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On the convergence of the regularized entropy-based moment method for kinetic equations

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Abstract. The entropy-based moment method is a well-known discretization for the velocity variable in kinetic equations which has many desirable theoretical properties but is difficult to implement with high-order numerical methods. The regularized entropy-based moment method was recently introduced to remove one of the main challenges in the implementation of the entropy-based moment method, namely the requirement of the realizability of the numerical solution. In this work we use the method of relative entropy to prove the convergence of the regularized method to the original method as the regularization parameter goes to zero and give convergence rates. Our main assumptions are the boundedness of the velocity domain and that the original moment solution is Lipschitz continuous in space and bounded away from the boundary of realizability. We provide results from numerical simulations showing that the convergence rates we prove are optimal.

1. Introduction

Kinetic equations model systems consisting of large numbers of particles that interact with each other or with a background medium and arise in a wide variety of applications including rarefied gas dynamics [7], neutron transport [19], radiative transport [21], and semiconductors [20]. The numerical solution of kinetic equations remains an area of active research. In this work, we consider the entropy-based moment method [18], which is a discretization of the velocity variable in the kinetic equation. It has many desirable theoretical properties but is computationally expensive and challenging to implement.

Recently in [1] a regularized version of the entropy-based moment equations was proposed to simplify the implementation of numerical methods for the entropy-based moment equations. These regularized entropy-based moment equations require the selection of a regularization parameter, and the authors in [1] proposed a rule for selecting the regularization parameter so that the error introduced by the regularization was of the order of the error in the spatiotemporal discretization. With this selection rule the authors produced numerical results which showed that the regularized equations could be used to compute accurate results of the original entropy-based moment equations.

In this work, we prove that exact solutions of the regularized entropy-based moment equations converge to the solutions of the original equations. We quantify the difference between these two solutions using the relative entropy, and the convergence rate is quadratic in the regularization parameter. If

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we assume that the solution of the regularized problem is 'a priori' known to live in some compact subset of the set of realizable states, the error bounds in relative entropy imply linear convergence in the regularization parameter in the L^2 norm. When no such a priori bounds are available, only L^1 convergence can be proven rigorously, see Corollary 2 for details.

In addition to the smoothness of the kinetic entropy function and some technical assumptions on the basis functions, we make the following standing assumptions on the kinetic model:

(i) boundedness of the set of velocities of the kinetic equation, and

(ii) Lipschitz continuity of the function defining the collision term in the original entropy-based moment equations.

We also assume periodic boundary conditions and that the solution of the original entropy-based moment equations is Lipschitz continuous and bounded away from the boundary of the set of realizable moment vectors.

The relative-entropy techniques we use are very similar to what was done in [3, 4, 12, 23]. In general, relative entropy estimates are a widely applicable tool for comparing thermomechanical theories having the form of hyperbolic balance laws that are endowed with a strictly convex entropy [11]. A general limitation of this methodology is that it requires the solution to the limiting system to be Lipschitz continuous - this property can (usually) only be expected for short times since shocks may form. There is recent progress in overcoming this limitation, at least in one space dimension, by using the relative entropy with shifts methodology that was developed by Vasseur and co-workers [17, 22]. However, this condition can most probably not be removed in two or more space dimensions since it is connected to non-uniqueness of entropy solutions to hyperbolic balance laws.

To present our results, we first introduce the entropy-based moment equations and their regularized version in Section 2 and precisely state our assumptions. Then, in Section 3, we introduce the technique of relative entropy and give a general version of our main result. We prove the estimates upon which our main result relies and give the subsets of the realizable set on which they hold in Section 4. Next we present the results of numerical experiments confirming the theoretically predicted rates of convergence in Section 5, and finally we draw conclusions and discuss directions for future work in Section 6.

2. Entropy-based moment equations and regularization

2.1. The kinetic equation

Kinetic equations evolve the *kinetic density function* $f\colon [0,\infty) \times X \times V \to [0,\infty)$ according to

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \mathcal{C}(f(t,x,\cdot))(v) \tag{2.1}$$

(when neglecting long-range interactions). The function f depends on time $t \in [0,\infty)$, position $x \in X \subseteq \mathbb{R}^d$, and a velocity variable $v \in V \subseteq \mathbb{R}^d$. The operator $\mathcal{C}$ introduces the effects of particle collisions; at each x and t, it is an integral operator in v. In order to be well-posed, (2.1) must be accompanied by appropriate initial and boundary conditions.

The results in this work depend strongly on the following assumption.

Assumption 1. *The set of velocities V is bounded.*

We will see the crucial consequences of this assumption in the next subsection. For any $g \in L^1(V)$ we use the notation

$$\langle g \rangle := \int_V g(v)\, dv. \tag{2.2}$$

We define

$$|V| := \langle 1 \rangle = \int_V dv\,. \tag{2.3}$$

Many of the constants below depend on $\max_{v\in V} \|v\|$, though for clarity of exposition we do not give this dependence explicitly.

We consider kinetic equations where the collision operator satisfies an entropy-dissipation property: Let $\eta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, be a strictly convex function and $\mathbb{F}(V) := \{g \in L^1(V) : \text{Range}(g) \subseteq D\}$. We call η the *kinetic entropy function.* Then the local entropy $\mathcal{H} : \mathbb{F}(V) \to \mathbb{R}$ is given by

$$\mathcal{H}(f) := \langle \eta(f) \rangle\,. \tag{2.4}$$

This entropy is dissipated if the collision operator $\mathcal{C}$ satisfies

$$\langle \eta'(g)\mathcal{C}(g) \rangle \le 0 \tag{2.5}$$

for all $g : V \to D$ such that the integral is defined. Furthermore, we assume that η is sufficiently smooth:

Assumption 2. *The kinetic entropy satisfies $\eta \in C^3(D)$, $\eta'' > 0$ on D, and* (2.5)*, i.e., η is an entropy dissipated by the kinetic equation* (2.1)*.*

2.2. The original entropy-based moment equations and realizability

The original entropy-based moment equations are a semidiscretization of the kinetic equation (2.1) in the velocity variable v. For an overview, see [18]. The velocity dependence of f at each point in time and space is replaced by the vector of moments

$$\mathbf{u}(t,x) := (u_0(t,x), u_1(t,x), \dots, u_N(t,x)) \in \mathbb{R}^{N+1}, \tag{2.6}$$

which contains the approximations of velocity integrals of f multiplied by the basis functions

$$\mathbf{m}(v) := (m_0(v), m_1(v), \dots, m_N(v)), \tag{2.7}$$

that is, $u_i(t,x) \simeq \langle m_i f(t,x,\cdot) \rangle$ for all $i \in \{0,1,\dots,N\}$. Usually the basis functions are polynomials. We make the following assumptions on the basis functions:

Assumption 3.

(i) *For every $i \in \{0,1,\dots,N\}$ we have $m_i \in L^\infty(V)$. Without loss of generality, we assume that $\|m_i\|_{L^\infty(V)}$ is bounded by one:*

$$\|m_i\|_{L^\infty(V)} = \sup_{v\in V} |m_i(v)| \le 1 \qquad \text{for } i \in \{0,1,\dots,N\}. \tag{2.8}$$

(ii) *The constant function is in the linear span of the basis functions. Without loss of generality we assume $m_0(v) \equiv 1$.*

For each moment vector, the entropy-based moment method reconstructs an ansatz for the kinetic density by solving the constrained optimization problem

$$\underset{g\in\mathbb{F}(V)}{\text{minimize}}\ \mathcal{H}(g) \qquad \text{subject to } \langle \mathbf{m} g \rangle = \mathbf{u} \tag{2.9}$$

(recall the definition of $\mathcal{H}$ in (2.4)). Under Assumption 1, this problem has a unique solution for every $\mathbf{u} \in \mathcal{R}$ [5, 16], where

$$\mathcal{R} := \{\mathbf{u} \,:\, \text{there exists a } g \in \mathbb{F}(V) \text{ such that } \langle \mathbf{m} g \rangle = \mathbf{u}\} \tag{2.10}$$

is the set of all *realizable* moment vectors, and consequently, the system of moment equations is well-defined and hyperbolic for all realizable moment vectors. [1]

The solution to (2.9) takes the form $G_{\hat{\boldsymbol{\alpha}}(\mathbf{u})}$, where

$$G_{\boldsymbol{\alpha}} := \eta_*'(\boldsymbol{\alpha} \cdot \mathbf{m}) \tag{2.11}$$

and $\hat{\boldsymbol{\alpha}} : \mathcal{R} \to \mathbb{R}^{N+1}$ maps a moment vector $\mathbf{u}$ to the solution of the dual problem

$$\hat{\boldsymbol{\alpha}}(\mathbf{u}) = \underset{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}}{\operatorname{argmax}} \left\{ \boldsymbol{\alpha} \cdot \mathbf{u} - \langle \eta_*(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle \right\}; \tag{2.12}$$

here η_* is the Legendre dual [2] of η (and thus η_*' is the inverse function of η'). The components of $\boldsymbol{\alpha}$ are the Lagrange multipliers for the primal problem (2.9).

Now we are ready to give the entropy-based moment equations:

$$\partial_t \mathbf{u} + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = \mathbf{r}(\mathbf{u}), \tag{2.13a}$$

where the flux function $\mathbf{f}$ and relaxation term $\mathbf{r}$ are given by

$$\mathbf{f}(\mathbf{u}) := \left\langle v \mathbf{m} G_{\hat{\boldsymbol{\alpha}}(\mathbf{u})} \right\rangle \qquad \text{and} \qquad \mathbf{r}(\mathbf{u}) := \left\langle \mathbf{m} \mathcal{C}(G_{\hat{\boldsymbol{\alpha}}(\mathbf{u})}) \right\rangle. \tag{2.13b}$$

Classical solutions of the moment equations (2.13) satisfy the entropy dissipation law

$$\partial_t h(\mathbf{u}) + \nabla_x \cdot j(\mathbf{u}) = h'(\mathbf{u}) \cdot \mathbf{r}(\mathbf{u}) \le 0 \tag{2.14}$$

for the entropy and entropy flux

$$h(\mathbf{u}) := \left\langle \eta(G_{\hat{\boldsymbol{\alpha}}(\mathbf{u})}) \right\rangle \quad \text{and} \quad j(\mathbf{u}) := \left\langle v \eta(G_{\hat{\boldsymbol{\alpha}}(\mathbf{u})}) \right\rangle. \tag{2.15}$$

Note that $h' = \hat{\boldsymbol{\alpha}}$, and that h is strictly convex as a consequence of Assumption 2 [14]. For readers unfamiliar with the derivations of the dual problem and the entropy dissipation law and related properties of the entropy-based moment equations, we review these in Appendix B.

A consequence of Assumption 1 is that $\hat{\boldsymbol{\alpha}}$ is a smooth bijection from $\mathcal{R}$ to $\mathbb{R}^{N+1}$. Its inverse is the function which gives the moment vector of the entropy ansatz corresponding to a given multiplier vector:

$$\hat{\mathbf{u}}(\boldsymbol{\alpha}) := \langle \mathbf{m} \eta_*'(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle. \tag{2.16}$$

This function plays a role in the analysis later.

We make the following assumption on the collision term:

Assumption 4. *The function $\mathbf{r}$ in the collision term of the original entropy-based moment equations* (2.13) *is Lipschitz continuous with constant $C_{\mathbf{r}}$ and satisfies* $\lim_{\mathbf{u} \to 0} \mathbf{r}(\mathbf{u}) = 0$.

This applies, for example, to linear collision operators like in the case of isotropic scattering. This assumption cannot be expected to apply for the Boltzmann collision operator, at least not globally, since it is quadratic. The assumption $\lim_{\mathbf{u} \to 0} \mathbf{r}(\mathbf{u}) = 0$ is natural since otherwise there would be collision effects in the moment equations when no particles are present.

Finally, we note that the flux $\mathbf{f}$ and the source $\mathbf{r}$ in (2.13) can only be defined when the optimization problem (2.9) is feasible, i.e., when the $\mathbf{u} \in \mathcal{R}$. When $D = [0, \infty)$, this set corresponds to the set of vectors which contain the moments of the nonnegative density functions, which are indeed the only moment vectors we want to consider since the kinetic density f should be nonnegative. However, it is in general difficult to guarantee that numerical solutions (particularly high-order ones) of (2.13) remain within the realizable set $\mathcal{R}$.

[1] When V is not bounded, such as in entropy-based moment equations for the Boltzmann equation for rarefied gas dynamics, where $V = \mathbb{R}^3$, there are important examples of realizable moment vectors for which the primal problem has no solution; see [14, 15, 16]. This is a significant open problem for entropy-based moment equations, and neither the regularization nor our work here can get around this issue.

[2] See, e.g., [10, §3.3.2] or [6, §3.3], where what we call the Legendre dual is called the conjugate function.

2.3. **The regularized entropy-based moment equations**

To work around the problem of realizability, the regularized entropy-based moment equations were proposed in [1]. Let $\gamma \in (0, \infty)$ be the regularization parameter. Then the regularized entropy-based moment equations are given by

$$\partial_t \mathbf{u} + \nabla_x \cdot \mathbf{f}_\gamma(\mathbf{u}) = \mathbf{r}_\gamma(\mathbf{u}), \tag{2.17a}$$

where

$$\mathbf{f}_\gamma(\mathbf{u}) := \left\langle v \mathbf{m} G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})} \right\rangle \qquad \text{and} \qquad \mathbf{r}_\gamma(\mathbf{u}) := \left\langle \mathbf{m} \mathcal{C}(G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})}) \right\rangle. \tag{2.17b}$$

The ansatz $G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})}$ has the same form as above (i.e., $G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})} = \eta_*'(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot \mathbf{m})$) but is the solution of the unconstrained optimization problem

$$\underset{g \in \mathbb{F}(V)}{\text{minimize}} \ \langle \eta(g) \rangle + \frac{1}{2\gamma} \left\| \langle \mathbf{m} g \rangle - \mathbf{u} \right\|^2, \tag{2.18}$$

which is feasible for any moment vector $\mathbf{u} \in \mathbb{R}^{N+1}$ (again under Assumption 1). The new multiplier vector $\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})$ is the solution of the corresponding dual problem

$$\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) := \underset{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}}{\operatorname{argmax}} \left\{ \boldsymbol{\alpha} \cdot \mathbf{u} - \langle \eta_*(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle - \frac{\gamma}{2} \|\boldsymbol{\alpha}\|^2 \right\} \tag{2.19}$$

and satisfies the first-order necessary conditions

$$\mathbf{u} = \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) + \gamma \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}). \tag{2.20}$$

Classical solutions of the regularized equations (2.17) satisfy

$$\partial_t h_\gamma(\mathbf{u}) + \nabla_x \cdot j_\gamma(\mathbf{u}) = h_\gamma'(\mathbf{u}) \cdot \mathbf{r}_\gamma(\mathbf{u}) \le 0,$$

where

$$h_\gamma(\mathbf{u}) := \left\langle \eta(G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})}) \right\rangle + \frac{1}{2\gamma} \left\| \left\langle \mathbf{m} G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})} \right\rangle - \mathbf{u} \right\|^2 \qquad \text{and} \qquad j_\gamma(\mathbf{u}) := \left\langle v \eta(G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})}) \right\rangle. \tag{2.21}$$

Analogously to the original case, we have $h_\gamma' = \hat{\boldsymbol{\alpha}}_\gamma$. In this work, we are mostly concerned with entropy solutions of the regularized equations, i.e. weak solutions of (2.17) satisfying the admissibility criterion

$$\partial_t h_\gamma(\mathbf{u}) + \nabla_x \cdot j_\gamma(\mathbf{u}) \le h_\gamma'(\mathbf{u}) \cdot \mathbf{r}_\gamma(\mathbf{u}). \tag{2.22}$$

Note that any Lipschitz continuous solution of (2.17) is automatically an entropy solution.

Like h, the entropy h_γ of the regularized equations is convex, and its Legendre dual $(h_\gamma)_*$ will prove to be useful in the analysis. The Legendre dual $(h_\gamma)_*$ and its first and second derivatives are given by [1]

$$(h_\gamma)_*(\boldsymbol{\alpha}) = \langle \eta_*(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle + \frac{\gamma}{2} \|\boldsymbol{\alpha}\|^2, \tag{2.23a}$$

$$(h_\gamma)_*'(\boldsymbol{\alpha}) = \langle \mathbf{m} \eta_*'(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle + \gamma \boldsymbol{\alpha}, \text{ and} \tag{2.23b}$$

$$(h_\gamma)_*''(\boldsymbol{\alpha}) = \left\langle \mathbf{m} \mathbf{m}^T \eta_*''(\boldsymbol{\alpha} \cdot \mathbf{m}) \right\rangle + \gamma I, \tag{2.23c}$$

where I is the $(N+1) \times (N+1)$ identity matrix. Note that $(h_\gamma)_*' \circ h_\gamma' = (h_\gamma)_*' \circ \hat{\boldsymbol{\alpha}}_\gamma = \mathrm{id}$, so we also have $h_\gamma'' = ((h_\gamma)_*'' \circ \hat{\boldsymbol{\alpha}}_\gamma)^{-1}$, where the inverse indicates the matrix inverse. One also immediately recognizes from the form of $(h_\gamma)_*''$ that $(h_\gamma)_*$ and thus h_γ are strictly convex for any $\gamma > 0$.

3. Relative entropy for convergence

Let $X \subset \mathbb{R}^d$ be a d-cube and $T \in (0, \infty)$. We consider the initial-value problem

$$\partial_t \mathbf{u} + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = \mathbf{r}(\mathbf{u}) \qquad (t,x) \in (0,T] \times X, \tag{3.1a}$$

$$\mathbf{u}(0,x) = \mathbf{u}^0(x) \qquad x \in X, \tag{3.1b}$$

where $\mathbf{u}^0$ are the given initial conditions, and we use periodic boundary conditions in space. For any $\gamma \in (0,\infty)$, the regularized moment equations are

$$\partial_t \mathbf{u}_\gamma + \nabla_x \cdot \mathbf{f}_\gamma(\mathbf{u}_\gamma) = \mathbf{r}_\gamma(\mathbf{u}_\gamma) \qquad (t,x) \in (0,T] \times X, \tag{3.2a}$$

$$\mathbf{u}_\gamma(0,x) = \mathbf{u}^0(x) \qquad x \in X, \tag{3.2b}$$

and we also take periodic boundary conditions. Notice that the initial conditions are the same.

Let us briefly recall the notions of weak solution and entropy solution for (3.1), those for (3.2) are analogous.

Definition 3.1. Let $\mathbf{u}^0(x) \in L^1(X, \mathbb{R}^{N+1})$ then, we call $\mathbf{u} \in L^\infty(0,T; L^1(X, \mathbb{R}^{N+1}))$ a weak solution of (3.1) if is satisfies $\mathbf{f}(\mathbf{u}) \in L^1((0,T) \times X, \mathbb{R}^{N+1})$ and $\mathbf{r}(\mathbf{u}) \in L^1((0,T) \times X, \mathbb{R}^{N+1})$

$$\int_0^T \int_X \mathbf{u}\partial_t \Phi + \mathbf{f}(\mathbf{u})\nabla\Phi + \mathbf{r}(\mathbf{u})\Phi \, dx\, ds + \int_X \mathbf{u}^0 \Phi(0,\cdot\,)\, dx = 0$$

for all $\Phi \in C_0^\infty([0,T), C_{per}^\infty(X, \mathbb{R}^{N+1}))$. We call a weak solution of (3.1) an entropy solution with respect to the pair (h,j) provided $h(\mathbf{u}) \in L^\infty(0,T; L^1(X))$, and $j(\mathbf{u}), h'(\mathbf{u})\mathbf{r}(\mathbf{u}) \in L^1((0,T) \times X)$ and

$$\int_0^T \int_X h(\mathbf{u})\partial_t \phi + j(\mathbf{u})\nabla\phi + h'(\mathbf{u})\mathbf{r}(\mathbf{u})\phi \, dx\, ds + \int_X h(\mathbf{u}^0)\phi(0,\cdot\,)\, dx \geq 0$$

for all $\phi \in C_0^\infty([0,T), C_{per}^\infty(X, [0,\infty)))$.

Following Dafermos [9], we introduce the relative entropy and relative entropy flux relative to h_γ:

Definition 3.2. Given moments $\mathbf{u}_\gamma$, $\mathbf{u} \in \mathbb{R}^{N+1}$ the *relative entropy* and *relative entropy flux* are given by

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) := h_\gamma(\mathbf{u}_\gamma) - h_\gamma(\mathbf{u}) - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\mathbf{u}_\gamma - \mathbf{u}) \text{ and} \tag{3.3}$$

$$j_\gamma(\mathbf{u}_\gamma|\mathbf{u}) := j_\gamma(\mathbf{u}_\gamma) - j_\gamma(\mathbf{u}) - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\mathbf{f}_\gamma(\mathbf{u}_\gamma) - \mathbf{f}_\gamma(\mathbf{u})), \tag{3.4}$$

respectively.

Note that using the relative entropy with respect to h_γ allows us (in principle) to use non-realizable moment vectors in both arguments of the relative entropy. For our subsequent convergence result, however, there will be a strong difference between the first and the second slot of the relative entropy. In particular, in our convergence results, Corollaries 1 and 3, we will require the function in the second slot of the relative entropy to have values only in some compact subset of the set of realizable vectors.

Lemma 3.3. *Let* $\mathbf{u}$ *be a Lipschitz continuous solution of* (3.1) *and let* $\mathbf{u}_\gamma$ *be an entropy solution of* (3.2)*, see Definition 3.1. Then, for almost all* $0 \leq t \leq T$ *the following inequality holds:*

$$\int_X h_\gamma(\mathbf{u}_\gamma(t,x)|\mathbf{u}(t,x))\, dx \leq -\int_0^t \int_X (\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})(s,x)) : \mathbf{f}_\gamma(\mathbf{u}_\gamma(s,x)|\mathbf{u}(s,x)) \\ - q_\gamma(\mathbf{u}_\gamma(s,x), \mathbf{u}(s,x)) - J_\gamma(\mathbf{u}_\gamma(s,x), \mathbf{u}(s,x)) : \nabla_x \mathbf{u}(s,x)\, dx\, ds \tag{3.5}$$

with

$$\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u}) := \mathbf{f}_\gamma(\mathbf{u}_\gamma) - \mathbf{f}_\gamma(\mathbf{u}) - \mathbf{f}'_\gamma(\mathbf{u})(\mathbf{u}_\gamma - \mathbf{u}) \tag{3.6}$$

and

$$q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) := (h'_\gamma(\mathbf{u}_\gamma) - h'_\gamma(\mathbf{u})) \cdot \mathbf{r}_\gamma(\mathbf{u}_\gamma) - \mathbf{r}(\mathbf{u}) \cdot (h''_\gamma(\mathbf{u})(\mathbf{u}_\gamma - \mathbf{u})) \tag{3.7}$$

and

$$J_\gamma(\mathbf{u}_\gamma, \mathbf{u}) := (\mathbf{u}_\gamma - \mathbf{u}) h''_\gamma(\mathbf{u})(\mathbf{f}'(\mathbf{u}) - \mathbf{f}'_\gamma(\mathbf{u})) \tag{3.8}$$

Proof. The proof follows the proof of [9, Thm. 5.2.1]. We need, however, to account for the difference of flux functions in (3.2) and (3.1). Therefore, we provide a brief proof that is to be understood in the dual space of $W_0^{1,\infty}([0,T) \times X, [0,\infty))$, i.e. nonnegative Lipschitz continuous functions with compact support.

$$\begin{aligned}
&\partial_t h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + \nabla_x \cdot j_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \\
&= \partial_t h_\gamma(\mathbf{u}_\gamma) + \nabla_x \cdot j_\gamma(\mathbf{u}_\gamma) - h'_\gamma(\mathbf{u}) \cdot \partial_t \mathbf{u} - j'_\gamma(\mathbf{u}) : \nabla_x \mathbf{u} - (h''_\gamma(\mathbf{u})\partial_t \mathbf{u}) \cdot (\mathbf{u}_\gamma - \mathbf{u}) \\
&\quad - (h''_\gamma(\mathbf{u}) \cdot \nabla_x \mathbf{u}) \cdot (\mathbf{f}_\gamma(\mathbf{u}_\gamma) - \mathbf{f}_\gamma(\mathbf{u})) - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\partial_t \mathbf{u}_\gamma + \nabla_x \cdot \mathbf{f}_\gamma(\mathbf{u}_\gamma)) \\
&\quad + \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\partial_t \mathbf{u} + \mathbf{f}'_\gamma(\mathbf{u}) : \nabla_x \mathbf{u}) && \text{(3.9a)} \\
&\le h'_\gamma(\mathbf{u}_\gamma) r_\gamma(\mathbf{u}_\gamma) - h''_\gamma(\mathbf{u})(-\mathbf{f}'(\mathbf{u}) : \nabla_x \mathbf{u} + \mathbf{r}(\mathbf{u})) \cdot (\mathbf{u}_\gamma - \mathbf{u}) \\
&\quad - (h''_\gamma(\mathbf{u}) \cdot \nabla_x \mathbf{u}) \cdot (\mathbf{f}_\gamma(\mathbf{u}_\gamma) - \mathbf{f}_\gamma(\mathbf{u})) - h'_\gamma(\mathbf{u}) \mathbf{r}_\gamma(\mathbf{u}_\gamma) && \text{(3.9b)} \\
&= -(\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) : \mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) + J_\gamma(\mathbf{u}_\gamma, \mathbf{u}) : \nabla_x \mathbf{u}, && \text{(3.9c)}
\end{aligned}$$

where in (3.9b) we have used

$$\begin{aligned}
&\partial_t h_\gamma(\mathbf{u}_\gamma) + \nabla_x \cdot j_\gamma(\mathbf{u}_\gamma) \le h'_\gamma(\mathbf{u}_\gamma) \cdot \mathbf{r}_\gamma(\mathbf{u}_\gamma), && \text{(3.10a)} \\
&- h'_\gamma(\mathbf{u}) \cdot \partial_t \mathbf{u} - j'_\gamma(\mathbf{u}) : \nabla_x \mathbf{u} + \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\partial_t \mathbf{u} + \mathbf{f}'_\gamma(\mathbf{u}) : \nabla_x \mathbf{u}) = 0, && \text{(3.10b)} \\
&\partial_t \mathbf{u} = -\mathbf{f}'(\mathbf{u}) : \nabla_x \mathbf{u} + \mathbf{r}(\mathbf{u}), \text{ and} && \text{(3.10c)} \\
&\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\partial_t \mathbf{u}_\gamma + \nabla_x \cdot \mathbf{f}_\gamma(\mathbf{u}_\gamma)) = h'_\gamma(\mathbf{u}) \mathbf{r}_\gamma(\mathbf{u}_\gamma). && \text{(3.10d)}
\end{aligned}$$

where the equation (3.10b) expresses the fact that Lipschitz continuous solutions satisfy the entropy inequality as an equality. We have also used the commutation property

$$h''_\gamma \mathbf{f}'_\gamma = (\mathbf{f}'_\gamma)^T h''_\gamma$$

that follows by taking the derivative of the compatibility relation $h'_\gamma \mathbf{f}'_\gamma = j'_\gamma$ [1].
Now we fix $0 \le t \le T$ and for $\varepsilon > 0$ we test

$$\partial_t h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + \nabla_x \cdot j_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \le -(\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) : \mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) + J_\gamma(\mathbf{u}_\gamma, \mathbf{u}) : \nabla_x \mathbf{u},$$

with $\psi_\varepsilon \in W_0^{1,\infty}([0,T) \times X, [0,\infty))$

$$\psi_\varepsilon(s,x) := \begin{cases} 1 & s < t \\ 1 - \frac{s-t}{\varepsilon} & t < s < t + \varepsilon \\ 0 & s > t + \varepsilon \end{cases}$$

Sending ε to zero, we obtain (3.5) for all t that are Lebesgue points of the map $[0,T) \to \mathbb{R}$, $t \mapsto \int_X h_\gamma(\mathbf{u}_\gamma(t,x)|\mathbf{u}(t,x))\,dx$. Note that $\int_X h_\gamma(\mathbf{u}_\gamma(0,x)|\mathbf{u}(0,x))\,dx = 0$ since $\mathbf{u}$, $\mathbf{u}_\gamma$ satisfy the same initial condition. ∎

At this point, with relative-entropy methods it is typical to look to bound the integrand of the right-hand side using $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ so that Grönwall's inequality can be used.

It turns out that such bounds can be obtained if we make some assumption on the original solution $\mathbf{u}$ which bounds it away from the boundary of the realizable set $\mathcal{R}$. The exact form this assumption takes depends on the entropy; we will make this precise in Section 4.

Theorem 3.4. *Let there be a subset $\mathcal{S}$ of the set of realizable vectors $\mathcal{R}$ such that there exists $\gamma_0 > 0$ and constants $C_J, C_\mathbf{f}, C_q > 0$ and $D_J, D_\mathbf{f}, D_q > 0$ so that*

$$\begin{aligned} \|J_\gamma(\mathbf{u}_\gamma, \mathbf{u})\| &\leq C_J h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_J\gamma^2 \\ \|\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})\| &\leq C_\mathbf{f} h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_\mathbf{f}\gamma^2 \qquad \forall \mathbf{u}_\gamma \in \mathbb{R}^{N+1}, \mathbf{u} \in \mathcal{S}, \gamma \in (0, \gamma_0). \\ q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) &\leq C_q h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_q\gamma^2 \end{aligned} \tag{3.11}$$

Let $\mathbf{u}$ be a Lipschitz continuous solution of (3.1)*, i.e. there exists $C_u > 0$ such that $\|\nabla_x \mathbf{u}\|_{L^\infty([0,T]\times X)} \leq C_u$, satisfying $\mathbf{u}(t, x) \in \mathcal{S}$ for all $(t, x) \in [0, T] \times X$. Let $\{\mathbf{u}_\gamma\}_{\gamma\in(0,\gamma_0)}$ be a family of entropy solutions of* (3.2)*. Furthermore assume that there is a $C_{\hat{\boldsymbol{\alpha}}} > 0$ such that $\|\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|_{L^\infty([0,T]\times X)} \leq C_{\hat{\boldsymbol{\alpha}}}$ uniformly in γ. Then for γ sufficiently small,*

$$\int_X h_\gamma(\mathbf{u}_\gamma(T, x)|\mathbf{u}(T, x))\, dx \leq \exp(CT)DT\gamma^2, \tag{3.12}$$

where $C := C_{\hat{\boldsymbol{\alpha}}} C_\mathbf{f} + C_u C_J + C_q$ and $D := C_{\hat{\boldsymbol{\alpha}}} D_\mathbf{f} + C_u D_J + D_q$.

Proof. Inserting (3.11) into (3.5) implies

$$\begin{aligned} &\int_X h_\gamma(\mathbf{u}_\gamma(t, x)|\mathbf{u}(t, x))\, dx \\ &\qquad \leq \int_0^t \int_X (C_{\hat{\boldsymbol{\alpha}}} C_\mathbf{f} + C_u C_J + C_q) h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + (C_{\hat{\boldsymbol{\alpha}}} D_\mathbf{f} + C_u D_J + D_q)\gamma^2\, dx\, ds. \end{aligned} \tag{3.13}$$

The assertion of the Theorem follows by applying Grönwall's lemma. ■

Remark 3.5. It is important to note that we need to bound the norms of $\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ and $J_\gamma(\mathbf{u}_\gamma, \mathbf{u})$ while it is sufficient to bound $q_\gamma(\mathbf{u}_\gamma, \mathbf{u})$ from above (no bound from below is needed). This is due to the fact that, in (3.5), $\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ and $J_\gamma(\mathbf{u}_\gamma, \mathbf{u})$ are multiplied by $\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})$ and $\nabla_x \mathbf{u}$, respectively, whose directions are unknown.

Remark 3.6. We will work with sets $\mathcal{S}$ such that the derivative $\hat{\boldsymbol{\alpha}}'_\gamma(\mathbf{u}) = ((h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})))^{-1}$ is uniformly bounded for $\mathbf{u} \in \mathcal{S}$ and $\gamma \in (0, \gamma_0)$. Thus $\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \in L^\infty([0, T] \times X)$ follows from $\nabla_x \mathbf{u} \in L^\infty([0, T] \times X)$ and the chain rule, and $\|\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|_{L^\infty([0,T]\times X)}$ is bounded as $\gamma \to 0$. Note also that the fact that there exists C_u such that $\|\nabla_x \mathbf{u}\|_{L^\infty([0,T]\times X)} \leq C_u$ follows from the assumption that $\mathbf{u}$ is Lipschitz continuos.

4. Estimates

In this section we give sets $\mathcal{S}$ over which the flux and source terms can be bounded as required in Theorem 3.4.

First, we give some basic properties of realizable moment vectors and the functions $\hat{\mathbf{u}}$, $\mathbf{f}$, and their regularized counterparts which will be used repeatedly when estimating the flux and source terms.

4.1. Basic properties

For basis functions satisfying Assumption 3, the components of any realizable moment vector $\mathbf{u} \in \mathcal{R}$ satisfy

$$|u_i| = |\langle m_i g\rangle| \leq \langle g\rangle = u_0, \qquad \text{for all } i \in \{0, 1, \ldots, N\}, \tag{4.1}$$

thus there exists a constant $C_0 \in (0, \infty)$ such that

$$\|\mathbf{u}\| \leq C_0 u_0 \qquad \text{for all } \mathbf{u} \in \mathcal{R}. \tag{4.2}$$

Under Assumption 1 there is a $C_1 \in (0, \infty)$ such that

$$\|\mathbf{f}'(\mathbf{u})\| \le C_1 \tag{4.3}$$

for all $\mathbf{u} \in \mathcal{R}$ [2, Lem. 3.1], so $\mathbf{f}$ is globally Lipschitz continuous on $\overline{\mathcal{R}}$. The same bound holds globally for the regularized flux, i.e.,

$$\|\mathbf{f}'_\gamma(\mathbf{u})\| \le C_1 \tag{4.4}$$

for all $\mathbf{u} \in \mathbb{R}^{N+1}$ (this also follows from the argument used in [2, Lem. 3.1]). Since $\mathbf{f}(0) = 0$ (in the limit sense, since vector 0 lies on the boundary of $\mathcal{R}$), we have

$$\|\mathbf{f}(\mathbf{u})\| \le C_1\|\mathbf{u}\|. \tag{4.5}$$

The map $\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma$ plays an important role in our analysis. This map returns the realizable moment vector that the regularization uses to compute the flux and source terms. It is globally Lipschitz continuous. [3] If we let C_2 be its Lipschitz constant, then in particular

$$\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))\| \le C_2\|\mathbf{w}\| + \|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(0))\| \tag{4.7}$$

In all cases we consider, $\lim_{\gamma\to 0} \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(0)) = 0$ (see Appendix C), so we define

$$C_3 := \sup_{\gamma\in(0,\gamma_0)} \|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(0))\| \tag{4.8}$$

and generally use

$$\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))\| \le C_2\|\mathbf{w}\| + C_3 \tag{4.9}$$

for appropriate values of γ. Furthermore, since $\mathbf{f}_\gamma = \mathbf{f} \circ \hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma$,[4] we can combine this bound with (4.5) to get

$$\|\mathbf{f}_\gamma(\mathbf{w})\| \le C_1\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))\| \le C_1(C_2\|\mathbf{w}\| + C_3). \tag{4.11}$$

We conclude this section by recalling a handy property of the regularized problem. The partial derivative of $\hat{\boldsymbol{\alpha}}_\gamma$ with respect to γ was computed in [1, Thm. 3]:

$$\frac{\partial}{\partial\gamma}\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}) = -h''_\gamma(\mathbf{w})\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}). \tag{4.12}$$

From the positive-definiteness of h''_γ we can immediately conclude that $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|^2$ is a decreasing function of γ for any $\mathbf{w} \in \mathbb{R}^{N+1}$. We can further conclude that $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\| \to 0$ as $\gamma \to \infty$: Since $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|$ is a decreasing function of γ, it remains bounded for $\gamma \in (\gamma_0, \infty)$, where $\gamma_0 \in (0, \infty)$. Thus the moment vector $\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))$ associated with the ansatz $G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})}$ is also bounded for $\gamma \in (\gamma_0, \infty)$, since $\hat{\mathbf{u}}$ is a continuous map. Let

$$u_{\gamma_0,\mathbf{w}} := \sup_{\gamma\in(\gamma_0,\infty)} \|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))\|. \tag{4.13}$$

---

[3] Indeed, the singular values of its Jacobian are bounded by one: Let $H := \langle \mathbf{m}\mathbf{m}^T \eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot \mathbf{m})\rangle$, and recall that it is symmetric positive definite. Then $(\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u}) = H(H + \gamma I)^{-1}$. For any eigenvalue-eigenvector pair $(\lambda, \mathbf{c})$ of H, we have

$$(H + \gamma I)^{-1} H^2 (H + \gamma I)^{-1}\mathbf{c} = \left(\frac{\lambda}{\lambda + \gamma}\right)^2 \mathbf{c}, \tag{4.6}$$

from which we concluded that the singular values have the form $\lambda/(\lambda + \gamma)$ and thus are bounded by one.

[4] To see this, start from (2.17b) and apply the fact that $\hat{\boldsymbol{\alpha}}$ is the inverse function of $\hat{\mathbf{u}}$ to get

$$\mathbf{f}_\gamma(\mathbf{u}) = \left\langle v\mathbf{m}G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})}\right\rangle = \left\langle v\mathbf{m}G_{\hat{\boldsymbol{\alpha}}(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})))}\right\rangle = \mathbf{f}(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))); \tag{4.10}$$

here we have used $\hat{\boldsymbol{\alpha}} \circ \hat{\mathbf{u}} = \mathrm{id}$, and the last equality comes from the definition of $\mathbf{f}(\mathbf{u})$ in (2.13b).

Then by rearranging the first-order necessary conditions (2.20) we have

$$\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\| = \frac{\|\mathbf{w} - \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))\|}{\gamma} \le \frac{\|\mathbf{w}\| + u_{\gamma_0,\mathbf{w}}}{\gamma}, \tag{4.14}$$

from which we conclude $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\| \to 0$ as $\gamma \to \infty$.

We can also use (4.12) to show that the entropy h_γ is a decreasing function of γ. Using the formula $h_\gamma(\mathbf{w}) = h(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w}))) + \frac{\gamma}{2}\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|^2$ (obtained by inserting (2.15) and (2.20) into (2.21)), we have, after some basic manipulations,

$$\frac{\partial}{\partial\gamma} h_\gamma(\mathbf{w}) = -\frac{1}{2}\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|^2 \le 0. \tag{4.15}$$

4.2. **Two classes of kinetic entropy functions**

To prove the estimates (3.11), we need to assume that the true solution $\mathbf{u}$ is bounded away from the boundary of realizability. This can be achieved in various ways, but the specific form of the assumption must depend on properties of the kinetic entropy function η.

We consider two kinds of kinetic entropy functions. The first is defined with the Maxwell–Boltzmann entropy,

$$\eta(z) = z\log(z) - z, \tag{4.16}$$

in mind. (The second term is purely for mathematical convenience.)

Definition 4.1. Let $\eta : (0, \infty) \to \mathbb{R}$ satisfy Assumption 2. We call η *superlinear* if

$$\mathrm{Range}(\eta') = \mathrm{Dom}(\eta_*) = \mathbb{R}.$$

Since η' is an increasing function, these entropies grow superlinearly as $z \to \infty$. Furthermore, since $0 \in \mathrm{Range}(\eta')$, the entropy η has a global minimum.

Remark 4.2. Note that we use the term *superlinear* even though the functions $\eta(z) = z^\alpha$ for $\alpha \in (1, \infty)$ do not belong to our class of superlinear entropy functions.

The second kind of entropy we consider includes the Bose–Einstein entropy,

$$\eta(z) = z\log(z) - (1+z)\log(1+z), \tag{4.17}$$

and the Burg entropy,

$$\eta(z) = -\log(z). \tag{4.18}$$

Definition 4.3. Let $\eta : (0, \infty) \to \mathbb{R}$ satisfy Assumption 2. We call η *sublinear* if

$$\mathrm{Range}(\eta') = \mathrm{Dom}(\eta_*) = (-\infty, 0)$$

with $\lim_{z\to\infty} \eta'(z) = 0$ and $\lim_{z\to 0} \eta'(z) = -\infty$.

These entropies are monotonically decreasing functions with no global minimum. The decay as $z \to \infty$ is sublinear.

Note that we have assumed $\mathrm{Dom}(\eta) = (0, \infty)$ in both definitions. Since for Lagrange dual functions $\mathrm{Range}(\eta'_*) = \mathrm{Dom}(\eta')$, it follows that the ansätze $G_{\boldsymbol{\alpha}}$ for both superlinear and sublinear kinetic entropy functions take only positive values.

4.3. **The superlinear case**

For superlinear entropies we consider the family of sets

$$\mathcal{R}^M := \{\hat{\mathbf{u}}(\boldsymbol{\alpha}) : \|\boldsymbol{\alpha}\| \le M\}, \tag{4.19}$$

for $M \in (0,\infty)$. For each M, we have $\mathcal{R}^M \subset\subset \mathcal{R}$, i.e., $\mathcal{R}^M$ is a compact subset of $\mathcal{R}$, and as $M \to \infty$, the set $\mathcal{R}^M$ approaches the full realizable set $\mathcal{R}$ (since under Assumption 1 we have $\mathcal{R} = \hat{\mathbf{u}}(\mathbb{R}^{N+1})$). We will show that the assumptions of Theorem 3.4 hold for superlinear entropies when $\mathcal{S} = \mathcal{R}^M$ for any M.

First we give some properties on $\mathcal{R}^M$. Since $\mathcal{R}^M$ is a compact set, both

$$u_M := \sup_{\mathbf{w}\in\mathcal{R}^M} \|\mathbf{w}\| \qquad \text{and} \qquad h_M := \sup_{\mathbf{w}\in\mathcal{R}^M} h(\mathbf{w}) \tag{4.20}$$

are finite. Since h_γ is a decreasing function of γ (recall (4.15)), we have

$$h_M = \sup_{\substack{\mathbf{w}\in\mathcal{R}^M \\ \gamma\in(0,\infty)}} h_\gamma(\mathbf{w}). \tag{4.21}$$

Another important consequence of restricting $\mathbf{u}$ to $\mathcal{R}^M$ is that h''_γ is bounded from above and below over $\mathcal{R}^M$. First recall that $h''_\gamma = ((h_\gamma)''_* \circ \hat{\boldsymbol{\alpha}}_\gamma)^{-1}$, so we work with $(h_\gamma)''_*$. Let $\mathbf{c}$ be the unit-length eigenvector associated with the largest eigenvalue of $(h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))$ for some $\mathbf{u} \in \mathcal{R}^M$. Then we have

$$\begin{aligned}
\lambda_{\max}((h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) &= \mathbf{c}\cdot\left(\left\langle \mathbf{m}\mathbf{m}^T \eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\cdot\mathbf{m})\right\rangle + \gamma I\right)\mathbf{c} &\quad (4.22a)\\
&= \left\langle (\mathbf{c}\cdot\mathbf{m})^2 \eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\cdot\mathbf{m})\right\rangle + \gamma &\quad (4.22b)\\
&\le |V|\left(\sup_{y\in[-M,M]} \eta''_*(y)\right) + \gamma_0 &\quad (4.22c)
\end{aligned}$$

for $\gamma \le \gamma_0$, where we have used that $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|$ is a decreasing function of γ (recall (4.12)). Note that, on $\mathcal{R}^M$, as $\gamma \to \infty$, all eigenvalues of h''_γ go to zero, and indeed the function h_γ becomes flat. Since the behavior of the regularized equations for large γ is not particularly interesting, we rule out these problems by considering only γ smaller than some arbitrary γ_0.

On the other hand, if we now let $\mathbf{c}$ be the unit-length eigenvector associated with the smallest eigenvalue of $(h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))$ for $\mathbf{u} \in \mathcal{R}^M$ we have

$$\lambda_{\min}((h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) = \left\langle (\mathbf{c}\cdot\mathbf{m})^2 \eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\cdot\mathbf{m})\right\rangle + \gamma \ge \lambda_{\min}(\langle \mathbf{m}\mathbf{m}^T\rangle) \inf_{y\in[-M,M]} \eta''_*(y) \tag{4.23}$$

(where $\lambda_{\min}(\langle \mathbf{m}\mathbf{m}^T\rangle) > 0$ because $\mathbf{m}$ span a basis). Note that Assumption 2 guarantees strict positivity of η''_* because $\eta''_*(y) = 1/\eta''(\eta'_*(y))$.

Thus for h''_γ we can conclude the existence of positive constants $\lambda_{\min,h'',M}$ and $C_{h'',M}$ such that

$$\begin{aligned}
\mathbf{v}\cdot h''_\gamma(\mathbf{u})\mathbf{v} &\ge \lambda_{\min,h'',M}\|\mathbf{v}\|^2 \\
\|h''_\gamma(\mathbf{u})\| &\le C_{h'',M}
\end{aligned} \qquad \text{for all } \mathbf{v}\in\mathbb{R}^{N+1},\ \mathbf{u}\in\mathcal{R}^M, \text{ and } \gamma\in(0,\gamma_0) \tag{4.24}$$

The flux terms.

Lemma 4.4. *Let η be a superlinear kinetic entropy function, $M \in (0,\infty)$, and $\gamma_0 \in (0,\infty)$. Then there exist positive constants $C_{\mathbf{f}}$, C_J, and D_J such that*

$$\begin{aligned}
\|\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})\| &\le C_{\mathbf{f}} h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \\
\|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| &\le C_J h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_J\gamma^2
\end{aligned} \qquad \forall \mathbf{u}_\gamma\in\mathbb{R}^{N+1}, \mathbf{u}\in\mathcal{R}^M, \gamma\in(0,\gamma_0). \tag{4.25}$$

These constants depend on M and γ_0 but are independent of $\mathbf{u}_\gamma$, $\mathbf{u}$, and γ.

In the proof of Lemma 4.4 we often use the following elementary lemma.

Lemma 4.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that $f''(z) > 0$ for all z, and let $a \in (0, \infty)$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$ be given. If $K \in (0, \infty)$ and $C \in (0, \infty)$ satisfy*

$$f(K) + c > 0, \quad f'(K) > 0, \quad \text{and} \quad C \geq \max\left\{ \frac{aK + b}{f(K) + c}, \frac{a}{f'(K)} \right\}, \tag{4.26}$$

then

$$az + b \leq C(f(z) + c) \tag{4.27}$$

holds for all $z \geq K$.

Proof. The condition $C \geq \frac{aK+b}{f(K)+c}$ implies that (4.27) holds for $z = K$ and $C \geq \frac{a}{f'(K)}$ implies, together with $f'' > 0$, that the z derivative of the right hand side of (4.27) is larger than the z derivative of its left hand side. ∎

In the proof of Lemma 4.4 we also use the following lemma to get a nonzero lower bound for the relative entropy.

Lemma 4.6. *Let $M \in (0, \infty)$, $L \in (M, \infty)$, and $\gamma_0 \in (0, \infty)$. Then*

$$C_{h,M,L} := \inf_{\substack{\mathbf{v} \in \mathbb{R}^{N+1} \setminus \mathcal{R}^L \\ \mathbf{u} \in \mathcal{R}^M \\ \gamma \in (0, \gamma_0)}} h_\gamma(\mathbf{v}|\mathbf{u}) \tag{4.28}$$

is strictly positive.

The proof of Lemma 4.6 can be found in Appendix D.

Proof of Lemma 4.4. We partition $\mathbb{R}^{N+1}$ into three subsets and consider $\mathbf{u}_\gamma$ on each of these three subsets, which we illustrate in Figure 4.1. On the first set, $\mathcal{R}^L$ for $L \in (M, \infty)$, we take advantage of the fact that $\mathbf{f}_\gamma$, J_γ, and the relative entropy all look like $\|\mathbf{u}_\gamma - \mathbf{u}\|^2$ locally, up to an $\mathcal{O}(\gamma^2)$ term. On the second set, $B_K \setminus \mathcal{R}^L$, where B_K is a norm ball in $\mathbb{R}^{N+1}$, we use the fact that neither $\mathbf{f}_\gamma$ nor J_γ nor q_γ blow up on compact sets. Finally, in the third set we use that $\mathbf{f}_\gamma$ and J_γ grow linearly in either $\|\mathbf{u}_\gamma\|$ or $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ for large $\|\mathbf{u}_\gamma\|$ while $h_\gamma(\mathbf{u}_\gamma)$ grows at least linearly in $\|\mathbf{u}_\gamma\|$ or $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$.

(i) We begin with $(\mathbf{u}_\gamma, \mathbf{u}) \in \mathcal{R}^L \times \mathcal{R}^M$ for an $L \in (M, \infty)$. (The reason for choosing $L > M$ is given in case (ii) below.) When $\mathbf{f}_\gamma$ is sufficiently smooth there exists an $\mathbf{w}$ on the line connecting $\mathbf{u}_\gamma$ and $\mathbf{u}$ such that

$$\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u}) = (\mathbf{f}_\gamma''(\mathbf{w})(\mathbf{u}_\gamma - \mathbf{u}))(\mathbf{u}_\gamma - \mathbf{u}). \tag{4.29}$$

In our case, $\mathbf{f}_\gamma$ indeed possesses the requisite smoothness: We can write $\mathbf{f}_\gamma$ as $\mathbf{f}_\gamma = \mathbf{g} \circ \hat{\boldsymbol{\alpha}}_\gamma$, where $\mathbf{g}(\boldsymbol{\alpha}) = \langle v \mathbf{m} G_{\boldsymbol{\alpha}} \rangle$. The function $\mathbf{g}$ is smooth and its derivatives are bounded over any bounded set. Note that $\mathbf{w} \in \text{Conv}(\mathcal{R}^L)$, and since $\mathcal{R}^L \subset\subset \mathcal{R}$ and $\mathcal{R}$ is convex, we also know $\text{Conv}(\mathcal{R}^L) \subset\subset \mathcal{R}$. Thus we define

$$\widetilde{L} := \sup_{\mathbf{w} \in \text{Conv}(\mathcal{R}^L)} \|\hat{\boldsymbol{\alpha}}(\mathbf{w})\| < \infty. \tag{4.30}$$

Since $\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})$ is continuous with respect to γ for $\gamma \in [0, \infty)$ (where $\hat{\boldsymbol{\alpha}}_{\gamma=0} = \hat{\boldsymbol{\alpha}}$ when $\mathbf{w} \in \mathcal{R}$) [1, §3.1] and $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|$ is a decreasing function of γ (recall (4.12)) we have

$$\widetilde{L} = \sup_{\substack{\mathbf{w} \in \text{Conv}(\mathcal{R}^L) \\ \gamma \in (0, \infty)}} \|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\|. \tag{4.31}$$

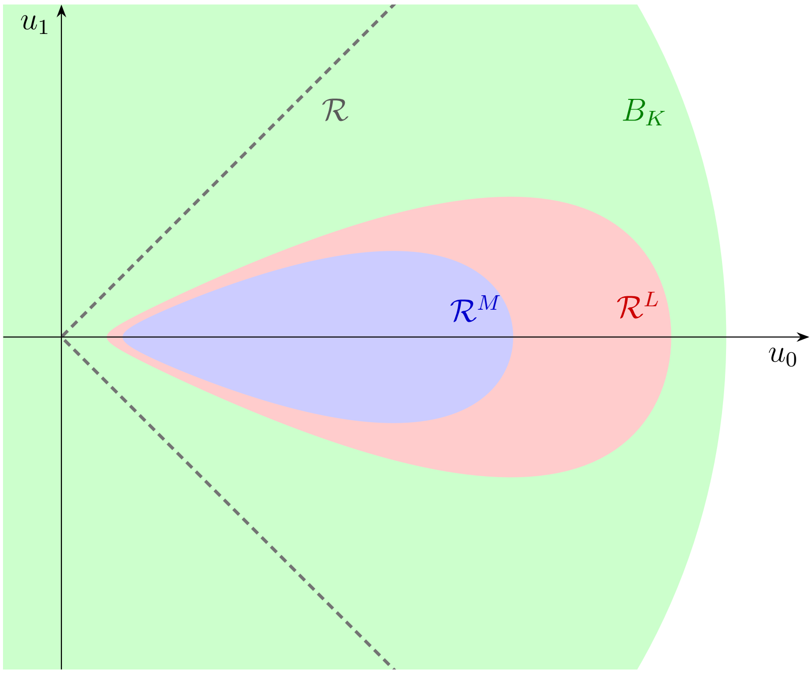

Figure 4.1. The sets used in the proof of Lemma 4.4. For this figure, we consider the M_1 case in slab geometry: $V = [-1, 1]$, $\mathbf{m}(v) = (1, v)$. In this case, the realizable set is given by $\mathcal{R} = \{(u_0, u_1) : |u_1| < u_0\}$, and we used $M = 1$, $L = 1.3$, and $K = 8$.

Thus it is clear that we only consider $\mathbf{g}$ and its derivatives within a bounded set. Furthermore, when bounded away from the boundary of $\mathcal{R}$, $\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})$ is a smooth function of $\mathbf{u}$, and this smoothness is uniform for $\gamma \in (0, \gamma_0)$. For example, $\hat{\boldsymbol{\alpha}}_\gamma' = h_\gamma''$ (recall (2.23c)), and from (4.24) we have $\|h_\gamma''(\mathbf{w})\| \le C_{h'', \widetilde{L}}$ for $\mathbf{w} \in \mathcal{R}^L$. The second derivative $\hat{\boldsymbol{\alpha}}_\gamma''$ can be similarly bounded, but we omit this more tedious computation. So we define

$$C_{\mathbf{f}'', L} := \sup_{\substack{\mathbf{w} \in \mathrm{Conv}(\mathcal{R}^L) \\ \gamma \in (0, \gamma_0)}} \|\mathbf{f}_\gamma''(\mathbf{w})\| \tag{4.32}$$

for some $\gamma_0 \in (0, \infty)$ and conclude

$$\|\mathbf{f}_\gamma(\mathbf{u}_\gamma | \mathbf{u})\| \le C_{\mathbf{f}'', L} \|\mathbf{u}_\gamma - \mathbf{u}\|^2 \tag{4.33}$$

for $(\mathbf{u}_\gamma, \mathbf{u}) \in \mathcal{R}^L \times \mathcal{R}^M$. We now turn to J_γ. The deciding factor in J_γ is $(\mathbf{f}'(\mathbf{u}) - \mathbf{f}_\gamma'(\mathbf{u}))$; to estimate it we use $\mathbf{f}_\gamma = \mathbf{f} \circ \hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma$, the Lipschitz continuity of $\mathbf{f}'$ on $\mathcal{R}^M$ (guaranteed by (4.32)), and the accuracy inequality

$$\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) - \mathbf{u}\| \le M\gamma \tag{4.34}$$

from [1, Thm. 2] as follows:

$$\begin{aligned}
\|\mathbf{f}'(\mathbf{u}) - \mathbf{f}_\gamma'(\mathbf{u})\| &= \|\mathbf{f}'(\mathbf{u}) - \mathbf{f}'(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})))(\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u})\| &(4.35a)\\
&= \|\mathbf{f}'(\mathbf{u}) - \mathbf{f}'(\mathbf{u})(\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u}) \\
&\quad + \mathbf{f}'(\mathbf{u})(\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u}) - \mathbf{f}'(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})))(\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u})\| &(4.35b)\\
&\le \|\mathbf{f}'(\mathbf{u})(I - (\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u}))\| \\
&\quad + \|(\mathbf{f}'(\mathbf{u}) - \mathbf{f}'(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))))(\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u})\| &(4.35c)
\end{aligned}$$

$$
\begin{aligned}
&\le C_1\gamma\|h''_\gamma(\mathbf{u})\| + C_{\mathbf{f}'',M}M\gamma\|(\hat{\mathbf{u}}\circ\hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u})\| &\text{(4.35d)}\\
&\le (C_1C_{h'',M} + C_{\mathbf{f}'',M}MC_2)\gamma &\text{(4.35e)}\\
&=: C_4\gamma, &\text{(4.35f)}
\end{aligned}
$$

where we define $C_4 := C_1C_{h'',M} + C_{\mathbf{f}'',M}MC_2$. In (4.35d) we have used

$$
I - (\hat{\mathbf{u}}\circ\hat{\boldsymbol{\alpha}}_\gamma)'(\mathbf{u}) = \gamma h''_\gamma(\mathbf{u}), \tag{4.36}
$$

which is a straightforward computation using $\hat{\mathbf{u}}' = h''_*$ and $\hat{\boldsymbol{\alpha}}'_\gamma(\mathbf{u}) = (h''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) + \gamma I)^{-1}$ (see (2.23c)). In this step we are also able to use $C_{\mathbf{f}'',M}$ because $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|$ is a decreasing function of γ (recall (4.12)).

With (4.35) and the bound on h''_γ from (4.24) we can bound J_γ by

$$
\begin{aligned}
\|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| &\le \|\mathbf{u}_\gamma - \mathbf{u}\|\|h''_\gamma(\mathbf{u})\|\|\mathbf{f}'(\mathbf{u}) - \mathbf{f}'_\gamma(\mathbf{u})\| &\text{(4.37a)}\\
&\le \|\mathbf{u}_\gamma - \mathbf{u}\|C_{h'',M}C_4\gamma &\text{(4.37b)}\\
&\le \frac{C_{h'',M}C_4}{2}\left(\gamma^2 + \|\mathbf{u}_\gamma - \mathbf{u}\|^2\right), &\text{(4.37c)}
\end{aligned}
$$

where for the last step we have applied Young's inequality. Finally, we must bound $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ similarly from below. This follows immediately from (4.24):

$$
h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) = (\mathbf{u}_\gamma - \mathbf{u})\cdot h''_\gamma(\mathbf{w})(\mathbf{u}_\gamma - \mathbf{u}) \ge \lambda_{\min,h'',\widetilde{L}}\|\mathbf{u}_\gamma - \mathbf{u}\|^2 \tag{4.38}
$$

for all $(\mathbf{u}_\gamma,\mathbf{u}) \in \mathcal{R}^L \times \mathcal{R}^M$ and $\mathbf{w}$ is the appropriate vector in $\mathrm{Conv}(\mathcal{R}^L)$ from the mean-value theorem. Altogether we have

$$
\|\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})\| \le \frac{C_{\mathbf{f}'',L}}{\lambda_{\min,h'',\widetilde{L}}}h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \tag{4.39}
$$

$$
\|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| \le \frac{C_{h'',M}C_4}{2\lambda_{\min,h'',\widetilde{L}}}h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + \frac{C_{h'',M}C_4}{2}\gamma^2 \tag{4.40}
$$

for $(\mathbf{u}_\gamma,\mathbf{u}) \in \mathcal{R}^L \times \mathcal{R}^M$.

(ii) We now consider $\mathbf{u}_\gamma \in B_K \setminus \mathcal{R}^L$ for the ball $B_K := \{\mathbf{w} \in \mathbb{R}^{N+1} : \|\mathbf{w}\| \le K\}$. Since B_K is a compact set, we know that the constants

$$
C_{\mathbf{f},K} := \sup_{\substack{\mathbf{u}_\gamma\in B_K\setminus\mathcal{R}^L\\ \mathbf{u}\in\mathcal{R}^M\\ \gamma\in(0,\gamma_0)}} \|\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})\| \tag{4.41}
$$

$$
C_{J,K} := \sup_{\substack{\mathbf{u}_\gamma\in B_K\setminus\mathcal{R}^L\\ \mathbf{u}\in\mathcal{R}^M\\ \gamma\in(0,\gamma_0)}} \|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| \tag{4.42}
$$

are finite for any $\gamma_0 \in (0,\infty)$. Indeed with (4.4), (4.11), (4.20), and (4.24) we immediately have the crude upper bounds

$$
C_{\mathbf{f},K} \le C_1((C_2+1)(K+u_M) + 2C_3) \quad \text{and} \quad C_{J,K} \le 2(K+u_M)C_{h'',M}C_1. \tag{4.43}
$$

To bound $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ from below we use $C_{h,M,L}$ from Lemma 4.6. (Note that here $L > M$ is crucial: this lower bound is not strictly positive for $L \le M$.) Altogether we have

$$
\|\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})\| \le \frac{C_{\mathbf{f},K}}{C_{h,M,L}}h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \qquad \text{and} \qquad \|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| \le \frac{C_{J,K}}{C_{h,M,L}}h_\gamma(\mathbf{u}_\gamma|\mathbf{u}). \tag{4.44}
$$

Up to now, K is arbitrary; in the next and final case we give a lower bound that K must satisfy.

(iii) Finally, for $\mathbf{u}_\gamma \in \mathbb{R}^{N+1} \setminus B_K$, how we proceed depends on which term in h_γ dominates: when $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ is large, we use the entropy term, otherwise we use the quadratic term. To distinguish these two subcases, let $\delta \in (0,1)$. The following arguments work for any value of $\delta \in (0,1)$; the particular choice of δ merely affects the value of the constants $C_{\mathbf{f}}$ and C_J.

(a) $\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| \le \delta\|\mathbf{u}_\gamma\|$

Here, we use the quadratic term of the relative entropy to dominate the linear term of the relative flux. We bound the relative flux by

$$\begin{aligned}\|\mathbf{f}_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| &\le C_1\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| + C_1\|\mathbf{u}_\gamma\| + C_1(C_2u_M + C_3) + C_1u_M \\ &\le C_1(1+\delta)\|\mathbf{u}_\gamma\| + C_1((C_2+1)u_M + C_3), \end{aligned} \tag{4.45}$$

J_γ by

$$\|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| \le 2C_1C_{h'',M}(\|\mathbf{u}_\gamma\| + u_M), \tag{4.46}$$

and the relative entropy by

$$\begin{aligned} h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) &\ge h(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))) + \frac{1}{2\gamma}(\|\mathbf{u}_\gamma\| - \|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\|)^2 - h_M - M(\|\mathbf{u}_\gamma\| + u_M) \\ &\ge |V|\eta_{\min} + \frac{(1-\delta)^2}{2\gamma_0}\|\mathbf{u}_\gamma\|^2 - h_M - M(\|\mathbf{u}_\gamma\| + u_M) \end{aligned} \tag{4.47}$$

where we use $\eta_{\min} := \min_{z\ge 0}\eta(z)$, and in the first inequality we have used that $\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|$ is a decreasing function of γ (recall (4.12)). Then the application of Lemma 4.5 with $z = \|\mathbf{u}_\gamma\|$ and $f(z) = (1-\delta)^2z^2/(2\gamma_0) - Mz$ gives the following conditions on K, $C_{\mathbf{f}}$, and C_J:

$$\frac{(1-\delta)^2}{\gamma_0}K^2 - MK > h_M + Mu_M - 2\eta_{\min}, \quad \frac{(1-\delta)^2}{\gamma_0}K > M, \tag{4.48}$$

$$\begin{aligned} C_{\mathbf{f}} &\ge \max\left\{\frac{C_1(1+\delta)K + C_1((C_2+1)u_M + C_3)}{\frac{(1-\delta)^2}{\gamma_0}K^2 - MK + |V|\eta_{\min} - h_M - Mu_M}, \frac{C_1(1+\delta)}{\frac{(1-\delta)^2}{\gamma_0}K - M}\right\}, \text{ and} \\ C_J &\ge \max\left\{\frac{2C_1C_{h'',M}(K+u_M)}{\frac{(1-\delta)^2}{\gamma_0}K^2 - MK + |V|\eta_{\min} - h_M - Mu_M}, \frac{2C_1C_{h'',M}}{\frac{(1-\delta)^2}{\gamma_0}K - M}\right\} \end{aligned} \tag{4.49}$$

(b) $\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| > \delta\|\mathbf{u}_\gamma\|$

In this case, we know that $\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ is not arbitrarily small and therefore formulate our bounds in terms of $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$. First, for the upper bounds we have, using (4.2),

$$\|\mathbf{f}_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| \le \left(C_1C_0 + C_1\frac{C_0}{\delta}\right)\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) + C_1(C_2u_M + C_3) + C_1u_M, \text{ and} \tag{4.50}$$

$$\|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| \le 2C_1C_{h'',M}\left(\frac{C_0}{\delta}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) + u_M\right). \tag{4.51}$$

For the lower bound on the entropy, notice that by Jensen's inequality

$$\langle\eta(g)\rangle \ge |V|\eta\left(\frac{1}{|V|}\langle g\rangle\right), \tag{4.52}$$

from which it follows that

$$h(\mathbf{u}) \ge |V|\eta\left(\frac{1}{|V|}u_0\right), \tag{4.53}$$

for any realizable $\mathbf{u}$. Thus for the relative entropy we have

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \ge |V|\eta\left(\frac{1}{|V|}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\right) - h_M - M\left(\frac{C_0}{\delta}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) + u_M\right). \tag{4.54}$$

Now, with the lower bound on $\hat{u_0}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$, namely,

$$\hat{u_0}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) \geq \frac{1}{C_0}\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| \geq \frac{\delta}{C_0}\|\mathbf{u}_\gamma\| \geq \frac{\delta K}{C_0} \tag{4.55}$$

we can apply Lemma 4.5 with $z = \hat{u_0}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ and $f(z) = |V|\eta(z/|V|) - MC_0 z/\delta$. Here the conditions on K, $C_\mathbf{f}$, and C_J become

$$|V|\eta\left(\frac{\delta K}{|V|C_0}\right) - MK > h_M + Mu_M, \quad \eta'\left(\frac{\delta K}{|V|C_0}\right) > \frac{MC_0}{\delta}, \tag{4.56}$$

$$\begin{aligned} C_\mathbf{f} &\geq \max\left\{\frac{C_1((\delta+1)K + (C_2+1)u_M + C_3)}{|V|\eta\left(\frac{\delta K}{|V|C_0}\right) - MK - h_M - Mu_M}, \frac{C_1C_0\left(1+\frac{1}{\delta}\right)}{\eta'\left(\frac{\delta K}{|V|C_0}\right) - \frac{MC_0}{\delta}}\right\}, \text{ and} \\ C_J &\geq \max\left\{\frac{2C_1C_{h'',M}(K+u_M)}{|V|\eta\left(\frac{\delta K}{|V|C_0}\right) - MK - h_M - Mu_M}, \frac{2C_0C_1C_{h'',M}}{\delta\left(\eta'\left(\frac{\delta K}{|V|C_0}\right) - \frac{MC_0}{\delta}\right)}\right\} \end{aligned} \tag{4.57}$$

Altogether we get (4.25) for

$$\begin{aligned} C_\mathbf{f} = \max\Bigg\{ & \frac{C_{\mathbf{f}'',L}}{\lambda_{\min,h'',\widetilde{L}}}, \frac{C_{\mathbf{f},K}}{C_{h,M,L}}, \frac{C_1(1+\delta)K + C_1((C_2+1)u_M + C_3)}{\frac{(1-\delta)^2}{\gamma_0}K^2 - MK + |V|\eta_{\min} - h_M - Mu_M}, \\ & \frac{C_1(1+\delta)}{\frac{(1-\delta)^2}{\gamma_0}K - M}, \frac{C_1((\delta+1)K + (C_2+1)u_M + C_3)}{|V|\eta\left(\frac{\delta K}{|V|C_0}\right) - MK - h_M - Mu_M}, \frac{C_1C_0\left(1+\frac{1}{\delta}\right)}{\eta'\left(\frac{\delta K}{|V|C_0}\right) - \frac{MC_0}{\delta}}\Bigg\}, \end{aligned} \tag{4.58}$$

$$\begin{aligned} C_J = \max\Bigg\{ & \frac{C_{h'',M}C_4}{2\lambda_{\min,h'',\widetilde{L}}}, \frac{C_{J,K}}{C_{h,M,L}}, \frac{2C_1C_{h'',M}(K+u_M)}{\frac{(1-\delta)^2}{\gamma_0}K^2 - MK + |V|\eta_{\min} - h_M - Mu_M}, \\ & \frac{2C_1C_{h'',M}}{\frac{(1-\delta)^2}{\gamma_0}K - M}, \frac{2C_1C_{h'',M}(K+u_M)}{|V|\eta\left(\frac{\delta K}{|V|C_0}\right) - MK - h_M - Mu_M}, \frac{2C_0C_1C_{h'',M}}{\delta\left(\eta'\left(\frac{\delta K}{|V|C_0}\right) - \frac{MC_0}{\delta}\right)}\Bigg\}, \text{ and} \end{aligned} \tag{4.59}$$

$$D_J = \frac{C_{h'',M}C_4}{2}, \tag{4.60}$$

where K satisfies (4.48) and (4.56). ■

The source term

Lemma 4.7. *Let η be a superlinear kinetic entropy function, $M \in (0,\infty)$, and $\gamma_0 \in (0,\infty)$. Then there exist positive constants C_q and D_q such that*

$$q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) \leq C_q h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_q\gamma^2 \quad \forall \mathbf{u}_\gamma \in \mathbb{R}^{N+1}, \mathbf{u} \in \mathcal{R}^M, \gamma \in (0,\gamma_0). \tag{4.61}$$

Proof. We use the same lower bounds just derived for the relative entropy $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ in Lemma 4.4. Thus we only need to give upper bounds of q_γ on the same decomposition of $\mathbb{R}^{N+1} \times \mathcal{R}^M$ used in the proof of Lemma 4.4.

(i) Let $(\mathbf{u}_\gamma, \mathbf{u}) \in \mathcal{R}^L \times \mathcal{R}^M$. To write down an upper bound of q_γ, we define the constants

$$C_{h'''} = \sup_{\substack{\mathbf{v}\in\mathrm{Conv}(\mathcal{R}^L)\\ \gamma\in(0,\gamma_0)}} \|h_\gamma'''(\mathbf{v})\| \quad \text{and} \quad r_M = \sup_{\mathbf{v}\in\mathcal{R}^M} \|\mathbf{r}(\mathbf{v})\|, \tag{4.62}$$

all finite by smoothness of h_γ and $\mathbf{r}$ and compactness of $\mathrm{Conv}(\mathcal{R}^L)$ and $\mathcal{R}^M$, and we use

$$\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) - \mathbf{u}\| \leq \|\mathbf{u}_\gamma - \mathbf{u}\| + M\gamma, \tag{4.63}$$

which follows from $\mathbf{u} \in \mathcal{R}^M$ [1, Thm. 2]. Then we rearrange q_γ and straightforwardly get the estimate

$$\begin{aligned} q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) &= (h'_\gamma(\mathbf{u}_\gamma) - h'_\gamma(\mathbf{u})) \cdot (\mathbf{r}_\gamma(\mathbf{u}_\gamma) - \mathbf{r}(\mathbf{u})) \\ &\qquad + (h'_\gamma(\mathbf{u}_\gamma) - h'_\gamma(\mathbf{u}) - h''_\gamma(\mathbf{u})(\mathbf{u}_\gamma - \mathbf{u})) \cdot \mathbf{r}(\mathbf{u}) && (4.64a) \\ &\le (C_{h'',\widetilde{L}} C_{\mathbf{r}} + r_M C_{h'''}) \|\mathbf{u}_\gamma - \mathbf{u}\|^2 + C_{h'',\widetilde{L}} C_{\mathbf{r}} M \gamma \|\mathbf{u}_\gamma - \mathbf{u}\| && (4.64b) \\ &\le (C_{h'',\widetilde{L}} C_{\mathbf{r}} + r_M C_{h'''}) \|\mathbf{u}_\gamma - \mathbf{u}\|^2 + \frac{C_{h'',\widetilde{L}} C_{\mathbf{r}} M}{2} \left(\gamma^2 + \|\mathbf{u}_\gamma - \mathbf{u}\|^2\right), && (4.64c) \end{aligned}$$

which appropriately mirrors (4.33) and (4.37).

(ii) Now for the case $\mathbf{u}_\gamma \in B_K \setminus \mathcal{R}^L$ (where again $B_K := \{\mathbf{w} \in \mathbb{R}^n : \|\mathbf{w}\| \le K\}$), we first use $h'_\gamma(\mathbf{u}_\gamma) \cdot \mathbf{r}_\gamma(\mathbf{u}_\gamma) \le 0$ to get

$$q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) \le -h'_\gamma(\mathbf{u}) \cdot \mathbf{r}_\gamma(\mathbf{u}_\gamma) - \mathbf{r}(\mathbf{u}) \cdot (h''_\gamma(\mathbf{u})(\mathbf{u}_\gamma - \mathbf{u})) \tag{4.65}$$

As with the flux, none of these terms blow up for $(\mathbf{u}_\gamma, \mathbf{u}) \in B_K \times \mathcal{R}^M$ for any finite K:

$$\begin{aligned} C_{q,K,\gamma_0} &:= \sup_{\substack{\mathbf{u}_\gamma \in B_K \setminus \mathrm{Conv}(\mathcal{R}^L) \\ \mathbf{u} \in \mathcal{R}^M \\ \gamma \in (0,\gamma_0)}} -h'_\gamma(\mathbf{u}) \cdot \mathbf{r}_\gamma(\mathbf{u}_\gamma) - \mathbf{r}(\mathbf{u}) \cdot (h''_\gamma(\mathbf{u})(\mathbf{u}_\gamma - \mathbf{u})) && (4.66a) \\ &\le M C_{\mathbf{r}} (C_2 K + C_3) + r_M C_{h'',M} (K + u_M). && (4.66b) \end{aligned}$$

(iii) For large $\mathbf{u}_\gamma$ we show, as with the flux, that q_γ grows linearly with $\|\mathbf{u}_\gamma\|$ when $\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ is small and linearly with $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ otherwise. Indeed, if for $\delta \in (0,1)$ we have $\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| \le \delta \|\mathbf{u}_\gamma\|$, then

$$q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) \le M C_{\mathbf{r}} \delta \|\mathbf{u}_\gamma\| + r_M C_{h'',M} (\|\mathbf{u}_\gamma\| + u_M), \tag{4.67}$$

i.e., linear growth in $\|\mathbf{u}_\gamma\|$ as in (4.45) and (4.46). On the other hand, when $\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| > \delta \|\mathbf{u}_\gamma\|$, then

$$q_\gamma(\mathbf{u}_\gamma, \mathbf{u}) \le M C_{\mathbf{r}} C_0 \hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) + r_M C_{h'',M} \left(\frac{C_0}{\delta} \hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) + u_M\right), \tag{4.68}$$

which is linear growth in $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))$ as in (4.50) and (4.51).

Now we simply need to change the numerators from the flux case above to get (4.61) for

$$\begin{aligned} C_q \ge \max \Bigg\{ &\frac{(C_{h'',\widetilde{L}} C_{\mathbf{r}} + r_M C_{h'''}) + \frac{C_{h'',\widetilde{L}} C_{\mathbf{r}} M}{2}}{\lambda_{\min,h'',\widetilde{L}}}, \frac{C_{q,K,\gamma_0}}{C_{h,M,L}}, \\ &\frac{M C_{\mathbf{r}} \delta K + r_M C_{h'',M}(K + u_M)}{\frac{(1-\delta)^2}{\gamma_0} K^2 - MK + |V| \eta_{\min} - h_M - C_0 v_{0,M}}, \frac{M C_{\mathbf{r}} \delta + r_M C_{h'',M}}{\frac{(1-\delta)^2}{\gamma_0} K - M}, \\ &\frac{M C_{\mathbf{r}} \delta K + r_M C_{h'',M}(K + u_M)}{|V| \eta\left(\frac{\delta K}{|V| C_0}\right) - MK - h_M - M u_M}, \frac{M C_{\mathbf{r}} C_0 + r_M C_{h'',M} \frac{C_0}{\delta}}{\eta'\left(\frac{\delta K}{|V| C_0}\right) - \frac{M C_0}{\delta}} \Bigg\} \end{aligned} \tag{4.69}$$

for some K and δ satisfying the same conditions as for the flux term, and

$$D_q := \frac{C_{h'',\widetilde{L}} C_{\mathbf{r}} M}{2}. \tag{4.70}$$

■

Lemmas 4.4 and 4.7 yield a more precise version of Theorem 3.4:

Corollary 1. *Let η be a superlinear kinetic entropy function and $\mathbf{u}$ a Lipschitz solution of the entropy-based moment equations (3.1), with C_u as in Theorem 3.4, for which there exists $M \in (0, \infty)$ such that $\mathbf{u}(t, x) \in \mathcal{R}^M$ for all $(t, x) \in [0, T] \times X$. Let $\{\mathbf{u}_\gamma\}_{\gamma \in (0, \gamma_0)}$ be a family of entropy solutions of (3.2).*

Then $\|\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|_{L^\infty([0,T]\times X)} \leq C_{h'',M} C_u$, and

$$\int_X h_\gamma(\mathbf{u}_\gamma(T, x)|\mathbf{u}(T, x))\, dx \leq \exp(CT) DT\gamma^2 \tag{4.71}$$

for $C := C_{h'',M} C_u C_{\mathbf{f}} + C_u C_J + C_q$ and $D := C_u D_J + D_q$, where the constants $C_{\mathbf{f}}$, C_J, C_q, D_J, and D_q are given by Lemmas 4.4 and 4.7.

Corollary 2. *Let η, $\mathbf{u}$, and $\mathbf{u}_\gamma$ satisfy the conditions of Corollary 1 and $L \in (M, \infty)$, and consider*

$$X_L := \{x \in X \,:\, \mathbf{u}_\gamma(T, x) \in \mathcal{R}^L\}.$$

Then, there exists a constant $C_{L^1} > 0$ such the following estimate holds:

$$\lambda_{\min, h'', \widetilde{L}} \|\mathbf{u}_\gamma(T, \cdot\,) - \mathbf{u}(T, \cdot\,)\|^2_{L^2(X_L)} + C_{L^1} \|\mathbf{u}_\gamma(T, \cdot\,) - \mathbf{u}(T, \cdot\,)\|_{L^1(X \setminus X_L)} \leq \exp(CT) DT\gamma^2 \tag{4.72}$$

Proof. For $\mathbf{u}_\gamma \in \mathcal{R}^L$ we have (cf. (4.38))

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \geq \lambda_{\min, h'', \widetilde{L}} \|\mathbf{u}_\gamma - \mathbf{u}\|^2 \tag{4.73}$$

For $\mathbf{u}_\gamma \in \mathbb{R}^{N+1} \backslash \mathcal{R}^L$, we can use the lower bounds on the relative entropy from the proof of Lemma 4.4 to show that the relative entropy grows at least linearly with $\mathbf{u}_\gamma$, i.e., that there exists a $C_{L^1} \in (0, \infty)$ such that

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \geq C_{L^1} \|\mathbf{u}_\gamma - \mathbf{u}\|. \tag{4.74}$$

First, for $\mathbf{u}_\gamma \in B_K \setminus \mathcal{R}^L$ (where K must be big enough that $\mathcal{R}^M \subset B_K$), since $\|\mathbf{u}_\gamma - \mathbf{u}\| \leq 2K$, we simply have

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \geq C_{h,M,L} \geq \frac{C_{h,M,L}}{2K} \|\mathbf{u}_\gamma - \mathbf{u}\|. \tag{4.75}$$

This gives the first upper bound on C_{L^1}.

For $\mathbf{u}_\gamma \in \mathbb{R}^{N+1} \backslash B_K$, we can quickly see that the lower bounds (4.47) and (4.54) both grow superlinearly in $\|\mathbf{u}_\gamma\|$. For the first subcase, we have

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \geq |V|\eta_{\min} + \frac{(1-\delta)^2}{2\gamma_0} \|\mathbf{u}_\gamma\|^2 - h_M - M(\|\mathbf{u}_\gamma\| + u_M) \tag{4.76}$$

Here we can apply Lemma 4.5 with $C = 1$, $b = 0$ and a playing the role of C_{L^1} to see that K and C_{L^1} must satisfy

$$C_{L^1} K \leq |V|\eta_{\min} + \frac{(1-\delta)^2}{2\gamma_0} K^2 - h_M - M(K + u_M) \quad \text{and} \quad C_{L^1} \leq \frac{(1-\delta)^2}{\gamma_0} K - M. \tag{4.77}$$

We note that clearly for any C_{L^1} one can always find a large enough K such that these conditions are satisfied.

For the second subcase, instead of using (4.54) directly we estimate the last term as in (4.47) to get

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \geq |V|\eta\left(\frac{1}{|V|} \hat{u_0}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\right) - h_M - M(\|\mathbf{u}_\gamma\| + u_M), \tag{4.78}$$

and then we require that so that η is monotone increasing and we apply (4.55) to get

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \geq |V|\eta\left(\frac{\delta}{|V|C_0} \|\mathbf{u}_\gamma\|\right) - h_M - M(\|\mathbf{u}_\gamma\| + u_M). \tag{4.79}$$

This inequality only holds if we are in the range of values where η is monotonically increasing, so K must be large enough that $(\delta K)/(|V|C_0) \geq \operatorname{argmin} \eta$. Again we can apply Lemma 4.5 as above, with

$C = 1$, $b = 0$ and a playing the role of C_{L^1} to get the conditions

$$C_{L^1}K \le |V|\eta\left(\frac{\delta K}{|V|C_0}\right) - h_M - M(K + u_M) \quad \text{and} \quad C_{L^1} \le \frac{\delta}{C_0}\eta'\left(\frac{\delta K}{|V|C_0}\right) - M \tag{4.80}$$

As above, since η grows superlinearly, for any C_{L^1} one can always find a large enough K such that these conditions are satisfied.

Thus we conclude that there exists a C_{L^1} such that

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \ge C_{L^1}\|\mathbf{u}_\gamma - \mathbf{u}\| \qquad \text{for all } (\mathbf{u}_\gamma, \mathbf{u}) \in \mathbb{R}^{N+1} \setminus \mathcal{R}^L \times \mathcal{R}^M. \tag{4.81}$$

■

4.4. **The sublinear case**

In contrast to the Maxwell–Boltzmann-like entropies, the entropies η we consider in this class are not bounded from below but $\lim_{z\to\infty}\eta'(z) = 0$. Consequently $\mathrm{Dom}(\eta_*) \subseteq (-\infty, 0)$, and the multipliers must satisfy $\boldsymbol{\alpha}\cdot\mathbf{m} < 0$ for all $v \in V$. For such entropies we replace the assumption in (4.19) with

$$\mathbf{u} \in \mathcal{R}^{M,m} := \left\{\hat{\mathbf{u}}(\boldsymbol{\alpha}) : \|\boldsymbol{\alpha}\| \le M \text{ and } \boldsymbol{\alpha}\cdot\mathbf{m}(v) \le -m \text{ for all } v \in V\right\}, \tag{4.82}$$

for some $M \in (0,\infty)$ and $m \in (0,\infty)$. Note that as $M \to \infty$ and $m \to 0$, the set $\mathcal{R}^{M,m}$ approaches the full realizable set $\mathcal{R}$. Related to the parameter m is

$$p_0 := -\sup_{\substack{\mathbf{w}\in\mathcal{R}^{M,m}\\ \gamma\in(0,\gamma_0)\\ v\in V}} \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{w})\cdot\mathbf{m}(v) > 0; \tag{4.83}$$

for some $\gamma_0 \in (0,\infty)$.

The additional condition parameterized by m in $\mathcal{R}^{M,m}$ ensures that the ansatz $G_{\boldsymbol{\alpha}}$ in the sublinear case is bounded away from zero. In the superlinear case, the ansätze $G_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \mathcal{R}^M$ are already bounded away from zero, but this is not so for the sublinear case, where $\boldsymbol{\alpha}$ must also fulfill $\boldsymbol{\alpha}\cdot\mathbf{m}(v) < 0$ for all $v \in V$.

Note that $u_{M,m}$ and $h_{M,m}$ can be defined as in (4.20):

$$u_{M,m} := \sup_{\mathbf{w}\in\mathcal{R}^{M,m}} \|\mathbf{w}\| \qquad \text{and} \qquad h_{M,m} := \sup_{\mathbf{w}\in\mathcal{R}^{M,m}} h(\mathbf{w}); \tag{4.84}$$

both are finite. We can also derive similar bounds on h''_γ. Let $\mathbf{c}$ be the unit-length eigenvector associated with the largest eigenvalue of $(h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))$ for some $\mathbf{u} \in \mathcal{R}^{M,m}$

$$\lambda_{\max}((h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) = \mathbf{c}\cdot\left(\left\langle \mathbf{m}\mathbf{m}^T\eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\cdot\mathbf{m})\right\rangle + \gamma I\right)\mathbf{c} \tag{4.85a}$$

$$= \left\langle (\mathbf{c}\cdot\mathbf{m})^2\eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\cdot\mathbf{m})\right\rangle + \gamma \tag{4.85b}$$

$$\le |V|\left(\sup_{y\in[-M,-p_0]} \eta''_*(y)\right) + \gamma_0 \tag{4.85c}$$

for $\gamma \le \gamma_0$. Similarly, if we now let $\mathbf{c}$ be the unit-length eigenvector associated with the smallest eigenvalue of $(h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))$ for $\mathbf{u} \in \mathcal{R}^{M,m}$ we have

$$\lambda_{\min}((h_\gamma)''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) = \left\langle (\mathbf{c}\cdot\mathbf{m})^2\eta''_*(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\cdot\mathbf{m})\right\rangle + \gamma \ge |V|\inf_{y\in[-M,-p_0]} \eta''_*(y). \tag{4.86}$$

Thus for the sublinear case there exist positive constants $\lambda_{\min,h'',M,m}$ and $C_{h'',M,m}$ such that

$$\begin{aligned} \mathbf{v}\cdot h''_\gamma(\mathbf{u})\mathbf{v} &\ge \lambda_{\min,h'',M,m}\|\mathbf{v}\|^2 \\ \|h''_\gamma(\mathbf{u})\| &\le C_{h'',M,m} \end{aligned} \qquad \text{for all } \mathbf{v} \in \mathbb{R}^{N+1},\ \mathbf{u} \in \mathcal{R}^M, \text{ and } \gamma \in (0,\gamma_0); \tag{4.87}$$

(cf. the superlinear case (4.24)).

Lemma 4.8. *Let η be a sublinear kinetic entropy function, $M \in (0,\infty)$, $m \in (0,\infty)$, and $\gamma_0 \in (0,\infty)$. Then there exist positive constants $C_{\mathbf{f}}$, C_J, D_J, C_q and D_q such that*

$$\begin{aligned} \|\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})\| &\le C_{\mathbf{f}} h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \\ \|J_\gamma(\mathbf{u}_\gamma,\mathbf{u})\| &\le C_J h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_J\gamma^2 \quad \forall \mathbf{u}_\gamma \in \mathbb{R}^{N+1}, \mathbf{u} \in \mathcal{R}^{M,m}, \gamma \in (0,\gamma_0). \\ q_\gamma(\mathbf{u}_\gamma,\mathbf{u}) &\le C_q h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) + D_q\gamma^2 \end{aligned} \tag{4.88}$$

Proof. Because it is so similar to the superlinear case, we only sketch the proof for the sublinear case.
For sublinear η, the estimates of $\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})$, $J_\gamma(\mathbf{u}_\gamma,\mathbf{u})$, $q_\gamma(\mathbf{u}_\gamma,\mathbf{u})$ and $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ on $B_K \times \mathcal{R}^{M,m}$ can be derived just as in the superlinear case for $B_K \times \mathcal{R}^M$, as well as the estimates of $\mathbf{f}_\gamma(\mathbf{u}_\gamma|\mathbf{u})$, $J_\gamma(\mathbf{u}_\gamma,\mathbf{u})$, and $q_\gamma(\mathbf{u}_\gamma,\mathbf{u})$ for large $\mathbf{u}_\gamma$ in (4.45), (4.46), (4.67), (4.50), (4.51), and (4.68). The lower bound of $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$, however, in (4.47) is no longer possible when η is not bounded from below, and Lemma 4.5 can no longer be applied to (4.54) because the right-hand side does not grow as $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) \to \infty$.
In the subcase $\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| \le \delta\|\mathbf{u}_\gamma\|$, we first apply a combination of the bounds on individual terms from above to get

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \ge |V|\eta\left(\frac{1}{|V|}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\right) + \frac{(1-\delta)^2}{2\gamma_0}\|\mathbf{u}_\gamma\|^2 - h_{M,m} - M\|\mathbf{u}_\gamma\| - Mu_{M,m}. \tag{4.89a}$$

Now we recognize that $\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) \le \|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\|$ and then use that η is a monotonically decreasing function to conclude

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) \ge |V|\eta\left(\frac{\delta}{|V|}\|\mathbf{u}_\gamma\|\right) + \frac{(1-\delta)^2}{2\gamma_0}\|\mathbf{u}_\gamma\|^2 - h_{M,m} - M\|\mathbf{u}_\gamma\| - Mu_{M,m}. \tag{4.89b}$$

Now thanks to the convexity of η, we can apply Lemma 4.5 with

$$f(z) = |V|\eta\left(\frac{\delta z}{|V|}\right) + \frac{(1-\delta)^2 z^2}{2\gamma_0} - Mz. \tag{4.90}$$

In the other subcase, where $\|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\| > \delta\|\mathbf{u}_\gamma\|$, we use the assumption (4.82) and the first-order necessary condition (2.20) to get

$$\begin{aligned} h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) &= h(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))) + \frac{\gamma}{2}\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)\|^2 - h(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) - \frac{\gamma}{2}\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|^2 \\ &\qquad - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) + \gamma\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma) - \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) - \gamma\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})) && (4.91a) \\ &= h(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))) - h(\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot (\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) - \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}))) \\ &\qquad + \frac{\gamma}{2}\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma) - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|^2 && (4.91b) \\ &\ge |V|\eta\left(\frac{1}{|V|}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\right) - h_{M,m} - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) - Mu_{M,m} && (4.91c) \\ &= |V|\eta\left(\frac{1}{|V|}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\right) - h_{M,m} - \langle \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}) \cdot \mathbf{m}\eta_*'(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)\cdot\mathbf{m})\rangle - Mu_{M,m} && (4.91d) \\ &\ge |V|\eta\left(\frac{1}{|V|}\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma))\right) - h_{M,m} + p_0\hat{u}_0(\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)) - Mu_{M,m}. && (4.91e) \end{aligned}$$

In the last step we have used $\eta_*' \ge 0$. We are again ready to apply Lemma 4.5 with $f(z) = |V|\eta(z/|V|) + p_0 z$ to derive conditions on K, $C_{\mathbf{f}}$, and C_J to achieve the desired estimate (4.88). ■

This immediately gives the following corollary.

Corollary 3. *Let η be a sublinear kinetic entropy function and $\mathbf{u}$ a Lipschitz continuous solution of the entropy-based moment equations (3.1), with C_u as in Theorem 3.4, for which there exist $M \in (0,\infty)$*

and $m \in (0,\infty)$ *so that* $\mathbf{u}(t,x) \in \mathcal{R}^{M,m}$ *for all* $(t,x) \in [0,T] \times X$*. Let* $\{\mathbf{u}_\gamma\}_{\gamma\in(0,\gamma_0)}$ *be a family of entropy solutions of* (3.2) *for* $\gamma \in (0,\gamma_0)$*.*

Then $\|\nabla_x \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|_{L^\infty([0,T]\times X)} \le C_{h'',M,m} C_u =: C_{\hat{\boldsymbol{\alpha}}}$*, and*

$$\int_X h_\gamma(\mathbf{u}_\gamma(T,x)|\mathbf{u}(T,x))\,dx \le \exp(CT)DT\gamma^2 \tag{4.92}$$

for $C := C_{\hat{\boldsymbol{\alpha}}} C_{\mathbf{f}} + C_u C_J + C_q$ *and* $D := C_u D_J + D_q$*, where the constants* $C_{\mathbf{f}}$*,* C_J*,* C_q*,* D_J*, and* D_q *are given by Lemma 4.8.*

Under the assumptions of Corollary 3 a result analogous to that of Corollary 2 is easy to prove.

5. Numerical Results

We consider the toy problem from [1, 13]. There the authors considered the moment equations for the linear kinetic equations in slab geometry (see e.g., [19]):

$$\partial_t f + v\partial_x f = \sigma_{\mathrm{s}}\left(\frac{1}{2}\langle f\rangle - f\right). \tag{5.1}$$

where $V = [-1,1]$ and $\sigma_{\mathrm{s}} \in [0,\infty)$. For the spatial domain we take $X = [0,1]$.

Remark 5.1. More generally, the kinetic equation includes terms for absorption of particles by a background medium as well as a source term, as in [1, 13]. In some test cases, absorption effects can push the moment solution towards the boundary of $\mathcal{R}$ and create challenges for the entropy-based moment method. However, for clarity of exposition we have left absorption terms out of our analysis because (a) we can push our test problems towards the boundary of $\mathcal{R}$ simply using the initial conditions and (b) the terms resulting from the absorption effects can be straightforwardly incorporated into our theoretical analysis and do not affect our main results.

The corresponding entropy-based moment equations are

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \sigma_{\mathrm{s}} R\mathbf{u}, \tag{5.2}$$

where $R = \operatorname{diag}\{0,-1,\dots,-1\}$. For the basis functions we used the Legendre polynomials. We used the Maxwell–Boltzmann entropy (4.16). The collision term $\mathbf{r}(\mathbf{u}) = \sigma_{\mathrm{s}} R\mathbf{u}$ clearly satisfies Assumption 4.

For numerical computations we used the fourth-order Runge–Kutta discontinuous Galerkin (RKDG) method as in [1] with 160 spatial cells and no slope limiter. With this spatial resolution the numerical solutions were accurate enough to observe the convergence in γ.

The initial conditions we used are constructed as follows. Let $\omega(x) := \frac{1}{2}M_0(1+\cos(2\pi x))$ be a periodic function which we use to define the multiplier vector

$$\boldsymbol{\beta}(x) = \begin{pmatrix} \log\left(\frac{\omega(x)}{2\sinh(\omega(x))}\right) \\ \omega(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.3}$$

Then the initial conditions are given by

$$\mathbf{u}^0(x) = \langle \mathbf{m}\exp(\boldsymbol{\beta}(x)\cdot\mathbf{m})\rangle. \tag{5.4}$$

Note that β_0 is chosen such that the zeroth order moment of the initial condition is one, i.e., $u_0^0(x) \equiv 1$. The solution $\mathbf{u}(t,x)$ of the original entropy-based moment equations with these initial conditions satisfies the assumption $\mathbf{u}(t,x) \in \mathcal{R}^M$ of Corollary 1 for $M \approx M_0$. Indeed the maximum value of $\|\hat{\boldsymbol{\alpha}}(\mathbf{u}(t,x))\|$ in space tends to decrease as time advances depending on the value of σ_{s}: the larger σ_{s}

is, the faster the norms of the multipliers decrease in time. For $\sigma_s = 0$, the value of $\max_x \|\hat{\boldsymbol{\alpha}}(\mathbf{u}(t,x))\|$ is nearly constant in time.

We ran the solutions until the final time $T = 0.1$. We used various values of N up to 15 and found that the results did not depend qualitatively on the value of N. We tried several values of σ_s from zero to one and here did observe that the results depended on the value of σ_s: for very small values of σ_s the solutions appear not to enter the regime of second-order convergence until γ is very small, on the order of 10^{-11}. Such values of γ are so small, that for these solutions the error due to the numerical optimizer started to dominate errors due to the regularization.

We remind the reader that in order to evaluate the flux function $\mathbf{f}_\gamma$ we compute the multiplier vector $\hat{\boldsymbol{\alpha}}_\gamma$ by numerically solving the dual problem (2.19). We used the numerical optimizer described in [1] but found that for problems with $\sigma_s = 0$ the value of tolerance τ on the norm of the dual gradient used in the stopping criterion, namely $\tau = 10^{-7}$, was not small enough to observe convergence in γ for the very small values of γ where the equations enter the regime of second-order convergence. The difficulty here is that, in our experience, one cannot reliably bring the norm of the dual gradient below 10^{-7} when the norm of the multipliers at the solution is about ten or bigger.

But for smaller values of M_0, we found that using a combination of the smaller tolerance $\tau = 10^{-8}$ as well as modifying the optimizer to make efforts to further reduce the norm of the dual gradient when possible allowed us to decrease the numerical errors from the optimizer enough so that we could observe near second-order convergence. This modification is described in pseudocode in Algorithm 1 and works as follows: The optimizer runs as usual until the norm of the dual gradient is smaller than τ. Then, the optimizer continues to take up to $\ell_{\max}$ additional iterations to bring the norm of the dual gradient under the smaller tolerance $\tau_d \in (0, \tau)$, which we call the *desired* tolerance. If the optimizer is unable to bring the norm of the dual gradient under τ_d in $\ell_{\max}$ additional iterations, the optimizer still exits successfully (as long as the current multiplier vector still satisfies the original stopping criterion). In all of the results reported here, we used $\tau_d = 10^{-11}$ and $\ell_{\max} = 10$.

Algorithm 1 The optimizer with modified stopping criterion

$k \leftarrow 0$
$\ell \leftarrow 0$
`acceptable_tolerance_achieved` $\leftarrow$ false
while $k < k_{\max}$ **do**
  if $(\|\hat{\mathbf{u}}(\boldsymbol{\alpha}_k) + \gamma\boldsymbol{\alpha}_k - \mathbf{u}\| < \tau_d)$ or $(\|\hat{\mathbf{u}}(\boldsymbol{\alpha}_k) + \gamma\boldsymbol{\alpha}_k - \mathbf{u}\| < \tau$ and $\ell > \ell_{\max})$ **then**
    return $\boldsymbol{\alpha}_k$
  end if
  if `acceptable_tolerance_achieved` $=$ false and $\|\hat{\mathbf{u}}(\boldsymbol{\alpha}_k) + \gamma\boldsymbol{\alpha}_k - \mathbf{u}\| < \tau$ **then**
    `acceptable_tolerance_achieved` $\leftarrow$ true
  end if
  Compute search direction $\mathbf{d}_k$
  Perform backtracking line search to determine backtracking parameter ξ_k
  $\boldsymbol{\alpha}_{k+1} \leftarrow \boldsymbol{\alpha}_k + \xi_k\mathbf{d}_k$
  $k \leftarrow k + 1$
  if `acceptable_tolerance_achieved` $=$ true **then**
    $\ell \leftarrow \ell + 1$
  end if
end while

The results are given in Tables 5.1 to 5.3, which include the errors between the solution of the original equations and the solutions of the regularized equations measured in the relative entropy as well as in the L^2 and L^∞ norms. The error measured in the relative entropy is

$$\mathcal{H}_\gamma(\mathbf{u}_\gamma|\mathbf{u}) := \int_X h_\gamma(\mathbf{u}_\gamma(T,x)|\mathbf{u}(T,x))\,dx, \tag{5.5}$$

where we compute $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ numerically using the formula

$$h_\gamma(\mathbf{u}_\gamma|\mathbf{u}) = \left\langle \eta(G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma)}|G_{\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})})\right\rangle + \frac{\gamma}{2}\|\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u}_\gamma) - \hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})\|^2, \tag{5.6}$$

where

$$\eta(G_{\boldsymbol{\alpha}}|G_{\boldsymbol{\beta}}) := \eta(G_{\boldsymbol{\alpha}}) - \eta(G_{\boldsymbol{\beta}}) - \eta'(G_{\boldsymbol{\beta}})(G_{\boldsymbol{\alpha}} - G_{\boldsymbol{\beta}}) \tag{5.7a}$$

$$= \eta(G_{\boldsymbol{\alpha}}) - \eta(G_{\boldsymbol{\beta}}) - (\boldsymbol{\beta}\cdot\mathbf{m})(G_{\boldsymbol{\alpha}} - G_{\boldsymbol{\beta}}). \tag{5.7b}$$

(For the second line we have used $G_{\boldsymbol{\beta}} = \eta_*'(\boldsymbol{\beta}\cdot\mathbf{m})$ and $\eta' \circ \eta_*' = \mathrm{id}$.) Formula (5.6) for $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ can be deduced by inserting (2.21) into (3.3) and simplifying, and it ensures the positivity of $h_\gamma(\mathbf{u}_\gamma|\mathbf{u})$ despite errors due to the approximate computation of $\hat{\boldsymbol{\alpha}}_\gamma(\mathbf{u})$. The spatial integrals are computed using an eight-point Gauss quadrature on each of four subintervals in each spatial cell. The observed convergence order ν between solutions computed with γ_1 and γ_2 is given by

$$\frac{\mathcal{H}_{\gamma_1}(\mathbf{u}_{\gamma_1}|\mathbf{u})}{\mathcal{H}_{\gamma_2}(\mathbf{u}_{\gamma_2}|\mathbf{u})} = \left(\frac{\gamma_1}{\gamma_2}\right)^\nu \tag{5.8}$$

The L^2 norm is computed using the same spatial quadrature, and the L^∞ norm is approximated by taking the maximum over these spatial quadrature points.

In Tables 5.1 and 5.2, second-order convergence is clear in the relative entropy until the value of the relative entropy reaches about 10^{-17}, which is below machine precision. These tables include varying values of M_0 and N. For $\sigma_\mathrm{s} \geq 10^{-5}$, we observed second-order convergence for all values of N up to 15 that we tried and for M_0 up to 200. For values of M_0 larger than 200, it is too difficult to satisfy the smaller optimization tolerance $\tau = 10^{-8}$. In all cases, we observe first-order convergence in the L^2 norm as well as the L^∞ norm.

For $\sigma_\mathrm{s} = 0$, we were only able to solve the equations for smaller values of M_0 and observed second-order convergence for a smaller range of values of γ. These results can be found in Table 5.3, where we have included results from additional values of γ between 10^{-9} and 10^{-11} to highlight the regime of second-order convergence. In Table 5.3, we see that the observed convergence orders increase monotonically to 1.99 and stay there until the value of the relative entropy goes below machine precision and the L^∞ norm is smaller than the optimization tolerance $\tau = 10^{-8}$. Indeed, at the final time, the optimizer was not able to solve many of the problems to the desired tolerance $\tau_d = 10^{-11}$, and in many of these problems, the tolerance τ is only barely fulfilled. Therefore errors on the order of 10^{-8} in the L^2 and L^∞ norms are not surprising, and this error of course also affects the computation of the relative entropy.

6. Conclusions

The regularized entropy-based moment method for kinetic equations keeps many of the desirable properties of the original entropy-based moment method but removes the requirement that the moment vector of the solution remains realizable. This facilitates the design and implementation of high-order numerical methods for the regularized moment equations. However, the regularized equations require the selection of a regularization parameter, and the error caused by regularization needs to be accounted for and balanced with other error sources. Our contribution is to rigorously prove the convergence as the regularization parameter goes to zero expected by formal arguments and to provide convergence rates. Numerical experiments show that these rates are indeed optimal.

γ	$\mathcal{H}_\gamma$	ν	L^2	ν	L^∞	ν
10^{-3}	3.977e-05	–	2.010e-03	–	3.939e-03	–
10^{-4}	5.319e-07	1.87	2.115e-04	0.98	3.970e-04	1.00
10^{-5}	5.439e-09	1.99	2.123e-05	1.00	3.948e-05	1.00
10^{-6}	5.454e-11	2.00	2.131e-06	1.00	3.971e-06	1.00
10^{-7}	5.504e-13	2.00	2.136e-07	1.00	3.667e-07	1.03
10^{-8}	5.557e-15	2.00	2.147e-08	1.00	4.422e-08	0.92
10^{-9}	5.886e-17	1.98	2.336e-09	0.96	5.671e-09	0.89
10^{-10}	2.082e-18	1.45	3.110e-10	0.88	1.002e-09	0.75

Table 5.1. Convergence test: $N = 9$, $\sigma_s = 1$, $M_0 = 100$. Parameters of the numerical optimizer: $\tau = 10^{-8}$, $\tau_d = 10^{-11}$, $\ell_{\max} = 10$.

γ	$\mathcal{H}_\gamma$	ν	L^2	ν	L^∞	ν
10^{-3}	1.196e-03	–	7.040e-03	–	2.040e-02	–
10^{-4}	2.769e-04	0.64	2.490e-03	0.45	1.587e-02	0.11
10^{-5}	3.442e-05	0.91	7.394e-04	0.53	6.236e-03	0.41
10^{-6}	1.691e-06	1.31	1.336e-04	0.74	2.020e-03	0.49
10^{-7}	2.033e-08	1.92	1.475e-05	0.96	2.168e-04	0.97
10^{-8}	2.043e-10	2.00	1.481e-06	1.00	2.164e-05	1.00
10^{-9}	2.130e-12	1.98	1.504e-07	0.99	2.199e-06	0.99
10^{-10}	2.362e-14	1.96	1.569e-08	0.98	2.287e-07	0.98
10^{-11}	2.380e-16	2.00	1.639e-09	0.98	2.415e-08	0.98

Table 5.2. Convergence test: $N = 5$, $\sigma_s = 0.01$, $M_0 = 150$. Parameters of the numerical optimizer: $\tau = 10^{-8}$, $\tau_d = 10^{-11}$, $\ell_{\max} = 10$.

γ	$\mathcal{H}_\gamma$	ν	L^2	ν	L^∞	ν
10^{-3}	9.833e-06	–	1.112e-03	–	1.738e-03	–
10^{-4}	8.398e-07	1.07	1.400e-04	0.90	2.146e-04	0.91
10^{-5}	6.116e-08	1.14	1.634e-05	0.93	2.483e-05	0.94
10^{-6}	3.136e-09	1.29	1.797e-06	0.96	2.636e-06	0.97
10^{-7}	1.092e-10	1.46	1.885e-07	0.98	2.674e-07	0.99
10^{-8}	2.255e-12	1.69	1.911e-08	0.99	2.679e-08	1.00
10^{-9}	2.639e-14	1.93	1.916e-09	1.00	2.679e-09	1.00
$10^{-9.25}$	8.420e-15	1.98	1.077e-09	1.00	1.509e-09	1.00
$10^{-9.5}$	2.676e-15	1.99	6.060e-10	1.00	8.477e-10	1.00
$10^{-9.75}$	8.499e-16	1.99	3.409e-10	1.00	4.776e-10	1.00
10^{-10}	2.704e-16	1.99	1.919e-10	1.00	3.017e-10	0.80
$10^{-10.25}$	8.711e-17	1.97	1.087e-10	0.99	1.886e-10	0.82
$10^{-10.5}$	2.950e-17	1.88	6.189e-11	0.98	1.391e-10	0.53
$10^{-10.75}$	9.893e-18	1.90	3.652e-11	0.92	1.182e-10	0.28
10^{-11}	4.185e-18	1.49	2.256e-11	0.84	9.941e-11	0.30
10^{-12}	1.801e-18	0.37	1.376e-11	0.21	6.393e-11	0.19

Table 5.3. Convergence test: $N = 15$, $\sigma_s = 0$, $M_0 = 8$. Parameters of the numerical optimizer: $\tau = 10^{-8}$, $\tau_d = 10^{-11}$, $\ell_{\max} = 10$.

Our results hold for wide classes of entropy functions including the Maxwell–Boltzmann entropy and the Bose–Einstein entropy. Our analysis relies on some key assumptions: The solution to the original moment equations needs to be Lipschitz and bounded away from the boundary of the set of realizable states.

Relaxing our assumptions would of course strengthen our results. One would like to be able to work with kinetic equations with unbounded velocity domains, but here the original moment equations have fundamental problems [14, 15, 16] which remain in the regularized equations. Nevertheless, the Euler equations, which are a case of the entropy-based moment method, do not have these problems and would be an interesting starting point for extending our analysis. One would also like to allow the solution to have values arbitrarily close to or even on the boundary of the realizable set, but not enough work has been done to consider the behavior of the moment equations near or on the boundary of the realizable set, such as in [8]. It is not even known whether the realizable set is invariant under the time evolution of the original entropy-based moment equations. Finally, requiring a Lipschitz continuous solution to the limiting system is typical for relative entropy estimates, see [9], and in multiple space dimensions this is connected with non-uniqueness of entropy solutions for certain moment systems such as the Euler equations.

Appendix A. Constants

Here we list some the constants which play the most significant roles throughout the paper. Each constant is a strictly positive real number.

C_0**:** Used to control the norm of a realizable moment vector using its zeroth entry: $\|\mathbf{u}\| \le C_0 u_0$ for all $\mathbf{u} \in \mathcal{R}$, where u_0 is the zeroth component of $\mathbf{u}$. Introduced in (4.2).

C_1**:** Global Lipschitz constant of $\mathbf{f}$. See (4.3).

C_2**:** Global Lipschitz constant of $\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma$; see (4.7).

C_3**:** $\sup_{\gamma \in (0,\gamma_0)} \|\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(0)\|$, see (4.8). This is used to get the affine bound in (4.9).

C_4**:** Used when bounding J_γ for $(\mathbf{u}_\gamma, \mathbf{u}) \in \mathcal{R}^L \times \mathcal{R}^M$. Equal to $C_1 C_{h'',\max,M} + C_{\mathbf{f}'',M} M C_2$; see (4.35).

u_M**:** Upper bound on $\|\mathbf{u}\|$ in $\mathcal{R}^M$; see (4.21).

h_M**:** Upper bound on $h(\mathbf{u})$ in $\mathcal{R}^M$; see (4.21).

$\lambda_{\min,h'',M}$**:** Lower bound on the smallest eigenvalue of h''_γ over $\mathcal{R}^M$ and $\gamma \in (0, \gamma_0)$; see (4.24) and for the corresponding constant in the sublinear case (4.87).

$C_{h'',M}$**:** Upper bound on $\|h''_\gamma\|$ over $\mathcal{R}^M$ and $\gamma \in (0, \gamma_0)$; see (4.24) and for the corresponding constant in the sublinear case (4.87).

$C_{\mathbf{f}'',L}$**:** Bound on the $\mathbf{f}''_\gamma$ over $\mathrm{Conv}(\mathcal{R}^L)$ and $\gamma \in (0, \gamma_0)$; see (4.32).

$C_{h,M,L}$**:** Lower bound on $h_\gamma(\mathbf{v}|\mathbf{u})$ for $(\mathbf{v}, \mathbf{u}) \in (\mathbb{R}^{N+1} \backslash \mathcal{R}^L) \times \mathcal{R}^M$ see Lemma 4.6 and Appendix D.

$C_{\mathbf{r}}$**:** Lipschitz constant for $\mathbf{r}$; see Assumption 4.

Appendix B. Entropy relationships

In this appendix we quickly review the computations from [18] showing the relationships between h, h_*, $\hat{\mathbf{u}}$, and $\hat{\boldsymbol{\alpha}}$, in particular the key result that $h' = \hat{\boldsymbol{\alpha}}$. Start with the definition of the entropy h,

$$h(\mathbf{u}) := \min_{g \in \mathbb{F}(V)} \left\{ \langle \eta(g) \rangle : \langle \mathbf{m} g \rangle = \mathbf{u} \right\}, \tag{B.1}$$

i.e., the minimal value of the primal problem (2.9) as a function of the moment vector. The corresponding Lagrangian is given by

$$L(g, \boldsymbol{\alpha}) := \langle \eta(g) \rangle + \boldsymbol{\alpha} \cdot (\mathbf{u} - \langle \mathbf{m} g \rangle), \tag{B.2}$$

and thus the dual problem is

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}} \min_{g \in \mathbb{F}(V)} L(g, \boldsymbol{\alpha}) = \max_{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}} \boldsymbol{\alpha} \cdot \mathbf{u} - \langle \eta_*(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle, \tag{B.3}$$

cf. (2.12), where to get this equality one takes the minimization inside the integral and applies the definition of the Legendre dual of η. Because the duality gap is zero [14, Thm. 16] we have

$$h(\mathbf{u}) = \max_{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}} \left\{ \boldsymbol{\alpha} \cdot \mathbf{u} - \langle \eta_*(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle \right\}, \tag{B.4}$$

so h is the Legendre transformation of

$$h_*(\boldsymbol{\alpha}) := \langle \eta_*(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle. \tag{B.5}$$

Its derivative is readily computed:

$$h_*'(\boldsymbol{\alpha}) = \langle \mathbf{m} \eta_*'(\boldsymbol{\alpha} \cdot \mathbf{m}) \rangle =: \hat{\mathbf{u}}(\boldsymbol{\alpha}). \tag{B.6}$$

Now, recall that $\hat{\boldsymbol{\alpha}}(\mathbf{u})$ in (2.12) is defined to be the multiplier vector that solves the dual problem (B.3). Then the first-order necessary conditions for the dual problem imply

$$\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}(\mathbf{u})) = \mathbf{u}. \tag{B.7}$$

The reverse, i.e., $\hat{\boldsymbol{\alpha}}(\hat{\mathbf{u}}(\boldsymbol{\alpha})) = \boldsymbol{\alpha}$, is a consequence of the uniqueness of the solution to the dual problem (thanks to convexity). Thus $\hat{\boldsymbol{\alpha}}$ is the inverse function of $\hat{\mathbf{u}}$. Finally, since the derivative of Legendre duals are inverses of each other, we have

$$h' = (h_*')^{-1} = \hat{\mathbf{u}}^{-1} = \hat{\boldsymbol{\alpha}}. \tag{B.8}$$

Appendix C. The regularized solution for zero vector as $\gamma \to 0$

In this section we quickly consider

$$\lim_{\gamma \to 0} \hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(0)), \tag{C.1}$$

where $0 \in \mathbb{R}^{N+1}$. This comes up in Section 4.1 when deriving global estimates on the function $\hat{\mathbf{u}} \circ \hat{\boldsymbol{\alpha}}_\gamma$ using Lipschitz continuity.

For convenience we assume that the basis functions are orthogonal to each other, which since $m_0 \equiv 1$ (recall Assumption 3) in particular implies that $\langle m_i \rangle = \langle m_0 m_i \rangle = 0$ for all $i \in \{1, \ldots, N\}$. As a consequence, most of the components of $\hat{\boldsymbol{\alpha}}_\gamma(0)$ are easy to determine. Consider the first-order necessary conditions:

$$0 = \langle m_i \eta_*'(\hat{\boldsymbol{\alpha}}_\gamma(0) \cdot \mathbf{m}) \rangle + \gamma \hat{\alpha}_{\gamma,i}(0), \qquad i \in \{0, 1, \ldots, N\}. \tag{C.2}$$

If we set $\hat{\alpha}_{\gamma,i}(0) = 0$ for $i \in \{1, \dots, N\}$, then the entropy ansatz is constant in v, and by orthogonality of $\{m_i\}$, we see that the first-order necessary conditions are satisfied for $i \in \{1, \dots, N\}$. It remains to determine the zeroth component $\hat{\alpha}_{\gamma,0}(0)$, for which we need to solve

$$0 = |V|\eta_*'(\hat{\alpha}_{\gamma,0}(0)) + \gamma\hat{\alpha}_{\gamma,0}(0). \tag{C.3}$$

From this equation, it is clear that $\hat{\alpha}_{\gamma,0}(0) < 0$ and thus that $\hat{\alpha}_{\gamma,0}(0) \to -\infty$ monotonically (recall (4.12)) as $\gamma \to 0$. The limiting value must be unbounded because $0 \notin \mathcal{R}$. Now we recall that for the η considered in this work we have $\mathrm{Range}(\eta_*') = \mathrm{Dom}(\eta') = (0, \infty)$, and furthermore that η_*' is a monotonically increasing function because η_* is convex. Therefore $\eta_*'(\hat{\alpha}_{\gamma,0}(0)) \to 0$ as $\gamma \to 0$. It follows that also $\hat{\mathbf{u}}(\hat{\boldsymbol{\alpha}}_\gamma(0)) \to 0$.

Appendix D. Proof of Lemma 4.6

Let $L \in (M, \infty)$. We want to show

$$C_{h,M,L} := \inf_{\substack{\mathbf{v}\in\mathbb{R}^{N+1}\setminus\mathcal{R}^L \\ \mathbf{u}\in\mathcal{R}^M \\ \gamma\in(0,\gamma_0)}} h_\gamma(\mathbf{v}|\mathbf{u}) > 0. \tag{D.1}$$

The basic idea is that, by strict convexity of $h_\gamma(\mathbf{v}|\mathbf{u})$ in its first argument, it only achieves its minimum value, zero, when $\mathbf{v} = \mathbf{u}$. But this is ruled out on $(\mathbf{v}, \mathbf{u}) \in (\mathbb{R}^{N+1} \setminus \mathcal{R}^L) \times \mathcal{R}^M$ because $\overline{\mathbb{R}^{N+1} \setminus \mathcal{R}^L} \cap \mathcal{R}^M = \emptyset$.

We can get a more explicit bound as follows. Let $Q \in (M, L)$. We claim that for every $\mathbf{v} \in \mathbb{R}^{N+1}\setminus\mathcal{R}^L$ and $\mathbf{u} \in \mathcal{R}^M$ there exists a $\lambda_Q \in (0, 1)$ such that

$$\mathbf{w}_Q := (1 - \lambda_Q)\mathbf{u} + \lambda_Q\mathbf{v} \in \mathcal{R}^L \setminus \mathcal{R}^Q. \tag{D.2}$$

For $\mathbf{v} \in \mathcal{R} \setminus \mathcal{R}^L$ this straightforward, because the function

$$f(\lambda) = \|\hat{\boldsymbol{\alpha}}((1 - \lambda)\mathbf{u} + \lambda\mathbf{v})\| \tag{D.3}$$

is a continuous function with $f(0) \le M$ and $f(1) \ge L$. For $\mathbf{v} \in \mathbb{R}^{N+1} \setminus \mathcal{R}$, by convexity of $\mathcal{R}$ there exists a unique $\lambda_\mathcal{R} \in (0, 1)$ such that $(1 - \lambda_\mathcal{R})\mathbf{u} + \lambda_\mathcal{R}\mathbf{v} \in \partial\mathcal{R}$. But then, since $\mathcal{R}^L \subset\subset \mathcal{R}$, there must also be a $\lambda_L \in (0, \lambda_\mathcal{R})$ such that $(1 - \lambda_L)\mathbf{u} + \lambda_L\mathbf{v} \in \mathcal{R} \setminus \mathcal{R}^L$, and so the first argument can be applied again.

Now, for any $(\mathbf{w}, \mathbf{u}) \in (\mathcal{R}^L \setminus \mathcal{R}^Q) \times \mathcal{R}^M$ we have

$$h_\gamma(\mathbf{w}|\mathbf{u}) \overset{(4.38)}{\ge} \lambda_{\min,h'',\widetilde{L}}\|\mathbf{w} - \mathbf{u}\|^2 \ge \lambda_{\min,h'',\widetilde{L}} \inf_{\substack{\mathbf{w}\in\mathcal{R}^L\setminus\mathcal{R}^Q \\ \mathbf{u}\in\mathcal{R}^M}} \|\mathbf{w} - \mathbf{u}\|^2 =: C_L > 0 \tag{D.4}$$

where the strict positivity follows from $\mathcal{R}^M \cap \overline{\mathcal{R}^L \setminus \mathcal{R}^Q} = \emptyset$.

Finally, let $(\mathbf{v}, \mathbf{u}) \in (\mathbb{R}^{N+1} \setminus \mathcal{R}^L) \times \mathcal{R}^M$ and $\mathbf{w}_Q$ be as in (D.2). By convexity of the relative entropy in its first argument, we have

$$(1 - \lambda_Q)h_\gamma(\mathbf{u}|\mathbf{u}) + \lambda_Q h_\gamma(\mathbf{v}|\mathbf{u}) \ge h_\gamma(\mathbf{w}_Q|\mathbf{u}) \ge C_L. \tag{D.5}$$

But with $h_\gamma(\mathbf{u}|\mathbf{u}) = 0$ and $\lambda_Q \in (0, 1)$, we immediately have $h_\gamma(\mathbf{v}|\mathbf{u}) \ge C_L$, and by taking the infimum as in (D.1) we conclude $C_{h,M,L} \ge C_L$.

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