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S E M I N A I R E   S U R

L'ANALYSE DIOPHANTIENNE ET SES APPLICATIONS

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MULTIDIMENSIONAL TWO-PARTICLES PROBLEM WITH NON-CENTRAL POTENTIAL

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It is well-known that multidimensional two-body problem with the central potential (generalized Keplerian problem) is completely integrable [1]. On the other side, when the potential is non-central and e.g. has the form $U(x_1,\dots,x_n) = V(r) + \sum_{i=1}^{n} \lambda_i x_i^2$ i.e. there are non-central linear forces, the situation changes. Very general results of Poincaré [2] show in this case, that under weak assumptions the corresponding two-body system, with periodic $V(r)$, cannot have additional first integrals analytical both in $x_1,\dots,x_n$ and $\lambda_1,\dots,\lambda_n$. Usual examples corresponding to the case $V(r) = r^{-1}$ (the Newton case), $V(r) = r^3$ and $n = 2$ show the non-existence of first integrals even in the simplest cases. However in the case of quadratic $V(r)$ the system is completely integrable and for $V(r) = r^{-2}$ (the Jacobi case) the system possesses one additional algebraic first integral. In this note we show that for $V(r) = r^4$ and arbitrary linear non-central forces the Hamiltonian system for two n-dimensional particles is completely integrable. It should be remarked that for the potential $V(r) = r^4$ the system of three particles in n-dimensional space in the absence of non-central forces possesses one additional algebraic first integral, through for $n \geq 3$ it is still unknown whether the corresponding system is completely integrable [3]. This note continues our previous one [3] and uses the same ideas.

Let $n \geq 2$, $\lambda_1,\dots,\lambda_n$ be arbitrary complex numbers. We consider a Hamiltonian system describing the motion of two bodies in n-dimensional space interacting via central potential r^4 and additional linear non-central forces :

$$(1) \qquad H_1 = \sum_{i=1}^{n} (p_{i1}^2 + p_{i2}^2) + \Big(\sum_{i=2}^{n} (q_{i1} - q_{i2})^2\Big)^2 - \sum_{i=1}^{n} \lambda_i (q_{i1} - q_{i2})^2 \; .$$

As all the projections of centre of masses-momentum are integrals of the system (1), then the Hamiltonian (1) can be reduced, as usually, to the problem of the motion of a particle in non-central potential field. The corresponding Hamiltonian is :

$$(2) \qquad H = \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \Big(\sum_{i=1}^{n} q_i^2\Big) - \sum_{i=1}^{n} \lambda_i q_i^2 \; ,$$

and the system of equations has the form :

(3) $$\ddot{q}_i = \lambda_i q_i - q_i\left(\sum_{j=1}^{n} q_j^2\right) : i = 1,\ldots,n \quad .$$

As we have proposed in the paper [3] the system (3) can be investigated using quadratic moments. Taking into account the non-centrality of potential we use the following quadratic moments :

(4) $$I_m = \sum_{i=1}^{n} \lambda_i^m q_i^2 : m = 0,1,2,\ldots \quad .$$

Besides these quadratic moments we consider also the following ones :

(5) $$I_m^{(2)} = \sum_{i=1}^{n} \lambda_i^m q_i \dot{q}_i \quad ;$$
$$I_m^{(3)} = \sum_{i=1}^{n} \lambda_i^m \dot{q}_i^2 : m = 0,1,2,\ldots \quad .$$

These systems of moments satisfy the following systems of differential equations

<u>Lemma 1</u> : According to (3) we have

(6) $$\dot{I}_m = 2I_m^{(2)} \quad ; \quad \dot{I}_m^{(2)} = I_m^{(3)} + I_{m+1} - I_m \cdot I_o \quad ;$$
$$\dot{I}_m^{(3)} = 2I_{m+1}^{(2)} - 2I_m^{(2)} I_o : m = 0,1,2,\ldots \quad .$$

Now it is possible to define the system of the first integrals of (3) algebraic in λ_i, q_i, p_i (or λ_i, q_i, $\dot{q}_i$). In fact these integrals are simply algebraic in I_m, $I_m^{(2)}$, $I_m^{(3)}$.

<u>Theorem 2</u> : There are n independent first integrals of the system (2), algebraic in λ_i, q_i, p_i .

We shall not only prove the existence of these integrals, but we'll also provide a recursive scheme of obtaining the sequence of such integrals valid for any n and, in fact, even for an infinite number of variables.

From the given moments (4), (5) we can construct analogous moments with coefficients-polynomials in λ_i In fact if $P_\ell(x)$ is a polynomial of degree ℓ, then we denote

$$I^{(1)}(P_\ell) = \sum_{i=1}^{n} P_\ell(\lambda_i)q_i^2 \quad ,$$

(7) $$I^{(2)}(P_\ell) = \sum_{i=1}^{n} P_\ell(\lambda_i)q_i\,\dot{q}_i \quad ,$$

$$I^{(3)}(P_\ell) = \sum_{i=1}^{n} P_\ell(\lambda_i)\dot{q}_i^2 \quad .$$

It is clear that any moment $I^{(j)}(P_\ell)$ is a linear combination of moments $I_o^{(j)},\dots,I_\ell^{(j)}$. Thus from lemma 1 we obtain the following usefu identities :

<u>Lemma 3</u> : Let $P(x)$ be a polynomial and $Q(x) = P(x).x$. Then

$$\frac{d}{dt} I^{(1)}(P) = 2I^{(2)}(P) \quad ;$$

(8) $$\frac{d^2}{dt^2} I^{(1)}(P) = 2I^{(3)}(P) + 2I^{(1)}(Q) - 2I_o \,.\, I^{(1)}(P) \quad ;$$

$$\frac{d^3}{dt^3} I^{(1)}(P) = 8I^{(2)}(Q) - 8I_o \,.\, I^{(2)}(P) - 4I_o^{(2)} \,.\, I^{(1)}(P) \quad .$$

The identities (8) follow immediately from Lemma 1 and (7).

Let C_ℓ : $\ell = 0,1,2,\dots$ be an infinite sequence of constants. We define an infinite sequence of polynomials $P_\ell(x)$: $\ell = 0,1,2,\dots$ of degree ℓ by the following inductive procedure :

(9) $$P_{\ell+1}(x) = P_\ell(x)x + 8C_o\, P_\ell(x) + C_\ell \;:\; \ell = 0,1,2,\dots \quad .$$

We leave undefined the constant P_o -the polynomial of degree 0 (having a deep metaphysical meaning).

Now we are ready to define the basic sequence of polynomials, quadratic in q_i and of degree ℓ in λ_i : $\mathcal{R}_\ell$.

<u>Definition 4</u> : For a given sequence of polynomials $P_\ell(x)$ defined by (9) and undetermined P_o, in accordance with the notations of (7) we put

(10) $$\mathcal{R}_\ell = I^{(1)}(P_\ell) + 2C_\ell = \sum_{i=1}^{n} P_\ell(\lambda_i)q_i^2 + 2C_\ell \;:\; \ell = 0,1,2,\dots \quad .$$

We shall show below that, in fact, the constants C_ℓ correspond to the values of algebraic integrals of (3). To do this we define by induction the infinite sequence of polynomials in $I_m^{(j)}$ and C_o . Let us show that in fact this sequence coincides with $\mathcal{R}_\ell$. So we define this sequence S_ℓ : $\ell = 0,1,2,\dots$ by induction :

Definition 5 : We put $S_o = 1/2$; $S_1 = 1/4\, I_o + 2C_o$ and for $\ell = 1,2,\dots$

$$(11)\qquad \begin{aligned} S_{\ell+1} = {} & 2\sum_{k=0}^{\ell-1} S_k \frac{d^2}{dt^2} S_{\ell-k} - \sum_{k=1}^{\ell-1} \frac{d}{dt} S_k \cdot \frac{d}{dt} S_{\ell-k} + \\ & + 16 S_1 \sum_{k=0}^{\ell} S_k S_{\ell-k} - 4 \sum_{k=1}^{\ell} S_k S_{\ell-k+1} \quad . \end{aligned}$$

From the definition we obtain

Lemma 6 : For $\ell = 0,1,2,\dots$ S_ℓ is a polynomial in I_o, C_o, $I_m^{(j)}$: $j = 1,2,3$, $m = 0,1,2,\dots$. So S_ℓ is a polynomial in p_i, q_j and C_o.

Proof : From the formulaes (6) we obtain that the ring $Q[I_m^{(j)} : j = 1,2,3 \ ; \ m = 0,1,2,\dots]$ is mapped by the differentiation $\frac{d}{dt}$ into itself. Here and below we put for simplicity $I_m^{(1)} = I_m$. Thus we obtain in (11) $S_{\ell+1}$ as a polynomial in $S_o,\dots,S_\ell$, $I_m^{(j)}$ and the lemma is proved by induction.

In fact, the S_ℓ have a much more simple structure. In order to present it we use

Lemma 7 : The sequence S_ℓ : $\ell = 0,1,2,\dots$ satisfies the following recurrent relation

$$(12)\qquad \frac{d}{dt} S_{\ell+1} = \frac{1}{4} \frac{d^3}{dt^3} S_\ell + 4S_1 \frac{d}{dt} S_\ell + I_o^{(2)} S_\ell$$

for any $\ell = 0,1,2,\dots$.

Lemma 7 is true indeed for $\ell = 0$. There is one possibility to prove (12) just looking at (11). But in the Appendix A to the paper, we'll give a short proof of lemma 7 using Riccati equation (cf. Gelfand [4]).

On the other hand the system $\mathfrak{R}_\ell$ also satisfies (12) :

Lemma 8 : For any $\ell = 0,1,2,\dots$

$$(13)\qquad \frac{d}{dt} \mathfrak{R}_{\ell+1} = \frac{1}{4} \frac{d^3}{dt^3} \mathfrak{R}_\ell + (I_o + 8C_o) \frac{d}{dt} \mathfrak{R}_\ell + I_o^{(2)} \mathfrak{R}_\ell \quad .$$

Proof : By (8) and (10) we have

$$\frac{d}{dt}\Re_{\ell+1} = 2I^{(2)}(P_{\ell+1}) \quad ;$$

$$\frac{1}{4}\frac{d^3}{dt^3}\Re_\ell = 2I^{(2)}(P_\ell \cdot x) - 2I_o I^{(2)}(P_\ell) - I_o^{(2)} I^{(1)}(P_\ell) \quad ;$$

(14)

$$(I_o + 8C_o)\frac{d}{dt}\Re_\ell = 2I_o I^{(2)}(P_\ell) + 16C_o\ I^{(2)}(P_\ell) \quad ;$$

$$I_o^{(2)}\Re_\ell = I_o^{(2)} I^{(1)}(P_\ell) + 2C_\ell\ I_o^{(2)} \quad .$$

Taking into account (9) and (7) we obtain

$$I^{(2)}(P_{\ell+1}) = I^{(2)}(P_\ell \cdot x) + 2C_\ell\ I_o^{(2)} + 8C_o\ I^{(2)}(P_\ell) \quad .$$

Thus (14) gives us immediately (13). The lemma is proved.

Now we have the main lemma in the proof of Theorem 1 :

Lemma 9 : For some constants C_1, C_2, $C_3, \ldots$ we have $S_{\ell+1} = \Re_\ell$ for $\ell = 1,2,\ldots$.

Remark : If we put the undetermined constant P_o to be 1/4, then we have

(15) $$S_{\ell+1} = \Re_\ell \text{ for any } \ell = 0,1,2,\ldots \quad .$$

Proof : First of all, for $P_o = \frac{1}{4}$ we have $\Re_o = \frac{1}{4} I_o + 2C_o = S_1$. Thus (15) is valid for $\ell = 0$. Let us suppose that there are constants $C_1,\ldots,C_\ell$ such that

(16) $$S_{m+1} = \Re_m \quad \text{for } m = 0,1,\ldots,\ell \quad .$$

We define $C_{\ell+1}$ in such a way that $S_{\ell+2} = \Re_\ell$. By (9) $P_{\ell+1}(x)$ is already defined , so $I^{(1)}(P_\ell)$ is defined. According to (13),

$$\frac{d}{dt} I^{(1)}(P_\ell) = \frac{1}{4}\frac{d^3}{dt^3}\Re_\ell + 4S_1\frac{d}{dt}\Re_\ell + I_o^{(2)}\Re_\ell \quad .$$

Taking into account (12) and (16) for $m = \ell$ we obtain

(17) $$\frac{d}{dt} I^{(1)}(P_\ell) = \frac{d}{dt} S_{\ell+2} \quad .$$

So there is a constant $C_{\ell+1}$ such that $\Re_{\ell+1} = I^{(1)}(P_\ell) + 2C_{\ell+1}$ and

$$\Re_{\ell+1} = S_{\ell+2} \quad .$$

Lemma 9 is proved.

The constants C_ℓ : $\ell = 1,2,...$ are exactly the integrals of (3). It should be mentioned once more that we have never used the finiteness of dimension (3).

We shall give exact expressions for the constants C_m. In fact, from Lemma 9 and (10) it is easily seen that C_ℓ : $\ell = 1,2,...$ are algebraic first integrals of the system (3). We shall summarize the previous inductive schemes in order to obtain a general form for the integrals C_m.

<u>Definition 10</u> : We define by induction the system $\tilde{C}_\ell$: $\ell = 1,2,...$ of polynomials in p_i, q_i, λ_i using the system S_ℓ in the following way.

$$\tilde{C}_1 = \frac{1}{2}\{S_2 - \sum_{i=1}^{n} (\frac{1}{4}\lambda_i + 3C_o)q_i^2\} \quad ,$$

or with the notations of (7),

(18) $$\tilde{C}_1 = \frac{1}{2}\{S_2 - I^{(1)}(\tilde{P}_1)\} \quad , \quad \tilde{P}_1(x) = \frac{1}{4}x + 3C_o \quad .$$

Now we define by induction the polynomials $\tilde{P}_\ell(x)$ -the coefficients of which are combinations of C_o, $\tilde{C}_1,...,\tilde{C}_{\ell-1}$. We put

$$\tilde{P}_o(x) = \frac{1}{4} \quad ; \quad \tilde{P}_1(x) = \frac{1}{4}x + 3C_o$$

and if $\tilde{C}_1,...,\tilde{C}_\ell$ are defined, then for $\ell = 1,2,...$

(19) $$\tilde{P}_{\ell=1}(x) = \tilde{P}_\ell(x) \cdot x + 8C_o \tilde{P}_\ell(x) + \tilde{C}_\ell$$

Thus if all $\tilde{C}_1,...,\tilde{C}_\ell$ are already defined we obtain $\tilde{P}_{\ell+1}(x)$ from (19) and put

$$\tilde{C}_{\ell+1} = \frac{1}{2}\{S_{\ell+2} - I^{(1)}(\tilde{P}_{\ell+1})\} = \tag{20}$$
$$= \frac{1}{2}\{S_{\ell+2} - \sum_{i=1}^{n} (\tilde{P}_\ell(\lambda_i)\lambda_i + 8C_o\,\tilde{P}_\ell(\lambda_i) + \tilde{C}_\ell)q_i^2\} \quad .$$

From Definition 4 and Lemma 9 it is easily seen :

<u>Lemma 11</u> : All the polynomials $\tilde{C}_\ell$: $\ell = 1,2,\ldots$ are algebraic first integrals of (3).

For a given n it easily follows from Definitions 5 and 10 that $\tilde{C}_\ell$: $\ell = 1,2,\ldots,n$ are functionally independent. In fact the $\tilde{C}_\ell$ are in involution. This can be proved by algebraic arguments using the precise form of S_ℓ from (11) (cf. Appendix). It should be noted that $\tilde{C}_1$ is indeed the Hamiltonian H from (2). We give also the expression for the second integral $\tilde{C}_2$:

$$(\sum_{i=1}^{n} q_i^2)(\sum_{i=1}^{n} p_i^2) + (\sum_{i=1}^{n} q_i^2)(\sum_{i=1}^{n} \lambda_i\, q_i^2) - \tag{21}$$
$$- (\sum_{i=1}^{n} q_i\, p_i)^2 + 2(\sum_{i=1}^{n} \lambda_i\, p_i^2 - \sum_{i=1}^{n} \lambda_i^2\, q_i^2) \quad .$$

Similar formulaes can be easily written for any $\tilde{C}_\ell$.

APPENDIX A

<u>The proof of formula (12)</u>

We use as a generating function for S_ℓ the resolvent of Sturm-Liouville operator :

$$S(t,z) = \sum_{\ell=0}^{\infty} S_\ell(t)\, .\, Z^{-\ell-\frac{1}{2}} \quad . \tag{22}$$

Let us consider S(t,z) satisfying (at least formally) the following Riccati equation

$$-2SS_{tt} + (S_t)^2 + 4zS^2 - 16S_1S^2 = 1 \quad . \tag{23}$$

It easily follows from Definition 5 and (11) that S defined by (22) satisfies (23). Differentiating (23) we obtain the following linear in S third order equation

(24) $$-S_{ttt} - 8\frac{d}{dt}S_1 \cdot S - 16S_1S_t + 4zS_t = 0 \quad .$$

As $\frac{d}{dt}S_1 = \frac{1}{2}I_o^{(2)}$, then we obtain from (24)

(25) $$-S_{ttt} - 4I_o^{(2)}S - 16S_1S_t + 4zS_t = 0 \quad .$$

Substituting (22) into (25) we obtain for $\ell = 0,1,2,\ldots$

(26) $$\frac{d}{dt}S_{\ell+1} = \frac{1}{4}\frac{d^3}{dt^3}S_\ell + I_o^{(2)}S_\ell + 4S_1\frac{d}{dt}S_\ell \quad .$$

It is clear from definition 5 that (26) coincides with (12). This completes the proof of Lemma 7.

APPENDIX B

Connection between $\tilde{C}_1$ and H.

By (11) we have :

$$S_2 = \frac{1}{16}(I_o^{(1)2} + 2I_o^{(3)} + 2I_1^{(1)} + 48I_o^{(1)}C_o + 192C_o^2 + 8C_o)$$

and so $$+ 192C_o^2$$

$$\tilde{C}_1 = \frac{1}{32}I_o^{(1)2} + \frac{1}{16}I_o^{(3)} - \frac{1}{16}I_1^{(1)} + 6C_o^2 + \frac{1}{4}C_o = \frac{1}{16}H + 6C_o^2 + \frac{1}{4}C_o \quad .$$

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