SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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*Séminaire de probabilités (Strasbourg)*, tome 17 (1983), p. 221-224

<http://www.numdam.org/item?id=SPS_1983__17__221_0>

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# Skorokhod Imbedding via Stochastic Integrals

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Given a Brownian motion $L_t$ and a probability measure $\mu$ on $\mathbb{R}$ with mean 0, a Skorokhod imbedding of $\mu$ is a stopping time $T$ adapted to the sigma fields of $L_t$ such that $L_T$ has distribution $\mu$. We give here a new method of constructing such an imbedding using results from the representation of martingales as stochastic integrals.

We first construct a Brownian motion $N_t$ and a stopping time $W$ such that $N_W$ has law $\mu$. We then show how, given an arbitrary Brownian motion $L_t$, one can construct a stopping time $T$ such that $L_T$ has law $\mu$.

Define $p_t(y) = (2\pi t)^{-1/2} e^{-y^2/2t}$, $q_t(y) = \partial p_t(y)/\partial y = -(2\pi t)^{-1/2}(y/t)e^{-y^2/2t}$. Let $X_t$ be a Brownian motion, $\underline{\underline{F}}_t$ its filtration, and $g$ a real-valued function.

Lemma 1. Suppose $E|g(X_1)| < \infty$. Then

a) $\sup_{|y|\le y_0} \int g(z)|z-y|^k e^{-(z-y)^2/2t}dz < \infty$ for all positive $k$, all $y_0$, all $t<1$.

b) $g(X_1) = Eg(X_1) + \int_0^1 a(s,X_s)dX_s$, where $a(s,y) = \int q_{1-s}(z-y)g(z)dz$ for $s<1$; furthermore $\int_0^1 a^2(s,X_s)ds < \infty$, a.s.

c) $E(g(X_1)|\underline{\underline{F}}_s) = b(s,X_s)$ for $s<1$, where $b(s,y) = \int p_{1-s}(z-y)g(z)dz$.

Proof. a) follows from the formula for the normal density and the fact that $|z-y|^k e^{-(z-y)^2/2t} \le e^{-z^2/2}$ for $z$ large.

b) Suppose first that $g$ is bounded, has compact support, and is in $C^2$. By Clark's formula [1] applied to the functional $g(X_1)$,

$$g(X_1) = Eg(X_1) + \int_0^1 E[g'(X_1)|\underline{\underline{F}}_s]dX_s .$$

(Another derivation of this representation is to use Ito's lemma to take care of the case $g(x) = e^{iux}$ and then use linearity and a limiting process.)

By the Markov property, if $s < 1$ ,

$$E[g'(X_1)|\underline{F}_s] = \int g'(z)p_{1-s}(X_s - z)dz .$$

An integration by parts gives the result for such $g$ ; the result for general $g$ follows by a limit argument.

c) By the Markov property, if $s < 1$ ,

$$E[g(X_1)|\underline{F}_s] = \int g(z)p_{1-s}(X_s - z)dz . \qquad \square$$

Lemma 2. Suppose $g$ is nondecreasing and not identically constant. Then

a) On compact subsets of $[0,1) \times \mathbb{R}$ , $a(s,y)$ is bounded above, bounded below away from 0 , and uniformly Lipschitz in $s$ and $y$ .

b) For each $s < 1$ , $b(s,y)$ is continuous and strictly increasing as a function of $y$ .

c) For each $s < 1$ , let $B(s,\cdot)$ be the inverse of $b(s,\cdot)$ ; then on compact subsets of its domain, $B(s,y)$ is uniformly Lipschitz in $s$ and jointly continuous in $s$ and $y$ .

Proof. a) Suppose $|y| \le y_0, s \le s_0 < 1$ . $a(s,y)$ is bounded above by lemma 1a. An integration by parts argument shows that $a(s,y) = \int p_{1-s}(y-z)dg(z)$ , hence $a$ is bounded below. Using the definition of $a(s,y)$ , appropriate bounds on $\partial q_{1-s}/\partial s$ and $\partial q_{1-s}/\partial y$ , and lemma 1a gives the uniformly Lipschitz result.

b) The definition of $b$ shows that $b(s,\cdot)$ is continuous. Since we also have $b(s,y) = \int g(y+z)p_{1-s}(z)dz$ , it follows that $b(s,\cdot)$ is nondecreasing, and in fact, strictly increasing since $g$ is not constant. Note that this implies that the range of $b(s,\cdot)$ must be an open (possibly infinite) interval.

c) Since $b(s,\cdot)$ is continuous and strictly increasing, we can define its inverse $B(s,\cdot)$ on the range of $b(s,\cdot)$ . $B(s,y)$ will be continuous in $y$.

Integrating by parts,

$$\partial b/\partial y = \int p_{1-s}(y-z)dg(z) ,$$

which is uniformly $> 0$ for $s,y$ in a compact subset of $[0,1) \times \mathbb{R}$ . $\partial b/\partial s$ is bounded above on compact sets since $\partial p_{1-s}/\partial s$ is, using lemma 1a again.

We now show that $B$ is uniformly Lipschitz in $s$ , $s,y$ in a compact subset of the domain of $B$ . Let $w = B(s+h,y)$ , $x = B(s,y)$ , and suppose $w \le x$ ,

the other case being similar. Then

$$0 = b(s+h,w) - b(s,x) = b(s+h,w) - b(s,w) + b(s,w) - b(s,x) \le C|h| - c(x-w),$$

or $|x-w| \le C|h|/c$ ,

where $C$ and $c$ are upper and lower bounds for $\partial b/\partial s$ and $\partial b/\partial y$ , respectively. This proves that $B$ is uniformly Lipschitz in $s$ , and it follows immediately that $B$ is jointly continuous. □

Now let $\mu$ be a probability measure on $\mathbb{R}$ and suppose $\int|x|\, d\mu(x) < \infty$ and $\int x\, d\mu(x) = 0$ . Let $F(x) = \mu(-\infty,x]$ , let $F^{-1}(y) = \inf\{x: F(x) \ge y\}$ , let $\Phi(x) = \int_{-\infty}^{x} p_1(y)dy$ , and let $g(x) = F^{-1}(\Phi(x))$ . Then $g(X_1)$ has distribution $\mu$ and $Eg(X_1) = 0$ .

Define $M_t = \int_0^t a(s,X_s)dX_s$ , where $a(s,y)$ is given by lemma 1 for $s < 1$ , $a(s,y) = 1$ for $s \ge 1$ . Note $M_1 = g(X_1)$ has law $\mu$ , and if $s < 1$ , $M_s = b(s,X_s)$ . Let $R(t) = \int_0^t a^2(s,X_s)$ , define $S(t) = \inf\{r:R(r) \ge t\}$ , and let $N_t = M_{S(t)}$ . Since the quadratic variation of the continuous martingale $N$ is $t$ , $N$ is a Brownian motion.

$N_{R(1)} = M_1$ , which has law $\mu$ . Letting $W = R(1)$ , it suffices to show that $R(1)$ is a stopping time of the $N_t$ process.

Proposition 3. (cf. Yershov, [2]). $(W \ge u)$ is in the right continuous completion of $\sigma(N_s;\ s \le u)$ .

Proof. Since $W = R(1) = \lim_{s\to 1} R(s)$ by monotone convergence, it suffices to consider $R(s)$, $s < 1$ . $(R(s) \ge u) = (s \ge S(u))$ .

It is not hard to see that $S(t)$ satisfies the equation

$$\frac{dS(t)}{dt} = a^{-2}(S(t),\ X_{S(t)})$$

if $S(t) < 1$ . But $X_{S(t)} = B(S(t),\ M_{S(t)}) = B(S(t),\ N_t)$ . Thus, for each $\omega$ , $S(t)$ satisfies the ordinary differential equation

$$(1)\quad \frac{dS(t)}{dt} = a^{-2}(S(t),\ B(S(t),\ N_t)) .$$

For each $\omega$ , $\{(S(t),\ N_t):\ S(t) \le s\}$ is contained in a compact subset of the domain of $B$ . This, lemma 2, and a theorem on uniqueness of solutions of

differential equations [3, pp.1-6] show that there is a unique solution $S(t)$ to (1) up to the first $t$ for which $S(t) = s$. Moreover, this solution may be constructed via Picard iteration. But then $(s \geq S(u))$ is in the right continuous completion of $\sigma(N_s;\ s \leq u)$ as required. □

Suppose now that $L$ is any Brownian motion. We construct an $L$-measurable stopping time $T$ such that $L$ has law $\mu$. Let $V(t)$ be the unique solution to

$$\frac{dV(t)}{dt} = a^{-2}(V(t),\ B(V(t),\ L_t))$$

for each $\omega$. (Since $L_t$ has the same law as $N_t$, $\{(V(t),\ L_t):\ V(t) \leq s\}$ will be in a compact subset of the domain of $B$, a.s.) Let $U(t) = V^{-1}(t)$, $t < 1$ and let $T = U(1) = \sup_{s<1} U(s)$. Clearly the law of $(L,T)$ is the same as the law of $(N,W)$, and so $L_T$ has distribution $\mu$.

$T^{1/2}$ will satisfy certain moment conditions if $\mu$ does. For example, suppose $\Psi\colon [0,\infty) \to [0,\infty)$ is continuous, $\int \Psi(|x|)d\mu(x) < \infty$, and for some $\varepsilon > 0$, $\Psi^{1/(1+\varepsilon)}$ is convex and increasing. By Doob's inequality applied to the submartingale $\Psi^{1/(1+\varepsilon)}(|M_s|)$, $E \sup_{s\leq 1} \Psi(|M_s|) < \infty$ since $E\,\Psi(|M_1|) < \infty$. Then by Burkholder's inequality, $E\,\Psi(W^{1/2}) < \infty$.

If $Y_t$ is a d-dimensional Brownian motion, $d \geq 2$, it is known that there are measures $\mu$ for which one cannot find a stopping time $T$ with $\mu$ the law of $Y_T$: just take $\mu$ atomic and recall that d-dimensional Brownian motion does not hit points. However, one can always find $f : \mathbb{R} \to \mathbb{R}^d$ such that the law of $f(X_1)$ is $\mu$, $X_t$ a 1-dimensional Brownian motion. (The coordinate functions of $f$ are not assumed to be nondecreasing.) One can then use lemma 1 to find a vector-valued function $A\colon [0,1) \times \mathbb{R} \to \mathbb{R}^d$ such that $f(X_1) = Ef(X_1) + \int_0^1 A(s,\ X_s)dX_s$.

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