Unfortunately there is as yet no published account of what we require, namely the CMO formulae for the process $\tilde{B}(x,t)$. But an inspection of [7] gives a good idea of what to expect. So the proof is given in sketch only.

Lemma 4.4 If $0 \le s \le t$ then

$$< \tilde{L}(.,t) - \tilde{L}(.,s), \tilde{L}(.,s) >_x \equiv 0$$

Proof: The idea is to project

$$\int_0^\infty e^{-\lambda t}\tilde{L}(x,t)dt\int_0^\infty e^{-\lambda s}\tilde{L}(x,s)ds =$$

$$\lambda\mu\int_0^\infty e^{-\lambda t}d_t\tilde{L}(x,t)\int_0^\infty e^{-\lambda s}d_s\tilde{L}(x,s) \tag{4.d}$$

onto the σ-field $\mathcal{E}^a$. The relevant modifications to be made in [7] are that the resolvent of Brownian motion must be replaced throughout by the resolvent of $\tilde{B}(x,t)$. And instead of $\exp\{-\sqrt{2\lambda}\tilde{L}(a,t)\}$ for the conditional law of $\tau(a,t)$ (see Corollary 1.5) we use instead the evaluation $\exp\{-\sqrt{2\lambda}\tanh\sqrt{2\lambda}(x-a)\tilde{L}(a,t)\}$ taken from Lemma 3.3. The killed resolvent

$$\mathbf{E}_y[\int_0^{T_a(x)} e^{-\lambda t}f(\tilde{B}(x,t))dt] \qquad (a < y \le x)$$

can be computed as

$$\frac{2}{\sqrt{2\lambda}}\left(\frac{\sinh\sqrt{2\lambda}(y-a)}{\cosh\sqrt{2\lambda}(x-a)}\right)\int_y^x \cosh\sqrt{2\lambda}(x-z)f(z)dz$$

where we suppose (as will be the case here) that f is supported on $[y, x]$. In fact $f(z)dz$ needs to be replaced by the Dirac mass at x, which gives us

$$\frac{2}{\sqrt{2\lambda}}\left(\frac{\sinh\sqrt{2\lambda}(y-a)}{\cosh\sqrt{2\lambda}(x-a)}\right)$$

the derivative at the point $y = a+$ being given by $2\mathrm{sech}\sqrt{2\lambda}(x-a)$. As a further, almost trivial, simplification we will only consider the case $a > 0$. Then we can read off from Lemmas 3.1 and 3.2 of [7] that the projection of (4.d) is given by

$$\lambda\mu 2\mathrm{sech}^2\sqrt{2\lambda}(x-a)\int_0^\infty \exp\{-\lambda t-\sqrt{2\lambda}\tanh\sqrt{2\lambda}(x-a)\tilde{L}(a,t)\}d_t\tilde{L}(a,t)$$

$$\int_0^t \exp\{-\mu s-[\sqrt{2(\lambda+\mu)}\tanh\sqrt{2(\lambda+\mu)}(x-a)-\sqrt{2\lambda}\tanh\sqrt{2\lambda}(x-a)]\times$$

$$\tilde{L}(a,s)\}d_s\tilde{L}(a,s)+\frac{1}{\sqrt{2\lambda}}2\tanh\sqrt{2\lambda}(x-a)2\mathrm{sech}\sqrt{2(\lambda+\mu)}(x-a)\times$$

$$\int_0^\infty \exp\{-(\lambda+\mu)t-\sqrt{2(\lambda+\mu)}\tanh\sqrt{2(\lambda+\mu)}(x-a)\tilde{L}(a,t)\}d_t\tilde{L}(a,t).$$

Now we take the derivative of this in x at $x = a+$. Comparing this with what we get by using the Ito formula directly, using the results of the previous two lemmas, we obtain the required result.

Lemma 4.5 The process N has a bicontinuous version.

Proof: We restrict our considerations to the compact set $[z_1, z_2] \times [u_1, u_2]$ of the plane and we re-normalise by letting $N(z_1, .) \equiv 0$. Fix an integer $p \geq 2$ and note that

$$||N(x,s)-N(y,t)||_p \leq ||N(x,s)-N(x,t)||_p+||N(x,t)-N(y,t)||_p$$

For fixed t we know that $\{N(y,t), y \geq x\}$ has a continuous martingale version, and so we can use the inequality of Burkholder-Davis-Gundy to get (using Lemma 4.3) the estimate

$$||N(x,t)-N(y,t)||_p^p \leq C_p\mathbf{E}[(2\int_x^y \tilde{L}(a,t)da)^{p/2}] \leq C(y-x)^{p/2}$$

where we use Jensen's inequality and the fact that we know the law of $\tilde{L}(a,t)$. Next we use the fact that $\{N(x,s) - N(x,t), x \geq z_1\}$ has a continuous version also to get (by the same reasoning) using this time Lemma 4.4

$$||N(x,t) - N(x,s)||_p^p \leq C_p\mathbf{E}[\left(2\int_{z_1}^{x} \tilde{L}(a,t)da\right)^{p/2}] \leq C(t-s)^{p/4}$$

when p is a multiple of 4. However then we calculate that

$$||N(x,s) - N(y,t)||_p^p \leq C||(x,s) - (y,t)||^{p/6}$$

provided p is a multiple of 12. The claim now follows by the Kolmogoroff criterion which we stated in the first section.

This even gives us a Hölder condition on the paths of both N and D. The following is our main theorem.

Theorem 4.6 The semimartingale $\{\tilde{L}(x,t), x \in R\}$ possesses a bicontinuous version of its canonical $\mathcal{E}^x$ decomposition, this decomposition being given explicitly by the following

$$\tilde{L}(x,t) = N(x,t) + \int_a^x 1_{(y<0)}(y)dy - \frac{d}{dt}\int_a^x \tilde{L}^2(y,t)dy$$

Moreover the quadratic variation satisfies the following properties

$$<\tilde{L}(.,t)>_x - <\tilde{L}(.,t)>= 2\int_a^x \tilde{L}(y,t)dy$$

If $0 \leq r \leq s < t$ then

$$<\tilde{L}(x,t) - \tilde{L}(x,s), \tilde{L}(x,r)> \equiv 0$$

Proof: This is a consequence of the previous three lemmas. Note how the identification of $D(x,t)$ follows because we now know that this process has a bicontinuous version. For by the remark following Lemma 4.2

$$\int_0^t (D(x,s) - D(a,s))ds = t\int_a^x 1_{(y<0)}dy - \int_a^x dy\int_0^t d_s\tilde{L}^2(y,s)$$

So the identification is immediate by the fundamental theorem of calculus applied pathwise.

We now look briefly at the process $\tilde{B}(x,t)$. This is not so important as $\tilde{L}$ so we give less detail.

Theorem 4.7 The process

$$\left\{\int_0^\infty e^{-\lambda t}\tilde{B}(x,t)dt, x \in R\right\}$$

is an $\mathcal{E}^x$ semimartingale with drift term given by

$$-2\int^x da\int_0^\infty e^{-\lambda t}\tilde{L}(a,t)d_t\tilde{B}(a,t)$$

Proof: We begin by proving that the process is a quasimartingale on every compact interval $[x,y]$. So we let $x = x_0 < x_1 < \ldots < x_n = y$ be a partition and consider the sum

$$\mathbf{E}\left[\sum|\mathbf{E}\left[\int_0^\infty e^{-\lambda t}dt[\tilde{B}(x_{i+1},t) - \tilde{B}(x_i,t)|\mathcal{E}^{x_i}\right]|\right] \tag{4.e}$$

However from Lemma 3.6 we can estimate for $x \geq a$

$$|\mathbf{E}[\int_0^\infty e^{-\lambda t}\{\tilde{B}(x,t) - \tilde{B}(a,t)\}dt|\mathcal{E}^a]| =$$

$$|\int_0^\infty e^{-\lambda t}\mathbf{E}[\tilde{B}(x,t\wedge T_a(x)) - a]dt + \int_0^\infty e^{-\lambda t}dt\int_0^\infty \mathbf{E}[Y_s - 1_{[0,t]}(s)|\mathcal{E}^a]d_s\tilde{B}(x,s)|$$
$$\leq \frac{1}{\lambda(x-a)} + |\int_0^\infty e^{-\lambda t}\mathbf{E}[\exp\{-\lambda\int_a^x L(b,\tau(a,t))db\} - 1|\mathcal{E}^a]d_t\tilde{B}(x,t)|$$

The conditional expectation computes from Lemma 3.3, so by the inequality of Burkholder-Davis-Gundy (in semimartingale form [3]) we have the bound

$$\mathbf{E}[|\int_0^\infty e^{-\lambda t}\mathbf{E}[\exp\{-\lambda\int_a^x L(b,\tau(a,t))db\} - 1|\mathcal{E}^a]d_t\tilde{B}(x,t)|] \leq$$

$$C\{\mathbf{E}[\left(\int_0^\infty e^{-2\lambda t}\lambda^2(x-a)^2\tilde{L}^2(a,t)dt\right)^{1/2}] + \mathbf{E}[\int_0^\infty e^{-\lambda t}\lambda(x-a)\tilde{L}(a,t)d_t\tilde{L}(a,t)]\}$$

for x sufficiently close to a. We can bound the first term by using Jensen's inequality to take the expectation inside the square root. And because we get the bound $C(x-a)$, it follows that the sum at $(4.e)$ is bounded independently of the partition. This completes the proof that the process $\int_0^\infty e^{-\lambda t}\tilde{B}(x,t)dt$ is an $\mathcal{E}^x$ semimartingale. The calculation of the drift term is much the same as for $\tilde{L}$.

§5. Parameterised stochastic integrals and martingale representation

Recall that we have the decomposition

$$\tilde{L}(x,t) = N(x,t) + D(x,t)$$

where $N(.,t)$ is a martingale, and $D(.,t)$ has bounded variation. We introduce the following space of two-parameter processes. Denote by $\mathcal{H}$ the collection of equivalence classes of all processes $Z(x,t)$ which are measurable w.r.t. the product of the $\mathcal{E}^x$ predictable σ-algebra and the Borel σ-algebra on R^+ and which satisfy the condition

$$||Z||_2^2 = \mathbf{E}\left[2\int_0^\infty \int_{-\infty}^\infty Z^2(y,t)d_t\tilde{L}(y,t)dy\right] < +\infty$$

The equivalence relation is of course the usual one relative to this norm and it is clear that $\mathcal{H}$ is a Hilbert space. It is convenient to single out the collection $\mathcal{H}^a$ of elements of $\mathcal{H}$ which can be written in the form $Z(x,t) = \int_t^\infty K(x,s)ds$ where K is bicontinuous and $\mathcal{E}^x$ adapted with compact support in x. Notice that $K(x,t)$ is not required to be $\tilde{B}(x,t)$ adapted. Then by Theorem 1.9, since continuous processes are predictable, we can define the stochastic integral $\int_{-\infty}^x K(y,t)d_y\tilde{L}(y,t)$ for each fixed value of t. This enables us to produce a martingale as follows.

Lemma 5.1 For each $Z(x,t) = \int_t^\infty K(x,s)ds$ in $\mathcal{H}^a$ the process

$$\int_0^\infty dt \int_{-\infty}^x K(y,t)d_y\tilde{L}(y,t) - \int_0^\infty dt\int_{-\infty}^x 1_{(y<0)}K(y,t)dy$$
$$+\int_0^\infty\int_{-\infty}^x K(y,t)d_t\tilde{L}^2(y,t)dy$$

is a continuous square integrable $\mathcal{E}^x$ martingale whose increasing process is equal to $\int_0^\infty \int_{-\infty}^x Z^2(y,t)d_t\tilde{L}^2(y,t)dy$.

Proof: From Theorem 4.6 we can write the above expression as

$$\int_0^\infty dt \int_{-\infty}^x K(y,t)d_y N(y,t)$$

By Fubini this is a continuous martingale. For the calculation of the increasing process, we know from Theorem 4.6 that

$$< \int^\cdot K(y,t)d_y\tilde{L}(y,t), \int^\cdot K(y,s)d_y\tilde{L}(y,s) >_x = 2\int^x K(y,t)K(y,s)\tilde{L}(y,t\wedge s)dy$$

Now use the bilinearity of the bracket operation to find that the increasing process of the above martingale is given by

$$2\int_0^\infty dt \int_0^\infty ds \int^x K(y,t)K(y,s)\tilde{L}(y,t\wedge s)dy$$

The proof is completed by replacing $\tilde{L}(y,t\wedge s)$ by the corresponding integral and changing the order of integration (twice).

When Z is an element of $\mathcal{H}^a$ then we shall write the above martingale as $\int^x Z(y,t)\partial\tilde{L}(y,t)$. Obviously this does not depend on the choice of representation for Z. The next step is to extend this definition to all of $\mathcal{H}$.

Theorem 5.2 The above mapping extends uniquely to an isometry from $\mathcal{H}$ into the space of square integrable continuous $\mathcal{E}^x$ martingales.

Proof: By Lemma 5.1 if Z lies in $\mathcal{H}^a$ then the martingale $\int^x Z(y,t)\partial\tilde{L}(y,t)$ is continuous. However $\mathcal{H}^a$ is a dense collection in $\mathcal{H}$ so that if $\{Z_n\}$ is a sequence which converges to Z in the norm then the corresponding martingales form a Cauchy sequence in the martingale H_2 norm. Therefore this defines a unique continuous square integrable martingale.

It is convenient to use the notation $\int^x Z(y,t)\partial\tilde{L}(y,t)$ for the extended mapping also. Notice that we have now proved that the martingales of this type form a closed linear subspace of the square integrable $\mathcal{E}^x$ martingales. The hard part is to prove that it is a dense subspace. This is made easier by the fact that $\mathcal{H}^a$ suffices to represent the martingales arising from the conditional excursion formulae of Williams. First we introduce some more notation. As in [6] we shall write

$$K_t(n,\lambda,\mathbf{f}) = K_t(\lambda_1,\lambda_2,\ldots,\lambda_n; f_1, f_2,\ldots f_n)$$
$$= \int_0^t dt_n e^{-\lambda_n t_n} f_n(B_{t_n}) \int_0^{t_n} \ldots \int_0^{t_2} dt_1 e^{-\lambda_1 t_1} f_1(B_{t_1})$$

where the functions $\{f_n\}$ are always assumed to be continuous and to have compact support. Also it will be convenient to let $K_t(0,\lambda,\mathbf{f}) \equiv 1$. Note that here we do not require all the functions to vanish below a pre-determined level as in [6] so that the n^{th} order formulae will necessarily be more complicated. The point is of course that these random variables, with n, $\{\lambda_n\}$, and $\{f_n\}$ all varying, generate the σ-field $\mathcal{E}^\infty$. Their projection along the excursion filtration provides us with a dense family of martingales.

Our immediate task is to examine the first order conditional excursion formulae.

$$\mathbf{E}[\int_0^\infty e^{-\lambda t} f(B_s) ds | \mathcal{E}^x] = R_\lambda^x f(0) + \exp\{\sqrt{2\lambda} x^-$$
$$[R_\lambda^x f'(x+) \int_0^\infty \exp\{-\lambda s - \sqrt{2\lambda}\tilde{L}(x,s)\} d_s \tilde{L}(x,s) +$$
$$\int_0^\infty \exp\{-\lambda s - \sqrt{2\lambda}\tilde{L}(x,s)\} f(\tilde{B}(x,s)) ds] \qquad (5.a)$$

The term which now causes trouble is the absolutely continuous integral in $(5.a)$. The point is that, a priori, there is no reason to suppose that this does not contribute to the martingale part.

First recall from Corollary 1.5 the evaluation

$$\mathbf{E}[\exp\{-\lambda\tau(x,\xi)\} | \mathcal{E}^x] = \exp\{-\lambda\xi - \sqrt{2\lambda}\tilde{L}(x,\xi) - \sqrt{2\lambda} x^-\}$$

where $\xi \geq 0$ is any $\mathcal{E}^x$ measurable random variable.

Lemma 5.3 The martingale part of $(5.a)$ can be written in the form

$$C + \int^x Z(y,t) \partial \tilde{L}(y,t)$$

for some Z in $\mathcal{H}^a$.

Proof: Looking at $(5.a)$ as a function of x we see that the first term on the r.h.s. contributes only a drift. So we concentrate on the two integrals. For the first one we can integrate by parts and get

$$\exp\{\sqrt{2\lambda}x^-\}R_\lambda^x f'(x+)\frac{1}{\sqrt{2\lambda}}[1-\lambda\int_0^\infty \exp\{-\lambda t-\sqrt{2\lambda}\tilde{L}(x,t)\}dt]$$

So by Ito's formula and Fubini this contributes the martingale term

$$\lambda\int_0^\infty dt\int_{-\infty}^x \exp\{\sqrt{2\lambda}y^-\}R_\lambda^y f'(y+)\exp\{-\lambda t-\sqrt{2\lambda}\tilde{L}(y,t)\}d_yN(y,t)$$

By the occupation density formula the absolutely continuous integral can be written as

$$\int^x f(y)dy\int_0^\infty \mathbf{E}[\exp\{-\lambda\tau(y,t)\}|\mathcal{E}^x]d_t\tilde{L}(y,t)$$

which makes it clear that the martingale contribution comes from the conditional expectation part. Consequently, using Fubini, we see that it is given by

$$-\sqrt{2\lambda}\int_0^\infty dt\int_{-\infty}^x \exp\{-\sqrt{2\lambda}y^- - \lambda t-\sqrt{2\lambda}\tilde{L}(y,t)\}f(\tilde{B}(y,t))d_yN(y,t).$$

And this has the required form.

Theorem 5.4 Every martingale of the form $\mathbf{E}[K_\infty(n,\lambda,f)|\mathcal{E}^x]$ can be written as

$$C+\int_0^\infty dt\int_{-\infty}^x K(y,t)d_yN(y,t)$$

where $Z(x,t)=\int_t^\infty K(x,s)ds$ is in $\mathcal{H}^a$.

Proof: We only need to look at these for $n>1$. However the inductive step is rather messy so we proceed in a more descriptive way. Consider for the moment the situation where the process starts at x. Then, by [7] Lemma 3.1 or Theorem 3.1 we have

$$\mathbf{E}_x[\int_0^\infty e^{-\mu t}g(B_t)K_t(n,\lambda,\mathbf{f})dt|\mathcal{E}^x]=$$

$$R_\lambda^x f'(x+)\int_0^\infty dt\mathbf{E}_x[\exp\{-\mu\tau(x,t)\}K_{\tau(x,t)}(n,\lambda,\mathbf{f})|\mathcal{E}^x]d_t\tilde{L}(x,t)+$$

$$\int_0^\infty dt\mathbf{E}_x[\exp\{-\mu\tau(x,t)\}K_{\tau(x,t)}(n,\lambda,\mathbf{f})|\mathcal{E}^x]g(\tilde{B}(x,t))dt \qquad (5.b)$$

Now let us consider the conditional expectation part on the r.h.s. of (5.b). It will split into two terms, corresponding respectively to the excursions below and above the level x. The first term is given by

$$\int_0^t \mathbf{E}_x[\exp\{-\mu\tau(x,t) - \lambda_n\tau(x,s)\}K_{\tau(x,s)}(n-1,\lambda,\mathbf{f})|\mathcal{E}^x]f_n(\tilde{B}(x,s))ds$$

But since $s < t$ we can use the conditional independence result of Lemma 1.2 at time $\tau(x,s)$ to see that this equals

$$\int_0^t \exp\{-\mu(t-s) - \sqrt{2\mu}[\tilde{L}(x,t) - \tilde{L}(x,s)]\}$$

$$\mathbf{E}_x[\exp\{-(\mu+\lambda_n)\tau(x,s)\}K_{\tau(x,s)}(n-1,\lambda,\mathbf{f})|\mathcal{E}^x]f_n(\tilde{B}(x,s))ds$$

where we have evaluated the conditional expectation of $\tau(x,t) - \tau(x,s)$ by using Lemma 1.2 and Corollary 1.5. The point is that we have now moved the variable t outside the conditional expectation. The second term can be treated by applying [7] Lemma 3.2 so that again the variable t comes out of the conditional expectation. Continuing this procedure, using [7] Lemmas 3.1 and 3.2 when we need to look at the excursions above x and Lemma 1.2 when we look at the process below x, we eventually obtain that (5.b) can be expressed as a sum of a number of terms of the form

$$\int_0^\infty dI(0,t)\int_0^t dI(1,t_1)\ldots\int_0^{t_{n-1}} dI(n,t_n) \tag{5.c}$$

where each $dI(j,s)$ can be written in one of the two forms

$$dI(j,s) = F(x)\exp\{-\lambda s - \sqrt{2\lambda}\tilde{L}(x,s)\}d_s\tilde{L}(x,s)$$

or

$$dI(j,s) = F(\tilde{B}(x,s))\exp\{-\lambda s - \sqrt{2\lambda}\tilde{L}(x,s)\}ds \tag{5.d}$$

Now we want to remove the singular processes. This we can do by changing the order of integration, bringing each one in turn to the extreme r.h.s. of (5.c) and then integrating by parts. This is possible because the relevant term will never contain any explicit dependence on $\tilde{B}(x,s)$ and it yields a process of the form $F(x)\exp\{-\lambda s - \sqrt{2\lambda}\tilde{L}(x,s)\}$ plus an absolutely continuous integral. Hence we can assume that in (5.c) all integrands are of the form (5.d). Now look at the integrand in the multiple integral of (5.c) and apply Ito's formula to obtain its martingale part, just as we did for the first order case, only that now there will be up to n terms. This supplies us with a parameterised stochastic integral of the required form provided we use a stochastic Fubini theorem to bring the integrating processes to the front. Finally we look at the contribution from the initial excursion. We will decompose $(5.b)$ at time T_x so that the l.h.s. can now be written as

$$\mathbf{E}[\int_0^{T_x} e^{-\mu t} g(B_t) K_t(n,\lambda,\mathbf{f}) dt | \mathcal{E}^x] + \mathbf{E}[\int_{T_x}^{\infty} e^{-\mu t} g(B_t) K_t(n,\lambda,\mathbf{f}) dt | \mathcal{E}^x]$$

The first term does not have a martingale part, and it is independent of $\mathcal{E}^x$. The second term can be further decomposed by repeating the procedure on the first integral in $K_t(n,\lambda,\mathbf{f})$. This gives

$$\mathbf{E}[\int_{T_x}^{\infty} e^{-\mu t} g(B_t) dt \int_0^{T_x} e^{-\lambda_1 s} f_1(B_s) K_s(n-1,\lambda,\mathbf{f}) ds | \mathcal{E}^x] \quad +$$

$$\mathbf{E}[\int_{T_x}^{\infty} e^{-\mu t} g(B_t) dt \int_{T_x}^{t} e^{-\lambda_1 s} f_1(B_s) K_s(n-1,\lambda,\mathbf{f}) ds | \mathcal{E}^x] \qquad (5.e)$$

We can apply the strong Markov property at time T_x to reduce the first part of $(5.e)$ to the consideration of a first order conditional excursion formula, which we already know how to deal with. It is clear that we can repeat the procedure to get higher order formulae of the type we have just looked at and which we can therefore represent in the required form. The proof is now finished once we remark that by Corollary 2.2 (ii) the initial σ-field is trivial.

Corollary 5.5 The square integrable $\mathcal{E}^x$ martingales can all be represented as

$$C + \int^x Z(y,t) \partial \tilde{L}(y,t)$$

where Z is an element of $\mathcal{H}$.

Proof: By the theorem this holds for a dense set of $\mathcal{E}^x$ martingales. However the martingales which satisfy this representation property form a closed subspace of the square integrable $\mathcal{E}^x$ martingales.

REFERENCES

1. **Barlow, M.T. and Yor, M.** *(Semi-)martingale inequalities and local times, Zeit. für Wahrscheinlichkeitstheorie* **55**, 237-254 (1981).

2. **Davis, M.H.A. and Varaiya, P.** *The multiplicity of an increasing family of σ-fields*, *Ann. of Prob.* **2** 958-963 (1974).

3. **Dellacherie, C. et Meyer, P.A.** *Probabilités et potentiel (Théorie des martingales).* Herman, Paris, 1980.

4. **El-Karoui, N. et Chaleyat-Maurel, M.** *Un problème de réflexion et ses applications au temps local et aux équations différentielles stochastiques sur R - cas continu.* In: *Temps Locaux, Astérisque Vol. 52/3*, Soc. Math. France, 1977.

5. **Maisonneuve, B.** *Exit systems. Ann. Prob.* **3**, 399-411, (1975).

6. **McGill, P.** *Markov properties of diffusion local time : a martingale approach Adv. Appl. Prob.* **14** , 789-810 (1982).

7. **McGill, P.** *Calculation of some conditional excursion formulae. Zeit. für Wahrscheinlichkeitstheorie* **61**, 255-260 (1982).

8. **McGill, P.** *Time changes of Brownian motion and the conditional excursion theorem. In: Lect. Notes in Math. 1095* Springer-Verlag, Berlin and New York, 1984.

9. **McGill, P.** *Quelques propriétés du compensateur du processus des excursions browniennes. Comptes Rendus Acad. Sc.* **t. 298**, Série I, no. 15 357-359, (1984).

10. **Meyer, P.** *Flot d'une équation différentielle stochastique. In: Séminaire de Probabilités XV, 103-117,* Lecture Notes in Mathematics *Vol. 850* , Springer-Verlag, Berlin and New York, 1981.

11. **Stricker C. and Yor, M.** *Calcul stochastique dépendant d'un paramètre, Zeit. für Wahrscheinlichkeitstheorie* **45**, 109-133 (1978).

12. **Walsh, J.B.** *Excursions and local time. In: Temps Locaux, Astérisque Vol. 52/3,* Soc. Math. France, 1977.

13. **Walsh, J.B.** *Stochastic integration with respect to local time. In: Seminar on stochastic processes 1982,* Birkhaüser, Basel 1983.

14. **Williams, D.** *Markov properties of Brownian local time. Bull. Amer. Math. Soc.* **75**, 1055-56 (1969).

15. **Williams, D.** *Conditional excursion theory. In: Séminaire de Probabilité XIII, Lect. Notes in Math. 721*, Springer-Verlag, Berlin and New York, 1979.

16. **Williams, D.** *Diffusions, Markov processes, and martingales.* Wiley, Chichester, 1979.

17. **Yamada, T. and Ogura, Y.** *On the strong comparison theorems for the solutions of stochastic differential equations, Zeit. für* Wahrscheinlichkeitstheorie **56**, 3-19 (1981).