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PRECISED HARDY INEQUALITIES ON $\mathbf{R}^d$ AND ON THE HEISENBERG GROUP $\mathbf{H}^d$

HAJER BAHOURI, JEAN-YVES CHEMIN, AND ISABELLE GALLAGHER

1. Introduction

The aim of this text is to present a proof of a "precised" version of the Hardy inequalities [12], [13]. Those inequalities have a very big importance in Analysis (among other applications we can mention blow-up methods, or the study of pseudodifferential operators with singular coefficients). Many works have been devoted to those inequalities, and our goal is first to provide a new, elementary proof of the standard Hardy inequality, and then to prove a precised inequality in the spirit of the precised Sobolev inequality [11]. The setting will be both the classical $\mathbf{R}^d$ space, as well as the Heisenberg group $\mathbf{H}^d$ (for an application of the Hardy inequality to the Heisenberg group we refer for instance to [3]).

1.1. **The $\mathbf{R}^d$ case.** Let us recall the standard inequality in $\mathbf{R}^d$: let s be a real number in the interval $]0, d/2[$. There is a constant C such that for any function $u \in \dot{H}^s(\mathbf{R}^d)$, the following inequality holds:

$$\int_{\mathbf{R}^d} \frac{|u(x)|^2}{|x|^{2s}}\,dx \leq C\|u\|^2_{\dot{H}^s(\mathbf{R}^d)}, \tag{1.1}$$

where the space $\dot{H}^s(\mathbf{R}^d)$ denotes the homogeneous Sobolev space

$$\dot{H}^s(\mathbf{R}^d) = \{u \in \mathcal{S}'(\mathbf{R}^d) \,/\, \widehat{u} \in L^1_{loc}(\mathbf{R}^d) \quad \text{and} \quad \|u\|_{\dot{H}^s(\mathbf{R}^d)} < +\infty\} \tag{1.2}$$

while $\widehat{u}$ denotes the Fourier transform of u and

$$\|u\|^2_{\dot{H}^s(\mathbf{R}^d)} \stackrel{\text{def}}{=} \int_{\mathbf{R}^d} |\xi|^{2s}|\widehat{u}(\xi)|^2\,d\xi. \tag{1.3}$$

Remarks.

1) When $s = 1$ and $d \geq 3$, Hardy inequality derives immediately by integration by parts against the radial vector field $R = x \cdot \nabla$.
For non integer values of s, the proof is much more delicate and requires the proof of the L^2 continuity of the operator

$$\frac{(-\Delta)^{-\frac{s}{2}}}{|x|^s}\cdot$$

2) The basic tool of the proof presented in this work is the paradifferential calculus of J.-M. Bony [6].

3) Let us now have a look at the scaling properties of Hardy inequalities. It is easy to see that inequality (1.1) is invariant under the scaling $u_\lambda(x) \stackrel{\text{def}}{=} \lambda^{\frac{d}{2}-s}u(\lambda x)$ while it is not under translation and oscillation.

4) To obtain translation invariance, it suffices to consider the more general Hardy inequality

$$\sup_a \int_{\mathbf{R}^d} \frac{|u(x)|^2}{|x-a|^{2s}}\,dx \leq C\|u\|^2_{\dot{H}^s(\mathbf{R}^d)}, \tag{1.4}$$

5) The oscillation invariance holds for the precised Hardy inequality (1.12) proved in this text. The detailed proof of that result can be found in [1] (see also [2] for an announcement).

In order to state this "precised" inequality, let us recall the definition of homogeneous Besov spaces. It requires the definition of Littlewood-Paley operators. Let us then begin by recalling the basis of this theory (for more details, we can consult [6] or [7]).

Proposition 1. *Let us denote by $\mathcal{C}$ the ring of center 0, of small radius $3/4$ and great radius $8/3$. There exist two radial functions χ and φ the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(\mathcal{B}(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$ such that*

$$\forall \xi \in \mathbf{R}^d\,,\ \chi(\xi) + \sum_{q\geq 0}\varphi(2^{-q}\xi) = 1, \tag{1.5}$$

$$\forall \xi \in \mathbf{R}^d \setminus \{0\}\,,\ \sum_{q\in\mathbf{Z}}\varphi(2^{-q}\xi) = 1, \tag{1.6}$$

$$|p-q| \geq 2 \Rightarrow \mathit{supp}\ \varphi(2^{-q}\cdot) \cap \mathit{supp}\ \varphi(2^{-p}\cdot) = \emptyset, \tag{1.7}$$

$$q \geq 1 \Rightarrow \mathit{supp}\ \chi \cap \mathit{supp}\ \varphi(2^{-q}\cdot) = \emptyset. \tag{1.8}$$

Remarks.

1) If we denote by

$$\Delta_q u = \varphi(2^{-q}D)u = 2^{qd}\int h(2^q(x-y))u(y)dy.$$

where $h = \mathcal{F}^{-1}\varphi$, the operators Δ_q map L^p onto L^p and commute with the derivatives.

2) We shall denote by

$$S_q u = \sum_{p\leq q-1}\Delta_p u.$$

We can prove that

$$S_q u = \chi(2^{-q}D)u = 2^{qd}\int \widetilde{h}(2^q(x-y))u(y)dy.$$

where $\widetilde{h} = \mathcal{F}^{-1}\chi$.

3) Moreover the operators Δ_q satisfy the classical Bernstein inequalities

$$\forall \quad 1 \leq a \leq b \leq \infty, \quad \|\Delta_q u\|_{L^b} \leq C2^{dq(\frac{1}{a}-\frac{1}{b})}\|\Delta_q u\|_{L^a}, \tag{1.9}$$

$$(1.10) \qquad \forall \quad 1 \le a \le b \le \infty, \quad \|S_q u\|_{L^b} \le C 2^{dq(\frac{1}{a}-\frac{1}{b})} \|S_q u\|_{L^a},$$

and

$$(1.11) \qquad \forall \quad 1 \le p \le \infty, \quad \forall k \in \mathbf{N}, \quad \frac{1}{C_k} 2^{kq} \|\Delta_q u\|_{L^p} \le \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^p} \le C_k 2^{kq} \|\Delta_q u\|_{L^p}.$$

Definition 1. *Let $s \in \mathbf{R}$ be given, as well as p and r, two real numbers in the interval $[1, \infty]$. The Besov space $\dot{B}^s_{p,r}(\mathbf{R}^d)$ is the space of tempered distributions u such that*

- *The series $\sum_{-m}^{m} \Delta_q u$ converges to u in $\mathcal{S}'(\mathbf{R}^d)$.*
- $\|u\|_{\dot{B}^s_{p,r}(\mathbf{R}^d)} \stackrel{\text{def}}{=} \Big\| 2^{qs} \|\Delta_q u\|_{L^p(\mathbf{R}^d)} \Big\|_{\ell^r(\mathbf{Z})} < \infty.$

Remarks.

1) It is easy to see that for any real number s, the homogeneous Sobolev space $\dot{H}^s(\mathbf{R}^d)$ coincides with $\dot{B}^s_{2,2}$, and the norms given by (1.2) and Definition 1 are equivalent:

$$\|u\|_{\dot{H}^s(\mathbf{R}^d)} \sim \Big\| 2^{js} \|\Delta_j u\|_{L^2(\mathbf{R}^d)} \Big\|_{\ell^2(\mathbf{Z})}.$$

2) In the particular case where $u \in L^r(\mathbf{R}^d)$, the series $\sum_{-m}^{m} \Delta_q u$ converges to u in $\mathcal{S}'(\mathbf{R}^d)$. This is due to Bernstein inequality (1.10) which implies that

$$\|S_j u\|_{L^\infty} \le C 2^{j\frac{d}{r}} \|u\|_{L^r}.$$

The result we will prove is the following.

Theorem 1. *Let s be a real number in the interval $]0, d/2[$ and let p and q be two real numbers in $[1, \infty]$ such that*

$$2 \le q < \frac{2d}{d-2s} < p \le \infty.$$

There is a constant C such that, for any function $u \in \dot{B}^{s-d(\frac{1}{2}-\frac{1}{q})}_{q,2}(\mathbf{R}^d)$, the following inequality holds:

$$(1.12) \qquad \left(\int_{\mathbf{R}^d} \frac{|u(x)|^2}{|x|^{2s}} \, dx \right)^{\frac{1}{2}} \le C \|u\|^{\alpha}_{\dot{B}^{s-d(\frac{1}{2}-\frac{1}{q})}_{q,2}(\mathbf{R}^d)} \|u\|^{1-\alpha}_{\dot{B}^{s-d(\frac{1}{2}-\frac{1}{p})}_{p,2}(\mathbf{R}^d)}.$$

where $\alpha = \frac{q}{p-q}(p(\frac{1}{2} - \frac{s}{d}) - 1)$.

Remarks.

1) When $q = 2$ and $p = \infty$ we find the following inequality :

$$(1.13) \qquad \left(\int_{\mathbf{R}^d} \frac{|u(x)|^2}{|x|^{2s}} \, dx \right)^{\frac{1}{2}} \le C \|u\|^{\frac{2s}{d}}_{\dot{B}^{s-\frac{d}{2}}_{\infty,2}(\mathbf{R}^d)} \|u\|^{1-\frac{2s}{d}}_{\dot{H}^s(\mathbf{R}^d)}.$$

This inequality should be compared to the following similar result derived by P. Gérard, Y. Meyer and F. Oru in [11], in the case of the Sobolev inequalities.

$$(1.14) \qquad \|u\|_{L^p} \le C \|u\|^{\frac{2s}{d}}_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}(\mathbf{R}^d)} \|u\|^{1-\frac{2s}{d}}_{\dot{H}^s(\mathbf{R}^d)}.$$

with $\frac{1}{p}=\frac{1}{2}-\frac{s}{d}$.

2) The following classical result indicates the invariance of (1.12) under oscillations. We refer for instance to [1] for a proof.

Proposition 2. *Let σ be a real in the interval $]0,d[$ and let be f a function in $\mathcal{S}(\mathbf{R}^d)$. Then, there exists a constant C such that the oscillatory function $f_\varepsilon(x) \stackrel{\text{def}}{=} f(x)e^{ix\cdot\omega/\varepsilon}$, where ω belongs to S^{d-1}, satisfies $\|f_\varepsilon\|_{\dot{B}^{-\sigma}_{p,r}} \leq C\varepsilon^\sigma$.*

3) We will construct in what follows a fractal example putting in light that oscillations are not the sole responsible for smallness of the Besov norms.

4) Inequality (1.12) is optimum in the sense that it fails when $q=\frac{2d}{d-2s}=p$. More precisely, we have the following result (see [1] for a proof).

Proposition 3. *For any constant C, there is a function $u\in\dot{B}^0_{2q_c,2}(\mathbf{R}^d)$, where $q_c=\frac{d}{d-2s}$, such that*

$$\int_{\mathbf{R}^d}\frac{|u(x)|^2}{|x|^{2s}}dx \geq C\|u\|^2_{\dot{B}^0_{2q_c,2}(\mathbf{R}^d)}.$$

1.2. **The $\mathbf{H}^d$ case.** Before stating the precised Hardy inequality on the Heisenberg group $\mathbf{H}^d$, let us collect a few well known definitions and results on that group. The Heisenberg group $\mathbf{H}^d$ is the space $\mathbf{R}^{2d+1}$ endowed with the following product law:

$$(x,y,s)\cdot(x',y',s')=(x+x',y+y',s+s'+(y|x')-(y'|x)).$$

So $\mathbf{H}^d$ is a non commutative group. The Lie algebra of left invariant vector fields is spanned by the vector fields

$$X_j=\partial_{x_j}+y_j\partial_s\,,\ Y_j=\partial_{y_j}-x_j\partial_s \quad \text{with } j\in\{1,\cdots,d\}, \quad \text{and} \quad S=\partial_s=\frac{1}{2}[Y_j,X_j].$$

In all that follows, we shall denote by $\mathcal{Z}$ the family defined by $Z_j=X_j$ and $Z_{j+d}=Y_j$. One can associate Sobolev spaces to this system of vector fields through the following definition.

Definition 2. *Let k be a non negative integer, we denote by $H^k(\mathbf{H}^d)$ the inhomogeneous Sobolev space of order k which is the space of the functions u in $L^2(\mathbf{H}^d)$ (for the usual Lebesgue measure on $\mathbf{R}^{2d+1}$) such that*

$$Z_{j_1}\dots Z_{j_\ell}u\in L^2 \quad \textit{for any} \quad (j_m)_{1\leq m\leq \ell}\in\{1,\cdots,2d\}^\ell \quad \textit{with} \quad \ell\leq k.$$

and we have

$$\|u\|^2_{H^k(\mathbf{H}^d)} \stackrel{\text{def}}{=} \sum_{j_1,\dots,j_l,0\leq l\leq k}\|Z_{j_1}...Z_lu\|^2_{L^2(\mathbf{H}^d)}.$$

Remarks.

1) For any integer k, we can also define the homogeneous Sobolev norm of order k on the Heisenberg group and we have

$$\|u\|^2_{\dot{H}^k(\mathbf{H}^d)} \stackrel{\text{def}}{=} \sum_{j_1,\dots,j_k=1}^{2d}\|Z_{j_1}...Z_ku\|^2_{L^2(\mathbf{H}^d)}.$$

2) As in the $\mathbf{R}^d$ case, there are many ways of extending that definition to the case of indexes which are real numbers (and one obtains equivalent norms).

Let us point out that on the Heisenberg group $\mathbf{H}^d$, there is a notion of dilation defined for $a > 0$ by

$$\delta_a(z,s) = (az, a^2 s), \tag{1.15}$$

then the homogeneous dimension of $\mathbf{H}^d$ is $N = 2d + 2$. Finally, to state the precised Hardy inequality, let us introduce the Heisenberg distance to the origin ρ:

$$\rho(x,y,s) \stackrel{\text{def}}{=} ((x^2+y^2)^2 + s^2)^{\frac{1}{4}}.$$

Theorem 2. *Let s be a real number in the interval $]0, N/2[$, and let p and q be two real numbers in $[1,\infty]$ such that*

$$2 \leq q < \frac{2N}{N-2s} < p \leq \infty.$$

There is a constant C such that for any function $u \in \dot{B}^{s-d(\frac{1}{2}-\frac{1}{q})}_{q,2}(\mathbf{H}^d)$, the following inequality holds:

$$\left(\int_{\mathbf{H}^d} \frac{|u(w)|^2}{\rho(w)^{2s}}\, dw \right)^{\frac{1}{2}} \leq C \|u\|^{\alpha}_{\dot{B}^{s-d(\frac{1}{2}-\frac{1}{q})}_{q,2}(\mathbf{H}^d)} \|u\|^{1-\alpha}_{\dot{B}^{s-N(\frac{1}{2}-\frac{1}{p})}_{p,2}(\mathbf{H}^d)}. \tag{1.16}$$

where $\alpha = \frac{q}{p-q}(p(\frac{1}{2} - \frac{s}{N}) - 1)$ and $w = (x,y,s)$.

Remarks.

1) The proof of the precised inequality will proceed exactly as in the $\mathbf{R}^d$ case, once we recall that the paraproduct algorithm is also valid in $\mathbf{H}^d$ (see [5]) and prove the equivalent of Bernstein inequality (1.11) on the Heisenberg group. This inequality will be the subject of Section 3.2. One can note that this allows in particular to derive a (to our knowledge) new, elementary proof of the classical Hardy inequality in $\mathbf{H}^d$.

2) As in the $\mathbf{R}^d$ case, when $q = 2$ and $p = \infty$ we find the following inequality :

$$\left(\int_{\mathbf{H}^d} \frac{|u(w)|^2}{\rho(w)^{2s}}\, dw \right)^{\frac{1}{2}} \leq C \|u\|^{\frac{2s}{N}}_{\dot{B}^{s-\frac{d}{2}}_{\infty,2}(\mathbf{H}^d)} \|u\|^{1-\frac{2s}{N}}_{\dot{H}^s(\mathbf{H}^d)}, \tag{1.17}$$

which is in the spirit of the precised Sobolev inequality on the Heisenberg group proved in [5].

3) As in the $\mathbf{R}^d$ case again, it should be noted that Inequality (1.16) is invariant under oscillation. (For a detailed proof, we can consult [1]).

Structure of the paper. In Section 2 we start by presenting a short proof of Inequality (1.1), as well as the proof of Theorem 1 in Paragraph 2.2. Finally Paragraph 2.3 is devoted to the construction of an example showing another feature of Besov spaces, other than oscillations. Section 3 consists first in a more detailed presentation of the Heisenberg group $\mathbf{H}^d$, namely with the recollection of some results around the Littlewood-Paley decomposition and Bony's paraproduct algorithm. In Paragraph 3.2 is proved a new result concerning the Bernstein inequality on $\mathbf{H}^d$. Once those preliminaries are known, the proof of Theorem 2 follows exactly as in the $\mathbf{R}^d$ case, and is omitted.

Acknowledgements. We wish to thank A. Cohen for suggesting the idea of the construction presented in Section 2.3.

2. The $\mathbf{R}^d$ case

2.1. **Proof of Inequality (1.1).** In this section we wish to present an elementary proof of Inequality (1.1), which relies on Besov spaces. Inequality (1.1) is obviously a direct consequence of the two following classical propositions.

The first result consists in a product rule in Besov spaces. Using the paraproduct algorithm of J.-M. Bony [6], it is straightforward to prove the following proposition.

Proposition 4. *Let s be a real number in the interval $]0, d/2[$, and let f and g be two functions in $\dot{H}^s(\mathbf{R}^d)$. Then the product fg is an element of the Besov space $\dot{B}^{2s-\frac{d}{2}}_{2,1}(\mathbf{R}^d)$, and the following estimate holds*

$$\|fg\|_{\dot{B}^{2s-\frac{d}{2}}_{2,1}(\mathbf{R}^d)} \leq C\|f\|_{\dot{H}^s(\mathbf{R}^d)}\|g\|_{\dot{H}^s(\mathbf{R}^d)},$$

where the constant C only depends on s and on the dimension d.

The second classical result shows in what Besov space the function $x \mapsto |x|^{-2s}$ lies.

Proposition 5. *Let s be a real number in the interval $]0, d/2[$. Then the function $x \mapsto |x|^{-2s}$ belongs to the Besov space $\dot{B}^{d-2s}_{1,\infty}(\mathbf{R}^d)$.*

Putting together Proposition 4 and 5 clearly yields Inequality (1.1).

2.2. **Proof of Theorem 1.** Let us go now to the proof of Theorem 1. In order to do so we shall make more precise the product rule given in Proposition 4. Let us recall the paraproduct algorithm of J.-M. Bony [6]: we have

$$u^2 = 2T_u u + R(u,u), \quad \text{with} \quad T_u u \stackrel{\text{def}}{=} \sum_j S_{j-1}u\Delta_j u.$$

Let us start by recalling that

$$\|T_u u\|_{\dot{B}^{2s-d}_{\infty,1}(\mathbf{R}^d)} \leq C\|u\|^2_{\dot{B}^{s-\frac{d}{2}}_{\infty,2}(\mathbf{R}^d)} \leq C\|u\|^{2\alpha}_{\dot{B}^{s-d(\frac{1}{2}-\frac{1}{q})}_{q,2}(\mathbf{R}^d)}\|u\|^{2-2\alpha}_{\dot{B}^{s-N(\frac{1}{2}-\frac{1}{p})}_{p,2}(\mathbf{R}^d)}, \tag{2.1}$$

thanks to Sobolev embeddings.

To estimate the remainder term $R(u,u)$, we shall use the following elementary interpolation result.

Lemma 1 ([1]). *Let s be a real number in the interval $]0, d/2[$ and let p and q be two real numbers in $[1, \infty]$ such that*

$$2 \leq q < \frac{2N}{N-2s} < p \leq \infty.$$

There is a constant C such that for any functions f and g which belong to $L^p(\mathbf{R}^d) \cap L^q(\mathbf{R}^d)$, we have

$$\int_{\mathbf{R}^d} \frac{fg(x)}{|x|^{2s}}\, dx \leq C\|f\|^{\alpha}_{L^q(\mathbf{R}^d)}\|g\|^{\alpha}_{L^q(\mathbf{R}^d)}\|f\|^{1-\alpha}_{L^p(\mathbf{R}^d)}\|g\|^{1-\alpha}_{L^p(\mathbf{R}^d)}$$

with $\alpha = \frac{q}{p-q}(p(\frac{1}{2} - \frac{s}{d}) - 1)$.

The Hölder inequality then gives, by definition of $R(u,u)$,

$$\int_{\mathbf{R}^d} \frac{R(u,u)}{|x|^{2s}}\,dx \leq C \sum_{|\ell|\leq 1} \left(\sum_{j\in\mathbf{Z}} 2^{2j(s-d(\frac{1}{2}-\frac{1}{q}))} \|\Delta_j u\|_{L^q(\mathbf{R}^d)} \|\Delta_{j-\ell} u\|_{L^q(\mathbf{R}^d)} \right)^{\alpha}$$
$$\times \left(\sum_{j\in\mathbf{Z}} 2^{2j(s-d(\frac{1}{2}-\frac{1}{p}))} \|\Delta_j u\|_{L^p(\mathbf{R}^d)} \|\Delta_{j-\ell} u\|_{L^p(\mathbf{R}^d)} \right)^{1-\alpha}$$

and the result follows from the Cauchy-Schwartz inequality.

Theorem 1 is proved.

2.3. **Functions supported on a Cantor-like set.** In this section we will show that oscillations are not the sole reponsible for the smallness of a Besov norm. Below we present another situation, where a sequence of positive (in particular non oscillatory) functions converges towards zero in a negative Besov space, while remaining constant in a Lebesgue norm. More precisely we have the following results; we wil give the main ideas of the proof below, and refer to [1] for details.

Proposition 6. *There is a sequence* $(f_n)_{n\in\mathbf{N}}$ *of positive functions which saturate the precise Sobolev embedding (1.14).*

Proposition 7. *There is a sequence* $(f_n)_{n\in\mathbf{N}}$ *of positive functions such that* $\|f_n\|_{\dot{H}^s}$ *goes to infinity with* n *whereas the Hardy norm remains bounded:*

$$\sup_{n\in\mathbf{N}} \left(\int \frac{f_n^2(x)}{|x|^{2s}}\,dx \right)^{\frac{1}{2}} < +\infty.$$

The proof of that result is based on fractal ideas: basically the support of f_n is supported on a Cantor-type set, when n tends to ∞. In order to construct the family, let us consider the following application, which acts on smooth, compactly supported functions on the cube Q defined by $Q \stackrel{\text{def}}{=} [-\frac{1}{2}, \frac{1}{2}]^d$. Let T be the application defined by

$$T : \begin{cases} \mathcal{D}(Q) & \to \quad \mathcal{D}(Q) \\ \quad f & \mapsto \quad Tf \stackrel{\text{def}}{=} 2^d \displaystyle\sum_{J\in\{-1,1\}^d} f_J, \end{cases}$$

where $f_J(x) = f(4(x - x_J))$ and $x_J = \frac{3}{8}(J_1, \cdots, J_d)$. We have the following lemma.

Lemma 2. *The application* T *satisfies the following bounds, for all* $p \in [1, +\infty]$ *and all real numbers* s *such that* $s + d(1 - \frac{1}{p}) > 0$*:*

$$\begin{aligned} \|Tf\|_{L^p} &= 2^{d(1-\frac{1}{p})} \|f\|_{L^p}. \\ \|Tf\|_{\dot{B}^s_{p,r}} &\leq 2^{d(1-\frac{1}{p})+2s} \|f\|_{\dot{B}^s_{p,r}} + C\|f\|_{L^1}. \end{aligned}$$

Let us start by computing $\|Tf\|_{L^p}$. We can define $Q_J \stackrel{\text{def}}{=} \operatorname{supp} f_J$, and since the cubes Q_J and $Q_{J'}$ do not meet if $J \neq J'$, we have

$$\|Tf\|_{L^p}^p = 2^{pd} \sum_{J\in\{-1,1\}^d} \|f_J\|_{L^p}^p.$$

But $\|f_J\|_{L^p} = 2^{-2d/p}\|f\|_{L^p}$, so the result follows.

Let us now turn to the Besov norm $\|Tf\|_{\dot{B}^s_{p,r}}$. We have by definition

$$\|Tf\|_{\dot{B}^s_{p,r}} = (\sum_{j\in\mathbf{Z}} 2^{jsr}\|\Delta_j Tf\|_{L^p}^r)^{\frac{1}{r}}.$$

On the one hand, Bernstein's inequality, along with the fact that $s + d(1-\frac{1}{p}) > 0$ yields

$$(\sum_{j\leq 1} 2^{jsr}\|\Delta_j Tf\|_{L^p}^r)^{\frac{1}{r}} \leq C\|f\|_{L^1}.$$

The case when $j > 1$ is more delicate, and it is here that the special structure of the support of Tf will appear. Let us define the set, for $\delta > 0$ small enough,

$$Q^\delta \stackrel{\text{def}}{=} \{x \in \mathbf{R}^d \mid d(x, \bigcup_J Q_J) \leq \delta\}.$$

Then we write

$$(\sum_{j\geq 1} 2^{jsr}\|\Delta_j Tf\|_{L^p}^r)^{\frac{1}{r}} \leq I_1 + I_2,$$

where

$$I_1 \stackrel{\text{def}}{=} (\sum_{j\geq 1} 2^{jsr}\|\Delta_j Tf\|_{L^p({}^cQ^\delta)}^r)^{\frac{1}{r}} \quad \text{and} \quad I_2 \stackrel{\text{def}}{=} (\sum_{j\geq 1} 2^{jsr}\|\Delta_j Tf\|_{L^p(Q^\delta)}^r)^{\frac{1}{r}}.$$

We can write, for $x \in {}^cQ^\delta$

$$|\Delta_j Tf(x)| \leq C_N 2^{jd}2^{-jN}\delta^{-N}\int_{\cup Q_J} |h(2^j(x-y))||2^j(x-y)|^N|Tf(y)|\,dy.$$

We infer, under Young inequality and choosing N large enough,

$$I_1 \leq C_N\|f\|_{L^1}.$$

Finally let us estimate I_2. We notice that

$$Q^\delta = \bigcup_{J\in\{-1,1\}^d} B_J^\delta, \quad \text{where} \quad B_J^\delta \stackrel{\text{def}}{=} Q_J + \delta,$$

hence, for δ small enough, we have $B_J^\delta \cap B_{J'}^\delta = \emptyset$ if $J \neq J'$, and if x belongs to B_J^δ, then for any $J' \neq J$, we have $d(x, B_{J'}^\delta) \geq \delta$.

Therefore, we can write for $x \in Q^\delta$

$$\Delta_j(Tf)(x) = \sum_J \Delta_j(Tf)\mathbf{1}_{B_J^\delta}(x)$$

Now

$$\Delta_j(Tf)\mathbf{1}_{B_J^\delta}(x) = 2^d\Delta_j(f_J)\mathbf{1}_{B_J^\delta}(x) + 2^d\sum_{J'\neq J}\Delta_j(f_{J'})\mathbf{1}_{B_J^\delta}(x),$$

so that $\Delta_j(Tf) = A + B$, with

$$A = 2^d \sum_J \Delta_j(f_J)\mathbf{1}_{B_J^\delta} \quad \text{and} \quad B = 2^d \sum_J \sum_{J' \neq J} \Delta_j(f_{J'})\mathbf{1}_{B_J^\delta}.$$

On the one hand we have

$$B(x) = 2^d \sum_J \sum_{J' \neq J} 2^{jd} \int_{\mathbf{R}^d} h(2^j(x-y)) f_{J'}(y)\, dy \cdot \mathbf{1}_{B_J^\delta}(x)$$

and reasoning as above we find that

$$\Big(\sum_{j \geq 1} 2^{jsr} \|B\|_{L^p}^r\Big)^{\frac{1}{r}} \quad \leq \quad C\|f\|_{L^1}.$$

Going back to the estimate of I_2 we gather

$$I_2 \leq C\|f\|_{L^1} + 2^d \Big(\sum_{j \geq 1} 2^{jsr} \|\sum_J \Delta_j(f_J) \cdot \mathbf{1}_{B_J^\delta}\|_{L^p}^r\Big)^{\frac{1}{r}}.$$

Since the sets B_J^δ are disjoint, we get

$$\|\sum_J \Delta_j(f_J) \cdot \mathbf{1}_{B_J^\delta}\|_{L^p}^p = \sum_J \|\Delta_j(f_J)\|_{L^p(B_J^\delta)}^p.$$

As $f_J(x) = f(4(x - x_J))$, we deduce that

$$\|\Delta_j(f_J)\|_{L^p} = 2^{-2\frac{d}{p}} \|\Delta_{j-2}(f)\|_{L^p}.$$

Thus

$$\|\sum_J \Delta_j(f_J) \cdot \mathbf{1}_{B_J^\delta}\|_{L^p} \leq 2^{-\frac{d}{p}} \|\Delta_{j-2}(f)\|_{L^p},$$

which implies that

$$I_2 \leq C\|f\|_{L^1} + 2^{d(1-\frac{1}{p})+2s} \|f\|_{\dot{B}^s_{p,r}}.$$

This ends the proof of the lemma.

Lemma 2 allows us to construct the family of Propositions 6 and 7, simply by choosing

$$f_n \stackrel{\text{def}}{=} \frac{T^n f}{\|T^n f\|_{L^p}}$$

with $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$ and where $f \in \mathcal{D}(Q)$. We refer to [1] for details.

3. The case of the Heisenberg group

3.1. **Basic facts about the Heisenberg group.** To introduce the Littlewood-Paley theory on the Heisenberg group, we need to recall the definition of the Fourier transform in that framework. We refer to [18] and the references therein for more details. The Heisenberg group being non commutative, the Fourier transform on $\mathbf{H}^d$ is defined using irreducible unitary representations of $\mathbf{H}^d$. We shall choose here the Bargmann representations described by $(u^\lambda, \mathcal{H}_\lambda)$, with $\lambda \in \mathbf{R} \setminus \{0\}$, where $\mathcal{H}_\lambda$ are the spaces defined by

$$\mathcal{H}_\lambda = \{F \text{ holomorphic on } \mathbf{C}^d, \|F\|_{\mathcal{H}_\lambda} < \infty\},$$

where we define

$$\|F\|^2_{\mathcal{H}_\lambda} \stackrel{\text{def}}{=} \left(\frac{2|\lambda|}{\pi}\right)^d \int_{\mathbf{C}^d} e^{-2|\lambda||\xi|^2} |F(\xi)|^2 d\xi, \tag{3.1}$$

and u^λ is the map from $\mathbf{H}^d$ into the group of unitary operators of $\mathcal{H}_\lambda$ defined by

$$\begin{aligned} u^\lambda_{z,s} F(\xi) &= F(\xi - \overline{z}) e^{i\lambda s + 2\lambda(\xi \cdot z - |z|^2/2)} \quad \text{for} \quad \lambda > 0, \\ u^\lambda_{z,s} F(\xi) &= F(\xi - z) e^{i\lambda s - 2\lambda(\xi \cdot \overline{z} - |z|^2/2)} \quad \text{for} \quad \lambda < 0. \end{aligned}$$

Let us notice that $\mathcal{H}_\lambda$ equipped with the norm (3.1) is a Hilbert space and that the monomials

$$F_{\alpha,\lambda}(\xi) = \frac{(\sqrt{2|\lambda|}\,\xi)^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbf{N}^d, \tag{3.2}$$

constitute an orthonormal basis.

We associate the Fourier transform of an integrable function of $\mathbf{H}^d$ through the following definition.

Definition 3. *For $f \in L^1(\mathbf{H}^d)$, we define*

$$\mathcal{F}(f)(\lambda) = \int_{\mathbf{H}^d} f(z,s) u^\lambda_{z,s} dz ds.$$

The function $\mathcal{F}(f)$, with values in the bounded operators on $\mathcal{H}_\lambda$, is by definition the Fourier transform of f.

The convolution product of two functions f and g on $\mathbf{H}^d$ is defined by

$$f \star g(w) = \int_{\mathbf{H}^d} f(wv^{-1}) g(v) dv,$$

with the useful Young inequalities. Under the fact that, for any λ, the map

$$u^\lambda : \mathbf{H}^d \longrightarrow U(\mathcal{H}_\lambda)$$

is a group morphism, it is easy to verify that

$$\mathcal{F}(f \star g)(\lambda) = \mathcal{F}(f)(\lambda) \circ \mathcal{F}(g)(\lambda). \tag{3.3}$$

It turns out that for radial functions on the Heisenberg group, the Fourier transform becomes simplified and puts into light the quantity that will play the role of the frequency size. Let us first recall the concept of radial functions on the Heisenberg group.

Definition 4. *A function f defined on the Heisenberg group $\mathbf{H}^d$ is said to be radial if it is invariant under the action of the unitary group $U(d)$ of $\mathbf{C}^d$, which means that for any $u \in U(d)$, we have*

$$f(z,s) = f(u(z),s), \quad \forall (z,s) \in \mathbf{H}^d.$$

A radial function on the Heisenberg group can then be written under the form

$$f(z,s) = g(|z|,s).$$

The Fourier transform of radial functions of $L^2(\mathbf{H}^d)$, satisfies the following formulas:

$$\mathcal{F}(f)(\lambda) F_{\alpha,\lambda} = R_{|\alpha|}(\lambda) F_{\alpha,\lambda} \tag{3.4}$$

where

$$R_m(\lambda) = \begin{pmatrix} m+d-1 \\ m \end{pmatrix}^{-1} \int e^{i\lambda s} f(z,s) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} dz ds,$$

and where $L_m^{(p)}$ are Laguerre polynomials.

The key point in the construction of the Littlewood-Paley decomposition on $\mathbf{H}^d$ lies in the following proposition proved in [4] and [5].

Proposition 8. *For any function $Q \in \mathcal{D}(\mathbf{R})$ constant near the origin, the series*

$$f(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} Q((2m+d)\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda$$

converges in $\mathcal{S}(\mathbf{H}^d)$.

Now we are ready to define the Littlewood-Paley decomposition on $\mathbf{H}^d$. We will not give any proof but refer to the construction in [4] and [5] for all the details, and in particular for the proof of the following proposition.

Proposition 9 ([4], [5])**.** *Let us denote by $\mathcal{C}_0$ the ring $\left\{\tau \in \mathbf{R},\ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\right\}$ and by $\mathcal{B}_0$ the ball $\left\{\tau \in \mathbf{R},\ |\tau| \leq \frac{4}{3}\right\}$. Then there exists two radial functions $\widetilde{R}^*$ and R^* the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(\mathcal{B}_0)$ and to $\mathcal{D}(\mathcal{C}_0)$ such that*

$$\forall \tau \in \mathbf{R}, \quad \widetilde{R}^*(\tau) + \sum_{j \geq 0} R^*(2^{-2j}\tau) = 1 \tag{3.5}$$

and

$$\forall \tau \in \mathbf{R}^*, \quad \sum_{j \in \mathbf{Z}} R^*(2^{-2j}\tau) = 1, \tag{3.6}$$

and satisfying as well the support properties

$$|p-q| \geq 1 \Rightarrow \mathit{supp}\ R^*(2^{-2q}\cdot) \cap \mathit{supp}\ R^*(2^{-2p}\cdot) = \emptyset \tag{3.7}$$

$$\textit{and} \quad q \geq 1 \Rightarrow \mathit{supp}\ \widetilde{R}^* \cap \mathit{supp}\ R^*(2^{-2q}\cdot) = \emptyset. \tag{3.8}$$

Moreover, owing to Proposition 8, it can be proved that there are radial functions of $\mathcal{S}(\mathbf{H}^d)$, denoted ψ and φ such that

$$\mathcal{F}(\psi)(\lambda)F_{\alpha,\lambda} = \widetilde{R}^*_{|\alpha|}(\lambda)F_{\alpha,\lambda} \quad \text{and} \quad \mathcal{F}(\varphi)(\lambda)F_{\alpha,\lambda} = R^*_{|\alpha|}(\lambda)F_{\alpha,\lambda},$$

where we have noted $\widetilde{R}^*_m(\tau) = \widetilde{R}^*((2m+d)\tau)$ and $R^*_m(\tau) = R^*((2m+d)\tau)$. A simple computation shows finally that if we state

$$\varphi_j(z,s) = 2^{Nj}\varphi(2^j z, 2^{2j}s) \quad \text{and} \quad \psi_j(z,s) = 2^{Nj}\psi(2^j z, 2^{2j}s),$$

where $N \stackrel{\text{def}}{=} 2d+2$ is the homogeneous dimension of $\mathbf{H}^d$, then we have

$$\mathcal{F}(\varphi_j)(\lambda)F_{\alpha,\lambda} = R^*_{|\alpha|}(2^{-2j}\lambda) \quad \text{and} \quad \mathcal{F}(\psi_j)(\lambda)F_{\alpha,\lambda} = \widetilde{R}^*_{|\alpha|}(2^{-2j}\lambda).$$

Remarks.

1) Now as in the $\mathbf{R}^d$ case, we define the Littlewood-Paley operators Δ_j and S_j, for $j \in \mathbf{Z}$, by

$$\begin{aligned}\mathcal{F}(\Delta_j f)(\lambda)F_{\alpha,\lambda} &= R^*_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\\ \mathcal{F}(S_j f)(\lambda)F_{\alpha,\lambda} &= \widetilde{R}^*_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}.\end{aligned}$$

It is easy to see that

$$\Delta_j u = u \star 2^{Nj}\varphi(\delta_{2^j}\cdot)$$

and

$$S_j u = u \star 2^{Nj}\psi(\delta_{2^j}\cdot)$$

which implies that those operators map L^p into L^p for all $p \in [1,\infty]$ with norms which do not depend on j and commute with the left invariant vector fields on the Heisenberg group.

2) Along the same lines than the $\mathbf{R}^d$ case (Definition 1), we can define Besov spaces on the Heisenberg group (see [3]).

3) Paraproduct operators on the Heisenberg group similar to those defined by J.-M. Bony [6] are built in [5], although there is no simple formula for the Fourier transform of the product of two functions, and the definition of [6] turns out to be effective in this framework.

3.2. **Bernstein inequality on the Heisenberg group.** Using the complex system coordinates (z,s) obtained through the formula $z_j = x_j + iy_j$, we have another generator system of the Lie algebra of left invariant vector fields on the Heisenberg group $\mathbf{H}^d$ formed by the complex vector fields:

$$Z_j = \partial z_j + i\overline{z}_j\partial_s, \quad \overline{Z}_j = \partial\overline{z}_j - iz_j\partial_s, \quad \text{with } j \in \{1,\cdots,d\} \quad \text{and} \quad S = \partial_s = \frac{1}{2i}[\overline{Z}_j, Z_j].$$

Let us point out that when F is an element of the basis of the Hilbert space $\mathcal{H}_\lambda$ defined by(3.2), we have the following useful formulas, for any $j \in \{1,...,d\}$. If $\lambda > 0$,

$$\mathcal{F}(Z_j f)(\lambda)F_{\alpha,\lambda} = -\sqrt{2|\lambda|}\sqrt{\alpha_j+1}\mathcal{F}(f)(\lambda)F_{(\alpha_1,\ldots,\alpha_j+1,\ldots),\lambda}, \tag{3.9}$$

$$\mathcal{F}(\overline{Z}_j f)(\lambda)F_{\alpha,\lambda} = \sqrt{2|\lambda|}\sqrt{\alpha_j}\mathcal{F}(f)(\lambda)F_{(\alpha_1,\ldots,\alpha_j-1,\ldots),\lambda}, \tag{3.10}$$

$$\mathcal{F}(Z_j\overline{Z}_j f)(\lambda)F_{\alpha,\lambda} = -2|\lambda|(\alpha_j+1)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}, \tag{3.11}$$

$$\mathcal{F}(\overline{Z}_j Z_j f)(\lambda)F_{\alpha,\lambda} = -2|\lambda|\alpha_j\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}, \tag{3.12}$$

and with similar formulas if $\lambda < 0$.

To state the Bernstein inequality on the Heisenberg group, we need to define the concept of localization procedure in the frequency space on the framework of the Heisenberg group and to prove that the left invariant vector fields act in a particular way on distributions which are localized in frequency space in a ball or a ring. We will only state the definition in the case of smooth functions – otherwise one needs an additional regularization by convolution (see [3] or [5]).

Definition 5. *Let $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0,r_1,r_2)$ be a ring and $\mathcal{B}_r = \mathcal{B}(0,r)$ a ball of $\mathbf{R}$ center at the origin.*

- *A function u in $\mathcal{S}(\mathbf{H}^d)$ is said to be frequency localized in the ball $2^j\mathcal{B}_{\sqrt{r}}$, if*

$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+d)^{-1}2^{2j}\mathcal{B}}(\lambda)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda},$$

- *A function u in $\mathcal{S}(\mathbf{H}^d)$ is said to be frequency localized in the ring $2^j\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, if*
$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+d)^{-1}2^{2j}\mathcal{C}}(\lambda)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda}.$$

The following result is the analogue of Bernstein inequality in the classical case. It describes the cost of the left invariant derivatives of a frequency localized function.

Lemma 3. *Let $\mathcal{C}_{(r_1,r_2)}$ be a ring and $\mathcal{B}_r$ a ball of $\mathbf{R}$ center at the origin. For any non negative integer k, there exists a constant C_k so that, for any couple of real numbers (p,q) such that $q \geq p \geq 1$ and any function u of $L^p(\mathbf{H}^d)$, we have:*

- *If u is frequency localized in the ball $2^j\mathcal{B}_{\sqrt{r}}$, then we have*

$$\sup_{|\beta|=k} \|\mathcal{X}^\beta u\|_{L^q(\mathbf{H}^d)} \leq C_k 2^{jN(\frac{1}{p}-\frac{1}{q})+jk}\|u\|_{L^p(\mathbf{H}^d)}, \tag{3.13}$$

where $\mathcal{X}^\beta$ denotes a product of k left invariant vectors fields.

- *On the other hand, if u is frequency localized in the ring $2^j\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, then we have*

$$C_k^{-1}2^{jk}\|u\|_{L^p(\mathbf{H}^d)} \leq \sup_{|\beta|=k} \|\mathcal{X}^\beta u\|_{L^p(\mathbf{H}^d)} \leq C_k 2^{jk}\|u\|_{L^p(\mathbf{H}^d)}, \tag{3.14}$$

where $\mathcal{X}^\beta$ still denotes a product of k left invariant vectors fields.

Remarks

1) Estimate (3.13) was proved in [5].

2) Contrary to the case of the Laplacian $-\Delta_{\mathbf{H}^d}$ (see [3]), we do not dispose of a simple decomposition of the left invariant vector fields in the basis of the $F_{\alpha,\lambda}$. Therefore, the proof of the second result of this lemma is more delicate than the equivalent estimate in the classical case.

Let us prove Estimate (3.14) of Lemma 3. By density, it suffices to suppose that the function u is an element of $\mathcal{S}(\mathbf{H}^d)$. First, let us recall that thanks to (3.11), we have for any $k \in \{1,...,d\}$
$$\mathcal{F}(Z_k\overline{Z}_k u)(\lambda)F_{\alpha,\lambda} = -2|\lambda|(\alpha_k+1)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda}, \quad \text{when} \quad \lambda > 0,$$
and
$$\mathcal{F}(Z_k\overline{Z}_k u)(\lambda)F_{\alpha,\lambda} = -2|\lambda|\alpha_k\mathcal{F}(u)(\lambda)F_{\alpha,\lambda}, \quad \text{when} \quad \lambda < 0.$$
Now the frequency localization of u in the ring $2^j\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$ allows us to write
$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = R'_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda},$$
with $R'_{|\alpha|}(\lambda) = R'((2|\alpha|+d)\lambda)$, R' being a function of $\mathcal{D}(\mathbf{R}^*)$ whose value is 1 near $\mathcal{C}_{(r_1,r_2)}$. Therefore
$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = -\frac{2^{-2j}}{2|2^{-2j}\lambda|(\mathbf{1}_{\lambda>0}(\alpha_k+1)+\mathbf{1}_{\lambda<0}\alpha_k)}R'_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(Z_k\overline{Z}_k u)(\lambda)F_{\alpha,\lambda}.$$
The fact that $R' \in \mathcal{D}(\mathbf{R}^*)$ ensures, owing to Proposition 8, the existence of a radial function $h^k \in \mathcal{S}(\mathbf{H}^d)$ such that
$$\mathcal{F}(h^k)(\lambda)F_{\alpha,\lambda} = -\frac{1}{2|\lambda|(\mathbf{1}_{\lambda>0}(\alpha_k+1)+\mathbf{1}_{\lambda<0}\alpha_k)}R'_{|\alpha|}(\lambda)F_{\alpha,\lambda}.$$

Now, if we write $h_j^k(z,s) = 2^{jN} h^k(\delta_{2^j}\cdot)$, we get thanks to (3.3)

$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = 2^{-2j}\mathcal{F}(Z_k\overline{Z}_k u)(\lambda)(\mathcal{F}(h_j^k)(\lambda)F_{\alpha,\lambda}),$$

which implies that $u = 2^{-2j} Z_k\overline{Z}_k u \star h_j^k$. Thus Young's inequality leads to

$$\|u\|_{L^p(\mathbf{H}^d)} \leq C2^{-2j}\|Z_k\overline{Z}_k u\|_{L^p(\mathbf{H}^d)},$$

and then owing to (3.13)

$$\|u\|_{L^p(\mathbf{H}^d)} \leq C2^{-j}\|\overline{Z}_k u\|_{L^p(\mathbf{H}^d)}.$$

Along the same lines, we can prove that

$$\|u\|_{L^p(\mathbf{H}^d)} \leq C2^{-j}\|Z_k u\|_{L^p(\mathbf{H}^d)},$$

which leads by induction to the second Estimate (3.14).

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