SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

NORIHIKO KAZAMAKI

Note on a stochastic integral equation

Séminaire de probabilités (Strasbourg), tome 6 (1972), p. 105-108

<http://www.numdam.org/item?id=SPS_1972__6__105_0>

© Springer-Verlag, Berlin Heidelberg New York, 1972, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

NOTE ON A STOCHASTIC INTEGRAL EQUATION

by N.KAZAMAKI

In the present paper we shall consider the stochastic integral equation

$$(1) \qquad Z_t = x + \int_0^t f(Z_u)dM_u + \int_0^t g(Z_u)dU_u \ , \quad x \in R^1$$

where $M=(M_t), M_0=0$, is a locally square integrable martingale and $U=(U_t), U_0=0$, is a continuous increasing process.

Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space, given an increasing right continuous family $(\underline{\underline{F}}_t)$ of sub σ - fields of $\underline{\underline{F}}$. We suppose as usual that $\underline{\underline{F}}_0$ contains all the negligible sets. By a normal change of time $A=(\underline{\underline{F}}_t, a_t)$ we means a family of stopping times of the family $(\underline{\underline{F}}_t)$, finite valued, such that for $\omega \in \Omega$ the sample function $a_\cdot(\omega)$ is strictly increasing, $a_0(\omega)=0$, $a_\infty(\omega)=\lim_{t\to\infty} a_t(\omega)=\infty$ and continuous. We don't distinguish two processes X and Y such that for a.e $\omega \in \Omega$ $X_\cdot(\omega)=Y_\cdot(\omega)$. We assume that the reader knows the usual definitions.

THEOREM.- Assume that the family $(\underline{\underline{F}}_t)$ is quasi-left continuous. Then for coefficients f and g belonging to $C^1(R^1)$ and of bounded slope the equation (1) has one and only one solution.

PROOF.- From the quasi-left continuity of $(\underline{\underline{F}}_t)$, it follows that there exists a unique continuous increasing process $\langle M \rangle$ such that $M^2-\langle M \rangle$ is a local martingale. Define

$$(2) \qquad b_t=t+\langle M \rangle_t+U_t \ , \quad a_t= \inf(u;b_u>t) \ .$$

Then an easy computation shows that $A=(\underline{\underline{F}}_t, a_t)$ and $B=(\underline{\underline{F}}_{a_t}, b_t)$ are normal changes of time.

For every t, put

$$(3) \qquad Y_t = Z_{a_t} \ , \ N_t = M_{a_t} \ , \ V_t = U_{a_t} .$$

The process N is a square integrable martingale and V is the natural increasing process associated to N ; clearly V is, in fact, continuous. It is shown in [1] that we have

$$(4) \qquad \int_0^{a_t} f(Z_u)dM_u = \int_0^t f(Y_u)dN_u \ .$$

Thus , in order to show the existence of the unique solution of (1) , it suffices to consider the following stochastic integral equation

$$(1^*) \qquad Y_t = x + \int_0^t f(Y_u)dN_u + \int_0^t g(Y_u)dV_u \ .$$

For simplicity, the proof is spelled out for $0 \leqq t \leqq 1$ only. Without loss of generality, we may assume that $\max(\|f'\|_\infty \ , \ \|g'\|_\infty) \leqq 1/2$.

Define in succession

$$(5) \qquad \begin{aligned} Y_t^0 &= x \\ Y_t^n &= x + \int_0^t f(Y_u^{n-1})dN_u + \int_0^t g(Y_u^{n-1})dV_u \ , \ n=1,2,\ldots \end{aligned}$$

Put now

$$c_t^n = f(Y_t^n)-f(Y_t^{n-1}) \ , \ d_t^n = g(Y_t^n)-g(Y_t^{n-1}) \ .$$

As $t = b_{a_t} = a_t + \langle N \rangle_t + V_t$ by the definition of a_t , we have

$$\begin{aligned} D_n(t) = E[(Y_t^{n+1} - Y_t^n)^2] &\leqq 2E[(\int_0^t c_u^n \, dN_u)^2 + (\int_0^t d_u^n \, dV_u)^2] \\ &\leqq 2 \{ E[\int_0^t (c_u^n)^2 d\langle N \rangle_u] + E[V_t \int_0^t (d_u^n)^2 dV_u] \} \\ &\leqq 2 \{ E[\int_0^t (c_u^n)^2 du] + E[\int_0^t (d_u^n)^2 du] \} \\ &\leqq 2(\|f'\|_\infty^2 + \|g'\|_\infty^2) \int_0^t E[(Y_u^n - Y_u^{n-1})^2] \, du \\ &\leqq \int_0^t D_{n-1}(u) \, du \leqq \text{Const.} \cdot t^n/n! \ . \end{aligned}$$

Since the process $\left(\int_0^t c_u^n \, dN_u\right)$ is a martingale, the extension of Kolmogorov's inequality to martingales shows that for any $\varepsilon>0$

$$\begin{aligned}
\varepsilon^2 P\Big(\sup_{0\leq t\leq 1} \Big|\int_0^t c_u^n \, dN_u\Big| \geq \varepsilon\Big) &\leq E\Big[\Big(\int_0^1 c_u^n \, dN_u\Big)^2\Big] \\
&\leq E\Big[\int_0^1 (c_u^n)^2 du\Big] \\
&\leq \|f'\|_\infty^2 \int_0^1 D_{n-1}(u)\, du \\
&\leq \text{Const.} \times 1/n! \quad .
\end{aligned}$$

Similarly, we get by using the Schwarz inequality

$$\begin{aligned}
P\Big(\sup_{0\leq t\leq 1} \Big|\int_0^t d_u^n \, dV_u\Big| \geq \varepsilon\Big) &= P\Big(\sup_{0\leq t\leq 1} \Big[\int_0^t d_u^n dV_u\Big]^2 \geq \varepsilon^2\Big) \\
&\leq P\Big(\sup_{0\leq t\leq 1} V_t \cdot \int_0^t (d_u^n)^2 dV_u \geq \varepsilon^2\Big) \\
&\leq P\Big(\int_0^1 (d_u^n)^2 du \geq \varepsilon^2\Big) \\
&\leq \varepsilon^{-2} E\Big[\int_0^1 (d_u^n)^2 du\Big] \\
&\leq \text{Const.} \times \varepsilon^{-2}/n! \quad .
\end{aligned}$$

Thus $P\Big(\sup_{0\leq t\leq 1} |Y_t^{n+1} - Y_t^n| \geq 2\varepsilon\Big) \leq \text{Const.} \times \varepsilon^{-2}/n!$. Pick $\varepsilon^{-2} = (n-2)!$. Then $\varepsilon^{-2}/n!$ is the general term of a convergent sum, and so the Borel-Cantelli lemma shows that Y_t^n converges uniformly a.s for $0\leq t\leq 1$ to some random variable Y_t^* ; clearly Y_t^* is $\underline{F}_{a_t}$ - measurable and for a.s ω the sample function $Y_\cdot^*(\omega)$ is right continuous. Because of this, $f(Y_t^n)$ (resp. $g(Y_t^n)$) converges uniformly a.s to $f(Y_t^*)$ (resp. $g(Y_t^*)$). According to THEOREM 10 of [1] , $\int_0^t f(Y_u^n) dN_u$ converges uniformly in probability to $\int_0^t f(Y_u^*) dN_u$, i.e for each $\varepsilon>0$, $\lim_n P\Big(\sup_{0\leq t\leq 1} \Big|\int_0^t f(Y_u^n) dN_u - \int_0^t f(Y_u^*) dN_u\Big| > \varepsilon\Big) = 0$.

Thus, for some subsequence (n_k), we get

$$(6) \qquad \lim_k \sup_{0\leq t\leq 1} \Big|\int_0^t f(Y_u^{n_k}) \, dN_u - \int_0^t f(Y_u^*) \, dN_u\Big| = 0 \quad \text{a.s}$$

As $\int_0^t g(Y_u^n)dV_u$ converges uniformly a.s to $\int_0^t g(Y_u^*)\,dV_u$, we have

$$(7) \qquad Y_t^* = x + \int_0^t f(Y_u^*)\,dN_u + \int_0^t g(Y_u^*)dV_u \ .$$

This completes the proof of existence. We are now going to show its uniqueness. Let (Y_t^1) and (Y_t^2) be solutions of (1*). Then the random variable r defined by

$$r = \inf\,(t;\max_i |Y_t^i| \geqq n) \ .$$

is a stopping time of the family $(\underline{\underline{F}}_{a_t})$. We denote $Y_t^i I_{[t<r]}$ by $\hat{Y}_t^i$. Then for $t \leqq r$ we have

$$\hat{Y}_t^2 - \hat{Y}_t^1 = \int_0^t [f(\hat{Y}_u^2) - f(\hat{Y}_u^1)]dN_u + \int_0^t [g(\hat{Y}_u^2) - g(\hat{Y}_u^1)]dV_u \ .$$

From the definition of r , $\hat{D}(t)=E[(\hat{Y}_t^2-\hat{Y}_t^1)^2] \leqq 4n^2 < \infty$. On the other hand , $\hat{D}(t) \leqq \int_0^t \hat{D}(u)\,du$ as in the proof of existence. Thus $\hat{D}(t) \equiv 0$, and making $n \to \infty$ we obtain the uniqueness statement. Consequently $BY^*= (Y_{b_t}^*, \underline{\underline{F}}_t)$ is the unique solution of the equation (1). Hence the theorem is established.

REFERENCE

[1] N.KAZAMAKI ; Some properties of martingale integrals , Ann.Inst.Henri Poincaré, vol.VII, n°1, 1971.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN