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THE MOTION OF THE FREE SURFACE OF A LIQUID

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1. Introduction

We consider Euler's equations

$$(\partial_t + v^k\partial_k)v_j = -\partial_j p, \quad j = 1, ..., n \quad \text{in} \quad \mathcal{D}, \qquad \text{where} \quad \partial_i = \partial/\partial x^i \tag{1.1}$$

describing the motion of a perfect incompressible fluid in vacuum:

$$\operatorname{div} v = \partial_k v^k = 0 \quad \text{in} \quad \mathcal{D} \tag{1.2}$$

where $v = (v_1, ..., v_n)$ and $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ are to be determined. Here $v^k = \delta^{ki} v_i = v_k$ and we have used the summation convention that repeated upper and lower indices are summed over. We also require the boundary conditions on the free boundary $\partial\mathcal{D}$;

$$\begin{cases} p = 0, \quad \text{on} \quad \partial\mathcal{D} \\ (\partial_t + v^k\partial_k)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \end{cases} \tag{1.3}$$

The first condition says that the pressure vanishes outside the domain. The second condition says that the boundary should move with the velocity of the fluid particles at the boundary.

Given a simply connected bounded domain $\mathcal{D}_0 \subset \mathbb{R}^n$ and initial data v_0, satisfying the constraint $\operatorname{div} v_0 = 0$, we want to find a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and a vector field v solving (1.1)-(1.3) and satisfying the initial conditions

$$\begin{cases} \{x;\, (0, x) \in \mathcal{D}\} = \mathcal{D}_0 \\ v = v_0, \quad \text{on} \quad \{0\} \times \mathcal{D}_0 \end{cases} \tag{1.4}$$

Let $\mathcal{D}_t = \{x;\, (t, x) \in \mathcal{D}\}$ and, for each t, let N be the exterior unit normal to the free surface $\partial\mathcal{D}_t$. Christodoulou[C2] conjectured the initial value problem (1.1)-(1.4), is well posed in Sobolev spaces under the assumption

$$\nabla_N\, p \le -c_0 < 0, \quad \text{on} \quad \partial\mathcal{D}, \qquad \text{where} \quad \nabla_N = N^i \partial_{x^i}. \tag{1.5}$$

(1.5) is a natural *physical condition* since the pressure p has to be positive in the interior of the fluid. It is essential for the well posedness in Sobolev spaces. Taking the divergence of (1.1):

$$-\triangle p = (\partial_j v^k)\partial_k v^j, \qquad \text{in} \quad \mathcal{D}_t, \qquad p = 0, \quad \text{on} \quad \partial\mathcal{D}_t \tag{1.6}$$

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In the irrotational case (1.5) always hold. Then $(\operatorname{curl} v)_{ij} = \partial_i v_j - \partial_j v_i = 0$ so $\triangle p < 0$ and hence $p > 0$ and (1.5) hold by the strong maximum principle. Furthermore, Ebin[E1] showed that the equations are ill posed when (1.5) is not satisfied and the pressure is negative.

Wu[W1,W2] proved well posedness in Sobolev spaces in the irrotational case when the curl vanishes. Ebin[E2] announced an existence result when one adds surface tension to the boundary condition. In [CL] we proved *a priori* bounds in Sobolev spaces in the general case of non vanishing curl. Recently, in [L1, L2] we prove existence. This lecture outlines the energy estimates using the two different methods in [CL] respectively [L1,L2].

The incompressible perfect fluid is to be thought of as an idealization of a liquid. For small bodies like water drops surface tension should help holding it together and for larger denser bodies like stars its own gravity should play a role. Here we neglect the influence of such forces. Instead it is the incompressibility condition that prevents the body from expanding and it is the fact that the pressure is positive that prevents the body from breaking up in the interior. Let us also point out that, from a physical point of view one can alternatively think of the pressure as being a small positive constant on the boundary instead of vanishing. What makes this problem difficult is that the regularity of the boundary enters to highest order. Roughly speaking, the velocity tells the boundary where to move and the boundary is the zero set of the pressure that determines the acceleration. For more physical and historical background, including further references, we refer to [W2] and [CL].

2. Lagrangian coordinates.

We start by introducing Lagrangian coordinates in which the boundary becomes fixed. Let Ω be a domain in $\mathbf{R}^n$ and let $f_0 : \Omega \to \mathcal{D}_0$ be a diffeomorphism that is volume preserving; $\det(\partial f_0/\partial y) = 1$. Assume that $v(t,x)$ and $p(t,x)$, $(t,x) \in \mathcal{D}$ are given satisfying the boundary conditions (1.3). The Lagrangian coordinates $x = x(t,y) = f_t(y)$ are given by solving

$$\frac{dx}{dt} = v(t, x(t,y)), \qquad x(0,y) = f_0(y), \quad y \in \Omega \tag{2.1}$$

Then $f_t : \Omega \to \mathcal{D}_t$ is a volume preserving diffeomorphism, since $\operatorname{div} v = 0$, and the boundary becomes fixed in the new y coordinates. Let us introduce the notation

$$D_t = \frac{\partial}{\partial t}\Big|_{y=constant} = \frac{\partial}{\partial t}\Big|_{x=constant} + v^k \frac{\partial}{\partial x^k}, \tag{2.2}$$

for the material derivative and

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}. \tag{2.3}$$

In these coordinates Euler's equation (1.1) and the incompressibility condition (1.2) become

$$D_t^2 x_i = -\partial_i p, \qquad \text{and} \qquad \kappa = \det\left(\partial x/\partial y\right) = 1, \tag{2.4}$$

where $x = x(t,y)$, $p = p(t,y)$, and the boundary condition (1.3) and the initial condition (1.4) become

$$p\Big|_{\partial\Omega} = 0, \qquad \text{and} \qquad x\Big|_{t=0} = f_0, \qquad D_t\, x\Big|_{t=0} = v_0. \tag{2.5}$$

In fact, recall that $D_t \det(M) = \det(M)\operatorname{tr}(M^{-1} D_t M)$, for any matrix M depending on t so

$$D_t \det\left(\partial x/\partial y\right) = \det\left(\partial x/\partial y\right)\frac{\partial y^a}{\partial x^i}\frac{\partial D_t x^i}{\partial y^a} = \partial_i D_t x^i = \operatorname{div} D_t x = \operatorname{div} v = 0 \tag{2.6}$$

Note that p is uniquely determined as a functional of x by (2.4)-(2.5). In fact taking the divergence of Euler's equations (2.4) using (2.6) gives

$$\triangle p = -(\partial_i D_t x^j)(\partial_j D_t x^i). \tag{2.7}$$

3. Energy conservation.

Since $\det(\partial x/\partial y) = 1$ it follows from introducing Lagrangian coordinates,

$$\int_{\mathcal{D}_t} g\,dx = \int_{\Omega} g\,dy, \qquad \text{so} \quad \frac{d}{dt}\int_{\mathcal{D}_t} g\,dx = \int_{\mathcal{D}_t} D_t g\,dx \tag{3.1}$$

We note that if v is a solution of Euler's equations, $D_t v_i = -\partial_i p$ and p vanish on the boundary then

$$\frac{d}{dt}\int_{\mathcal{D}_t} |v|^2\,dx = 2\int_{\mathcal{D}_t} v^i D_t v_i\,dx = -2\int_{\mathcal{D}_t} v^i \partial_i p\,dx = 2\int_{\mathcal{D}_t} (\operatorname{div} v) p\,dx - 2\int_{\partial\mathcal{D}_t} v_N p\,dS = 0 \tag{3.2}$$

where $v_N = N_i v^i$ is the normal component of v. For further use we also note that

$$D_t \int_{\partial\mathcal{D}_t} g\,dS = \int_{\partial\mathcal{D}_t} (D_t g - g\nabla_N N^i D_t x_i)\,dS \tag{3.3}$$

if dS is the induced surface measure.

4. The linearized equations.

Differentiating (2.3) and using the formula for the derivative of the inverse of a matrix, $D_t M^{-1} = -M^{-1}(D_t M)M^{-1}$, gives

$$[D_t, \partial_i] = -(\partial_i D_t\, x^k)\partial_k \tag{4.1}$$

Differentiating (2.4), using (4.1) and (2.6) gives the linearized equations:

$$D_t^2 v_i - (\partial_k p)\partial_i v^k = -\partial_i q, \qquad \text{and} \qquad \operatorname{div} v = 0, \qquad q\Big|_{\partial\Omega} = 0 \tag{4.2}$$

where $v = D_t x$ and $q = D_t\, p$. Note that here q is determined as a functional of v. In fact using the divergence free condition for v and taking the divergence of (4.2) gives us an equation for q:

$$\triangle q = \partial_i\big((\partial_k p)\partial_i v^k\big) - 2(\partial_i D_t\, x^k)\big(\partial_k D_t\, v^i - (\partial_k D_t\, x^l)\partial_l v^i\big) + (\partial_i \partial_k p)\partial_k v_i \tag{4.3}$$

We can now think of x and p as given and solve (4.2)-(4.3) for v and q. It follows from (4.1) that if $\operatorname{curl} v_{ij} = \partial_i v_j - \partial_j v_i$ then

$$D_t \operatorname{curl} v_{ij} = \operatorname{curl}(D_t v) - (\partial_i D_t\, x^k)\partial_k v_j + (\partial_j D_t\, x^k)\partial_k v_i, \tag{4.4}$$

where $\operatorname{curl}(D_t v)$ vanishes if v is a solution of Euler's equations (2.4) and even if v is just a solution of (4.2) it can be controlled by terms similar to the other terms on the right hand side of (4.4). Similarly, the divergence of $D_t v$ can be controlled using (4.1) and (4.2). Hence we will be able to obtain estimates for the divergence and the curl of v and of $D_t v$. We will see in section 7 that this together with estimates for derivatives that are tangential at the boundary gives estimates for all first order derivatives ∂v.

In the next sections we will get energy estimates similar to the one in section 3. The main difference between (4.2) and (2.4) is the additional term in the left hand side. This term is an operator of order one acting on v so its not lower order and will have to be included in the energy. Note that since q and p vanishes on the boundary it follows that the ∂q and ∂p point in the normal direction and so by (4.2)

$$D_t^2 v_i - (\nabla_N p) N_k \partial_i v^k = -(\nabla_N q) N_i, \quad \text{on} \quad \partial\Omega \tag{4.5}$$

where N is the exterior unit conormal and $\nabla_N = N^i \partial_i$ is the normal derivative. The second term in the left will contribute with a positive term to the energy only because of the sign condition (1.5). The right hand side of (4.2) or (4.5) will be lower order but difficult to control. First, in sections 5 and 6, we will ignore this term and then afterwards in the following sections explain how to deal with it. Following [CL], in sections 8 and 9 ,we will deal with it by projecting the equation onto the tangent space of the boundary so the right hand side of (4.5) vanishes. Following [L1,L2], in sections 10 and 11, we will explain how to deal with it by projecting the equation along gradients onto divergence free vector fields so the right hand side of (4.2) vanishes again. These projections are quite different in nature since the first projection is local and the second is non-local.

5. Energy estimates in the irrotational case.

In the irrotational case $\partial_i v_k = \partial_k v_i$ and $\triangle v_i = 0$ so (4.5) become:

$$D_t^2 v_i - (\nabla_N p) \nabla_N v_i = -(\nabla_N q) N_i, \quad \text{on} \quad \partial\Omega, \qquad \text{and} \qquad \triangle v_i = 0, \quad \text{in} \quad \Omega \tag{5.1}$$

Since $\triangle v_i = 0$ for each i it follows that v is completely determined by its boundary values and so the normal derivative ∇_N can be considered as an operator on the boundary only. Hence the first equation in (5.1) makes sense restricted to the boundary. Furthermore, on function satisfying $\triangle v = 0$ the normal derivative is a positive symmetric operator on the boundary since by the divergence theorem

$$\int_{\partial\mathcal{D}_t} w \nabla_N v \, dS = \int_{\mathcal{D}_t} \nabla w \cdot \nabla v \, dx \tag{5.2}$$

The natural energy to use in the irrotational case is then

$$E(t) = \int_{\partial\mathcal{D}_t} |D_t v|^2 \nu \, dS + \int_{\partial\mathcal{D}_t} v \nabla_N v \, dS, \qquad \nu = (-\nabla_N p)^{-1} \tag{5.3}$$

This energy is positive because of the sign condition (1.5).

6. Energy estimates in the general case.

In the general case we can just replace the second term in the energy (5.3) by the right hand side of (5.2) to obtain:

$$E(t) = \int_{\partial\mathcal{D}_t} |D_t v|^2 \nu \, dS + \int_{\mathcal{D}_t} |\nabla v|^2 \, dx \tag{6.1}$$

Differentiating (6.1), using (3.1) and (3.3), we obtain

$$\frac{dE}{dt} = 2\int_{\partial\mathcal{D}_t} (D_t v^i) D_t^2 v_i \, \nu \, dS + 2\int_{\mathcal{D}_t} \delta^{ij}\delta^{kl} (\partial_i v_k) D_t \partial_j v_l \, dx + \int_{\partial\mathcal{D}_t} |D_t v|^2 (D_t \nu - \nu \nabla_N N^j D_t x_j) \, dS, \tag{6.2}$$

where the last term comes from that the measure on the boundary changes with time. Writing $D_t\partial_j v_l = \partial_l D_t v_j + D_t \operatorname{curl} v_{jl} + [D_t,\partial_l] v_j$ and using the divergence theorem we obtain

$$(6.3)\quad \frac{dE}{dt} = 2\int_{\partial\mathcal{D}_t} (D_t v^i)\big(D_t^2 v_i\,\nu + N_k\partial_i v^k\big)\,dS$$

$$+2\int_{\mathcal{D}_t} -(\partial_i \operatorname{div} v)D_t v^i + \delta^{ij}\delta^{kl}(\partial_i v_k)\big(D_t \operatorname{curl} v_{jl} - (\partial_l D_t\, x^n)\partial_n v_j\big)\,dx + \int_{\partial\mathcal{D}_t} |D_t v|^2 (D_t\nu - \nu\nabla_N N^j D_t x_j)\,dS$$

where the terms on the second row either vanish or are controlled by the energy itself using the second part of (4.2) and (4.4). Since $\nu^{-1}N_k = -\partial_k p$ it follows that

$$(6.4)\qquad \frac{dE}{dt} = 2\int_{\partial\mathcal{D}_t} (D_t v^i)\big(D_t^2 v_i\, - (\partial_k p)\partial_i v^k\big)\,\nu\, dS + O(E) = 2\int_{\partial\mathcal{D}_t} (D_t v^i)\partial_i q\,\nu\, dS + O(E)$$

by (4.2) or (4.5). The main difficulty is now to control this term, i.e. the right hand side of (4.5). In order to do this we have to modify the energy above. Furthermore in order to get estimates for higher derivatives we also have to obtain higher order energies.

7. Estimates of derivatives of a vector field by the divergence, the curl and tangential derivatives.

We claim that there is a constant C such that

$$(7.1)\qquad |\partial v| \le C\big(|\operatorname{div} v| + |\operatorname{curl} v| + \sum_{S\in\mathcal{S}} |Sv|\big)$$

where $\mathcal{S}$ is a set of vector fields that span the tangent space of the boundary at the boundary and the full tangent space in the interior.

Since $\mathcal{S}$ span the full tangent space in the interior when the distance to the boundary $d(y)\ge d_0$ we may assume that $d(y) < d_0$. Let $\Omega^a = \{y;\, d(y) > a\}$ and let $\mathcal{D}_t^a$ be the image of this set under mapping $y\to x(t,y)$. Let N the exterior unit normal to $\partial\mathcal{D}_t^a$. Then $q^{ij} = \delta^{ij} - N^i N^j$ is the inverse of the tangential metric. Since the tangential vector fields span the tangent space of the level sets of the distance function we have $q^{ij}a_i a_j \le C\sum_{S\in\mathcal{S}} S^i S^j a_i a_j$, where here $S^i = S^a\partial x^i/\partial y^a$. We claim that for any two tensor β_{ij}:

$$(7.2)\qquad \delta^{ij}\delta^{kl}\beta_{ki}\beta_{lj} \le C_n\big(\delta^{ij}q^{kl}\beta_{ki}\beta_{lj} + |\hat\beta|^2 + (\operatorname{tr}\beta)^2\big)$$

where $\hat\beta_{ij} = \beta_{ij} - \beta_{ji}$ is the anti symmetric part and $\operatorname{tr}\beta = \delta^{ij}\beta_{ij}$ is the trace. To prove (7.2) we may assume that β is symmetric and trace less. Writing $\delta^{ij} = q^{ij} + N^i N^j$ we see that the estimate for such tensors follows from the estimate $N^iN^jN^kN^l\beta_{ki}\beta_{lj} = (N^iN^k\beta_{ki})^2 = (q^{ik}\beta_{ki})^2 \le nq^{ij}q^{kl}\beta_{ki}\beta_{lj}$. (This inequality just says that $(\operatorname{tr}(Q\beta))^2 \le n\operatorname{tr}(Q\beta Q\beta)$ which is obvious if one writes it out and use the symmetry.)

8. Projection onto the tangential components.

Following [CL], we define the projection onto tangential components. For a $(0,r)$ tensor α let the projection onto the tangential components be given by

$$(8.1)\qquad (\Pi\alpha)_{i_1\dots i_r} = \Pi_{i_1}^{j_1}\cdots\Pi_{i_r}^{j_r}\alpha_{j_1\dots j_r},\qquad \text{where}\quad \Pi_i^j = \delta_i^j - N_i N^j$$

where N is the unit normal to the boundary. Then if q vanishes on the boundary it follows that the tangential derivative $\overline{\partial q} = (\Pi\partial)q$ also vanishes there. Furthermore

$$(\Pi\partial^2 q)_{ij} = \theta_{ij}\nabla_N q, \tag{8.2}$$

where $\theta_{ij} = \Pi_i^k \partial_k N_j$ is the second fundamental form of the boundary. In fact,

$$0 = \overline{\partial}_i\overline{\partial}_j q = \Pi_i^{\,i'}\partial_{i'}\Pi_j^{\,j'}\partial_{j'}q = \Pi_i^{\,i'}\Pi_j^{\,j'}\partial_{i'}\partial_{j'}q - (\overline{\partial}_i N_j)N^k\partial_k q - N_j(\overline{\partial}_i N^k)\partial_k q = (\Pi\partial^2 q)_{ij} - \theta_{ij}\nabla_N q$$

since $N_k\overline{\partial}_i N^k = \overline{\partial}_i(N_k N^k)/2 = 0$. Similarly, one can prove higher order versions of (8.2):

$$\Pi\partial^r q = O(\partial^{r-1}q) \tag{8.3}$$

We also define the tangential quadratic form

$$Q(\alpha,\alpha) = q^{i_1 j_1}\cdots q^{i_r j_r}\alpha_{i_1\dots i_r}\alpha_{j_1\dots j_r}, \quad \text{where} \quad q^{ij} = \delta^{ij} - \tilde{N}^i\tilde{N}^j \tag{8.4}$$

and $\tilde{N}$ is some extension of the normal to the interior, satisfying $|\tilde{N}| \le 1$. The tangential quadratic form is related to the projection by

$$\langle \Pi\alpha, \Pi\beta\rangle = Q(\alpha,\beta), \qquad \text{on} \quad \partial\Omega \tag{8.5}$$

Furthermore, one can prove higher order versions of (7.1)-(7.2):

$$|\partial^r v|^2 \le C\big(|\partial^{r-1}\operatorname{div} v|^2 + |\partial^{r-1}\operatorname{curl} v|^2 + \delta^{ij}Q(\partial^r v_i, \partial^r v_j)\big) \tag{8.6}$$

9. Energies using the projection onto the tangential components.

Following [CL], the basic idea is now to modify the energy (6.1) so it only contains tangential components on the boundary

$$E(t) = \int_{\partial\mathcal{D}_t} |\Pi D_t v|^2 \nu dS + \int_{\mathcal{D}_t} |(\tilde{\Pi}\partial)v|^2\, dx + \int_{\mathcal{D}_t} |\operatorname{curl} v|^2\, dx \tag{9.1}$$

where Π is the projection onto the tangent space of the boundary and $\tilde{\Pi}$ is some extension of the projection to the interior. Because the divergence vanishes and because the curl is controlled by the energy we can estimate all components of ∂v by the energy using (7.1)-(7.2).

When differentiating (9.1) we get a boundary term similar to the term on the right of (6.4) but with the projection onto the boundary:

$$\frac{dE}{dt} = 2\int_{\partial\mathcal{D}_t} \langle \Pi D_t v, \Pi\big(D_t^2 v - (\partial_k p)\partial v^k\big)\rangle\, \nu\, dS + O(E) = 2\int_{\partial\mathcal{D}_t} \langle \Pi D_t v, \Pi\partial q\rangle\, \nu\, dS + O(E) \tag{9.2}$$

Since $\partial q = (\nabla_N q)N$ is normal the projection of this to the tangent space of the boundary vanishes so the leading order term in the right hand side of (9.2) vanishes.

The higher order energy norms are now defined by

$$E_r(t) = \int_{\partial\mathcal{D}_t} Q(\partial^{r-1}D_t v, \partial^{r-1}D_t v)\, \nu dS + \int_{\mathcal{D}_t} \delta^{ij}Q(\partial^r v_i, \partial^r v_j)\, dx + \int_{\mathcal{D}_t} |\partial^{r-1}\operatorname{curl} v|^2\, dx \tag{9.3}$$

It follows from (8.6) that since $\operatorname{div} v = 0$;

$$\|\partial^r v\|_{L^2(\mathcal{D}_t)} \le C E_r. \tag{9.4}$$

Differentiating (9.3) using (4.2), (9.4) and (8.5) yields

(9.5)
$$\frac{dE_r}{dt} = 2\int_{\partial\mathcal{D}_t} Q\big(\partial^{r-1} D_t v, \partial^{r-1}(D_t^2 v - (\partial_k p)\partial v^k)\big)\, dS + O(E_r) = 2\int_{\partial\mathcal{D}_t} \langle \Pi\partial^{r-1} D_t v, \Pi \partial^r q\rangle\, \nu\, dS + O(E_r)$$

Using (8.3) one can estimate $\Pi\partial^r q$ by $\partial^{r-1} q$ on the boundary. Estimates for this in terms of the energy can be obtained from simple elliptic estimates for (4.3), see [CL]. Finally we obtain the energy estimate

$$\Big|\frac{dE_r}{dt}\Big| \le C E_r \tag{9.6}$$

which gives an energy bound.

Note that the projection plays an essential role here in the case of non vanishing curl. In fact, if $\partial^r v$ is in L^2 in the interior then the best we can get for the solution of (4.3) is that $\partial^r q$ is in L^2 in the interior and restricting to the boundary we loose half a derivative so we can not expect $\partial^r q$ to be in L^2 on the boundary.

10. Projection onto divergence free vector fields.

In [L1,L2] a different type of, non-local, projection is being used. The orthogonal projection onto divergence free vector fields in the inner product

$$\langle u, w\rangle = \int_{\mathcal{D}_t} \delta_{ij} u^i w^j\, dx \tag{10.1}$$

is given by

$$P u^i = u^i - \partial_i\, p_u, \qquad \triangle p_u = \operatorname{div} u, \quad p_u\Big|_{\partial\Omega} = 0. \tag{10.2}$$

In fact

$$\langle u, \partial q\rangle = \int_{\mathcal{D}_t} u^i \partial_i q\, dx = \int_{\partial\mathcal{D}_t} u^i N_i\, q\, dS - \int_{\mathcal{D}_t} (\partial_i u^i) q\, dx \tag{10.3}$$

vanishes if q vanishes on the boundary and u is divergence free.

11. Energies using the projection onto the tangential components.

Following [L1], let A be the operator on divergence free vector fields defined by

$$A w^i = P\big(-\partial_i(w^k \partial_k p)\big) \tag{11.1}$$

Then A is a positive symmetric operator, if condition (1.5) holds. In fact, if u and w are divergence free then

$$\langle u, Aw\rangle = -\int_{\mathcal{D}_t} u^i \partial_i(w^k\partial_k p)\, dx = -\int_{\partial\mathcal{D}_t} u_N w_N\, (\nabla_N p)\, dS, \qquad \text{where} \quad w_N = N_i w^i \tag{11.2}$$

Now, If we project (4.2) onto divergence free vector fields then the right hand side vanishes and we obtain

$$P(D_t^2 v^i) + Av^i = P\big(-(\partial_i\partial_k p)v^k\big) \tag{11.3}$$

where the right hand side clearly is lower order in v. For the equation (11.3) we now define a different type of energy

$$E(t) = \langle D_t v, D_t v\rangle + \langle v, Av\rangle \tag{11.4}$$

The main difficulty with proving existence, regularity and estimates for (11.3) is that A is of order one and positive but it is not elliptic. In fact A vanishes on compactly supported divergence free vector fields, and there are such vector fields outside the class of irrotational vector fields. However, this difficulty is being dealt with by applying tangential vector fields and using (7.1) together with that the curl and the divergence of Av vanishes, see [L1].

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