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KAM Tori and Quantum Birkhoff Normal Forms

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Abstract

This talk is concerned with the Kolmogorov-Arnold-Moser (KAM) theorem in Gevrey classes for analytic hamiltonians, the effective stability around the corresponding KAM tori, and the semi-classical asymptotics for Schrödinger operators with exponentially small error terms. Given a real analytic Hamiltonian H close to a completely integrable one and a suitable Cantor set Θ defined by a Diophantine condition, we find a family Λ_ω, $\omega \in \Theta$, of KAM invariant tori of H with frequencies $\omega \in \Theta$ which is Gevrey smooth with respect to ω in a Whitney sense. Moreover, we obtain a symplectic Gevrey normal form of the Hamiltonian in a neighborhood of the union Λ of the KAM tori which can be viewed as a Birkhoff normal form (BNF) of H around Λ. This leads to effective stability of the quasiperiodic motion near Λ. We investigate the semi-classical asymptotics of a Schrödinger type operator with a principal symbol H. We obtain semiclassical quasimodes with exponentially small error terms which are associated with the Gevrey family of KAM tori Λ_ω, $\omega \in \Theta$. To do this we construct a quantum Birkhoff normal form (QBNF) of the Schrödinger operator around Λ in suitable Gevrey classes starting from the BNF of H. As an application, we obtain a sharp lower bound for the counting function of the resonances which are exponentially close to a suitable compact subinterval of the real axis.

1 KAM tori and BNFs for analytic hamiltonians in Gevrey classes.

1.1 Classical KAM theorem.

Let $\mathbf{T}^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$, $n \geq 2$, denote by D a bounded domain in $\mathbf{R}^n$, and set $\mathbf{A}^n = \mathbf{T}^n \times D$, equipped with the standard symplectic two form $\sum d\varphi_j \wedge dI_j$. Consider in $\mathbf{A}^n$ a real analytic hamiltonian $H^0 : D \to \mathbf{R}$ which is independent of φ. The Hamiltonian vector field of H^0 is $X_{H^0} = \langle \nabla H^0(I), \partial/\partial\varphi \rangle$, hence, its flow is given by $\exp(tX_{H^0})(\varphi, I) = (\varphi + t\nabla H^0(I), I)$. Suppose H^0 is nondegenerate which means that the "frequency map" $\nabla H^0 : D \to \Omega := \nabla H^0(D)$ is a diffeomorphism, and denote by $\omega \to I(\omega)$ the inverse map. In the coordinates (φ, ω) in $\mathbf{T}^n \times \Omega$, the flow is given by $(t,\varphi,\omega)\to(\varphi+t\omega, I(\omega))$. Thus each torus $\Lambda^0_\omega := \mathbf{T}^n\times\{I(\omega)\}$ is invariant under X_{H^0}, and the restriction of the flow on it is a rotation $R_{t\omega}$ by tw.

The classical Kolmogorov-Arnold-Moser (KAM) theorem asserts that a large family of the invariant tori Λ^0_ω sustain small real analytic perturbations $H(\varphi,I)$ of $H^0(I)$, being just a little bit deformed. Fix $\kappa>0$ and $\tau>n-1$. The frequences ω of these tori satisfy the following Diophantine condition:

$$|\langle\omega,k\rangle| \;\geq\; \frac{\kappa}{(\sum|k_j|)^\tau}\,,\quad \text{for all } 0\neq k=(k_1,\ldots,k_n)\in\mathbf{Z}^n\,. \tag{1.1}$$

We denote by Ξ_κ the set of all $\omega\in\Omega$ satisfying (1.1) and also having distance $\geq\kappa$ to the boundary of Ω. Note that Ξ_κ is a Cantor set and the Lebesgue measure $\mathrm{vol}(\Xi_\kappa)>0$ if κ is small enough. The KAM theorem says that if H is a sufficiently "small" (with respect to κ) real analytic perturbation of H^0 , then for each $\omega\in\Xi_\kappa$ there is an analytic Lagrangian submanifold Λ_ω of $\mathbf{A}^n$ close to Λ^0_ω which is invariant under the flow of X_H and the restriction of $\exp(tX_H)$ to Λ_ω is conjugated to the rotation $R_{t\omega}$. Moreover, Λ_ω depend *smoothly* on $\omega\in\Xi_\kappa$ in the sense of Whitney. The Lagrangian manifolds $\Lambda_\omega\simeq\mathbf{T}^n$ are called KAM tori. It turns out that the family of KAM tori is *Gevrey smooth* with respect to the frequences ω.

1.2 KAM theorem for analytic hamiltonians in Gevrey classes.

We are going to show that for each $\tau'>\tau$ there exists a family of KAM tori Λ_ω, $\omega\in\Xi_\kappa$, which is $G^{\tau'+2}$-Gevrey regular in the sense of Whitney. For each $\mu\geq1$, we denote by $G^\mu(D)$ the space of all Gevrey functions in a domain $D\subset\mathbf{R}^n$ of index μ, namely $f\in G^\mu(D)$ if $f\in C^\infty(D)$ and for every compact subset Y of D there exists $C=C(Y)>0$ such that

$$\sup_{I\in Y}|\partial_I^\alpha f(I)| \;\leq\; C^{|\alpha|+1}\alpha!^{\,\mu}\,,\ \forall\,\alpha\in\mathbf{Z}^n_+\,,$$

where $\mathbf{Z}_+$ stands for the set of all nonnegative integers and $\alpha!^\mu=(\alpha_1!\ldots\alpha_n!)^\mu$, $\alpha=(\alpha_1,\ldots,\alpha_n)$. Evidently $G^1(D)$ coincides with the space of all analytic functions in D, while for $\mu>1$ there are nontrivial compactly supported G^μ functions. Given $\sigma,\mu\geq1$, we say that $R\in G^{\sigma,\mu}(\mathbf{T}^n\times D)$ if for every compact subset Y of D there exists $C=C(Y)>0$ such that

$$\sup_{(\varphi,I)\in\mathbf{T}^n\times Y}|\partial_\varphi^\beta\partial_I^\alpha R(\varphi,I)| \;\leq\; C^{|\alpha|+|\beta|+1}\beta!^{\,\sigma}\,\alpha!^{\,\mu}\ ,\ \forall\,\alpha,\,\beta\in\mathbf{Z}^n_+\,. \tag{1.2}$$

To formulate the "smallness" condition in the KAM theorem we extend H^0 holomorphically in a neighborhood of D. Let D^0 be a bounded domain in $\mathbf{R}^n$ and $r^0>0$. Let H^0 be a real analytic Hamiltonian in $D^0 + r^0 = \{z \in \mathbf{C}^n : |D^0 - z| \le r_0\}$, where $|D^0 - z| = \inf_{z' \in D^0} |z' - z|$. Suppose H^0 is *non-degenerate* in $D^0 + r^0$. This means that the map $D^0 \ni I \longrightarrow \nabla H^0(I) \in \nabla H^0(D^0) = \Omega^0$ is a diffeomorphism, the Hessian matrix $H^0_{zz}(z)$ of H^0 is non-degenerate in $D^0 + r^0$, and $|H^0_{zz}|_{D^0+r^0}$, $|(H^0_{zz})^{-1}|_{D^0+r^0} \le R$ for some $R > 0$, where $|\cdot|_{D^0+r^0}$ stands for the sup-norm in $D^0 + r^0$. Denote by $\psi_0 : \Omega^0 \to D^0$ the inverse map. Given $r^0 \ge r > 0$, $s > 0$, and a subdomain $D \subset D^0$, we set

$$\mathbf{T}^n + s \;=\; \{z \in \mathbf{C}^n/2\pi\mathbf{Z}^n : |\mathrm{Im}\, z| \le s\}\,,\ \mathbf{U}_{s,r} \;=\; \mathbf{U}_{s,r,D} \;=\; (\mathbf{T}^n + s) \times (D + r)\,,$$

the latter being equipped with sup-norm $|\cdot|_{s,r}$ and denote $\Omega = \nabla H^0(D)$.

As it was mentioned above, for each subdomain D of D^0 there exists $\widetilde{\kappa}(D) > 0$ such that $\mathrm{vol}\,(\Xi_\kappa) > 0$ for each $0 < \kappa < \widetilde{\kappa}(D)$. For such κ, we denote by Ω_κ the set of points of a positive Lebesgue density in Ξ_κ. In other words, $\omega \in \Omega_\kappa$ if for any neighborhood U of ω in Ω the Lebesgue measure of $U \cap \Xi_\kappa$ is positive. Obviously, Ω_κ and Ξ_κ have the same Lebesgue measure. The advantage of working with Ω_κ is that if R is a C^∞ function in Ω and $R = 0$ on Ω_κ then all the derivatives of R also vanish on Ω_κ. This will be used to obtain a BNF of H near a family of KAM tori with frequences in Ω_κ.

Fix the constant $\tau > n - 1$ in the small divisor condition (1.1) and chose $\tau' > \max\,(5/2, \tau)$ and $s > 0$. Notice that the condition $\tau' > 5/2$ is required only for dimensions $n \le 3$. Fix an integer $N \ge 1$.

Theorem 1.1 *Let H^0 be real analytic and non-degenerate in $D^0 + r^0$. Then there is $\delta > 0$ such that for any domain $D \subset D^0$, $0 < \kappa < \widetilde{\kappa}(D)$, $\kappa \le r \le r^0$, and any real analytic Hamiltonian H in $\mathbf{U}_{s,r,D}$ with $\delta_H := \kappa^{-2}|H - H^0|_{s,r} \le \delta$, there is a map $f : \mathbf{T}^n \times \Omega \to D$ of Gevrey class $G^{1,\tau'+2}$, such that each $\Lambda_\omega := \{(\theta,\, f(\theta,\omega)) : \theta \in \mathbf{T}^n\}$, $\omega \in \Omega_\kappa$, is a Lagrangian submanifold of $\mathbf{T}^n \times D$, invariant with respect to the flow of X_H, and the restriction of $\exp(tX_H)$ to Λ_ω is conjugated to the rotation $R_{t\omega}$ on $\mathbf{T}^n$. Moreover, for any $0 < q < 1$, there exists $L > 0$ independent of D, κ, r, δ_H, such that*

$$\left|D^\beta_\theta D^\alpha_\omega (f(\theta,\omega) - \psi_0(\omega))\right| \;\le\; L^{|\beta|+1}\,\kappa^{1-|\alpha|}\;\beta!\,\delta^q_H\ ,\ \forall\,(\theta,\omega) \,\in\, \mathbf{T}^n \times \Omega\ ,\ \beta \,\in\, \mathbf{Z}^n_+\,,\ |\alpha| \le N.$$

To prove the theorem we make use of the scheme proposed by Pöschel in [8].

1.3 BNF and effective stability

As a consequence we obtain a Birkhoff normal form (BNF) of H around the family of KAM tori Λ_ω, $\omega \in \Omega_\kappa$. It will be said that H admits a G^μ-BNF around the family of KAM tori with frequencies in Ω_κ if the following holds:

(BF) There exists a G^μ-diffeomorphism $\omega : D \to \Omega$ and an exact symplectic transformation $\chi_0 \in G^{1,\mu}(\mathbf{T}^n \times D, \mathbf{T}^n \times D)$ such that $H(\chi_0(\varphi, I)) = K_0(I) + R_0(\varphi, I)$, where $K_0 \in G^\mu(D)$ and $R_0 \in G^{1,\mu}(\mathbf{T}^n \times D)$ satisfy $D_I^\alpha R_0(\varphi, I) = 0$ and $D_I^\alpha(\nabla K_0(I) - \omega(I)) = 0$ for any $(\varphi, I) \in \mathbf{T}^n \times \omega^{-1}(\Omega_\kappa)$ and $\alpha \in \mathbf{Z}_+^n$.

Corollary 1.2 *Under the conditions of Theorem 1.1, the Hamiltonian H admits a G^μ-BNF, $\mu = \tau' + 2$, around the family of invariant tori Λ_ω with frequencies in Ω_κ. Moreover, the exact symplectic map χ_0 in (BF) has a generating function $\Phi \in G^{1,\mu}$, and the function Φ and the diffeomorphism ω in (BF) satisfy*

$$\left| D_\varphi^\beta D_I^\alpha (\Phi(\varphi, I) - \langle \varphi, I \rangle) \right| + \left| D_I^\alpha (\omega(I) - \nabla H^0(I)) \right| \leq L^{|\beta|+1} \kappa^{1-|\alpha|} \beta!\, \delta_H^q \ , \ \forall \beta \in \mathbf{Z}_+^n ,$$

for $(\varphi, I) \in \mathbf{T}^n \times D$ and $|\alpha| \leq N$, where $L > 0$ is independent of D, κ, r, δ_H.

Recall that a smooth function Φ in $\mathbf{T}^n \times D$ is a generating function of a symplectic map χ_0 of $\mathbf{T}^n \times D$ into itself, if $\det(\mathrm{Id} - \Phi_{\theta I}) \neq 0$, and

$$\chi_0 (\Phi_I(\theta, I), I) = (\theta, \Phi_\theta(\theta, I)) \ , \ (\theta, I) \in \mathbf{T}^n \times D .$$

Set $E_\kappa = \omega^{-1}(\Omega_\kappa)$ and denote by Y a compact neighborhood of E_κ in D. Then each torus $\mathbf{T}^n \times \{I\}$, $I \in E_\kappa$, is invariant under the flow of $\widetilde{H}(\varphi, I) = H(\chi_0(\varphi, I))$, and there we have $\exp(tX_{\widetilde{H}})(\varphi, I) = (\varphi + t\nabla K_0(I), I)$. Since R is G^μ with respect to $I \in D$, and its Taylor series vanishes at each $I \in E_\kappa$, there exist positive constants C_1 and c depending only on the constant C in (1.2) such that for every $\alpha, \beta \in \mathbf{Z}_+^n$ the following estimate holds

$$|\partial_\varphi^\beta \partial_I^\alpha R_0(\varphi, I)| \leq C_1^{|\alpha|+|\beta|+1} \beta!^{\,\sigma} \alpha!^{\,\mu} \exp\left(-c\,|E_\kappa - I|^{-\frac{1}{\mu-1}}\right) , \tag{1.3}$$

for any $(\varphi, I) \in \mathbf{T}^n \times Y$, $I \notin E_\kappa$, where $|E_\kappa - I| = \inf_{I' \in E_\kappa} |I' - I|$ is the distance to the compact set E_κ.

The symplectic normal form (BF) obtained in Corollary 1.2 leads immediately to effective stability of the quasiperiodic motion around the invariant tori.

Corollary 1.3 *There is $\widetilde{C}_0, \widetilde{C} > 0$ such that for each $0 < \varepsilon \le 1$, and any initial data $(\varphi_0, I_0) \in \mathbf{T}^n \times Y$ with $|E_\kappa - I_0| \le \varepsilon$ we have*

$$|\exp(tX_{\widetilde{H}})(\varphi_0, I_0) \ - \ (\varphi_0 + t\nabla K_0(I_0), I_0)| \ \le \ \widetilde{C}_0 \varepsilon \, ,$$

provided that

$$|t| \ \le \ \widetilde{C}\, \varepsilon \exp\left(\frac{c}{2} \varepsilon^{-1/(\tau'+1)} \right) .$$

Effective stability (of the action) for analytic perturbations of completely integrable Hamiltonians was first studied by Nekhoroshev. The Nekhoroshev theorem states that the variation of the action $I(t) - I_0$ on each orbit $(\varphi(t), I(t)) = \exp(tX_{H_\varepsilon})(\varphi_0, I_0)$ of an analytic Hamiltonian $H_\varepsilon = H_0 + O(\varepsilon)$ remains ε-small in a finite but exponentially long time interval $0 \le t \le T \exp(\varepsilon^a)$, $T > 0$, $a > 0$, if H_0 satisfies certain generic steepness conditions (see [5], [9], and the references there).

1.4 KAM tori near an elliptic equilibrium.

Consider the Hamiltonian $P_0(x, \xi) = |\xi|^2 + V(x)$, where $V(x) \ge V(0) = 0$ has a nondegenerate minimum at $x = 0$ and V is analytic in a neighborhood of 0. There is a linear change of the coordinates in x such that $P_0(x, \xi) = \sum_{j=1}^n \alpha_j^0 (x_j^2 + \xi_j^2)/2 + O(|(x, \xi)|^3)$ near the origin, where $\alpha_j^0 > 0$.

This is a special case of an analytic Hamiltonian P_0 with an elliptic equilibrium at some $\varrho_0 = (x_0, \xi_0)$ with the characteristic exponents $\pm i\alpha_1^0, \ldots, \pm i\alpha_n^0$, $\alpha_j^0 > 0$. Set $\alpha^0 = (\alpha_1^0, \ldots, \alpha_n^0)$. To apply Theorem 1.1 we exclude the resonances of order ≤ 4. In other words, we assume that $\langle \alpha^0, k \rangle \ne 0$ for each $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ with $0 < \sum |k_j| \le 4$. Then there exists an analytic polar symplectic change of the variables $(x, \xi) = \chi_1(\varphi, I)$, such that $H(\varphi, I) := P_0(\chi_1(\varphi, I))$ admits the following Birkhoff normal form

$$H(\varphi, I) = H^0(I) + O(|I|^{5/2}) \ , \ \text{as } |I| \to 0 \, , \ H^0(I) = P_0(\varrho_0) + \langle \alpha^0, I \rangle + \langle QI, I \rangle \, , \qquad (1.4)$$

where Q is a $n \times n$ matrix, $\varphi \in \mathbf{T}^n$ and I belongs to a proper open cone $\Gamma \subset \mathbf{R}_+^n$ with a vertex at 0. We suppose that H^0 is nondegenerated, which amounts to $\det Q \ne 0$.

To avoid the singularity of H at $I = 0$, we consider

$$D^0 = \{ I \in \mathbf{R}^n : \, |I_j| < C_1 a_0, \ j = 1, \ldots, n \} \, ,$$

where $0 < a_0 \ll 1$ and $C_1 > 1$. Obviously H^0 is nondegenerate in $D^0 + a_0$. For each $0 < a \leq a_0$ we set $D = D_a = \{I \in \Gamma : C_1^{-1} a \leq I_j \leq C_1 a,\ j = 1, \ldots, n\}$. Next we choose $\kappa = \kappa_a = \varepsilon a$, where $0 < \varepsilon \ll 1$ is fixed, then fix $0 < s \ll 1$, and take $r = a$. The perturbation H satisfies

$$|H - H^0|_{s,r} \ \leq \ \kappa^2 C_2 a^{1/2} \leq \kappa^2 \delta,$$

for each $0 < a \leq a_0$ choosing a_0 small enough. Hence, applying Theorem 1.1 and Corollary 1.2, we obtain:

Corollary 1.4 *Let ϱ_0 be an elliptic equilibrium of a real analytic Hamiltonian P_0 without resonances of order ≤ 4. Denote by $H(\varphi, I) = H^0(I) + H_1(\varphi, I)$ the corresponding BNF (1.4) and suppose that $\det Q \neq 0$. Then for each $0 < a \leq a_0$, $a_0 \ll 1$, there exists a symplectic diffeomorphism χ_0 of Gevrey class $G^{1,\mu}$, $\mu = \tau' + 2$, mapping $\mathbf{T}^n \times D$ into itself, a G^μ-diffeomorphism $\omega : D \to \Omega$, and $K \in G^\mu(D)$ such that the Hamiltonian $\widetilde{H} = H \circ \chi_0 \in G^{1,\mu}(\mathbf{T}^n \times D)$ has the form $\widetilde{H}(\varphi, I) \ = \ K_0(I) \ + \ R_0(\varphi, I)$, where R_0 and $\omega(I) - \nabla K_0(I)$ are flat on $\mathbf{T}^n \times E_\kappa$ and $E_\kappa = \omega^{-1}(\Omega_\kappa)$ respectively. The symplectic map χ_0 has a generating function $\Phi \in G^{1,\mu}(\mathbf{T}^n \times D)$ and for each $|\alpha| \leq 1$ and $0 < q < 1$, we have*

$$\left| D_\varphi^\beta D_I^\alpha (\Phi(\varphi, I) - \langle \varphi, I \rangle) \right| \ + \ \left| D_I^\alpha (\omega(I) - \nabla H^0(I)) \right|$$
$$\leq \ C^{|\alpha| + |\beta| + 1}\, a^{3q/2 - |\alpha|}\ \beta!\,,\ \forall \beta \in \mathbf{Z}_+^n\,,$$

for any $(\varphi, I) \in \mathbf{T}^n \times D$, where C does not depend on a.

2 Quantum Birkhoff normal forms and quasimodes with exponentially small errors.

2.1 Quantum Birkhoff normal forms.

Let M be either $\mathbf{R}^n$ or a compact real analytic manifold of dimension $n \geq 2$ and let

$$\mathcal{P}_h \ = \ \sum_{j=0}^{J} P_j(x, hD) h^j\,,\ 0 < h \leq h_0\,, \tag{2.1}$$

be a formally selfadjoint h-differential operator acting on half densities in $C^\infty(M, \Omega^{\frac{1}{2}})$, where $P_j(x, \xi)$ are polynomials of ξ with analytic coefficients, and $D = (D_1, \ldots, D_n)$, $D_j = -i\partial/\partial x_j$. The principal symbol of $\mathcal{P}_h$ is $P_0(x, \xi)$, $(x, \xi) \in T^*(M)$, and we suppose that the subprincipal symbol is zero. Our main example will be the Schrödinger operator $\mathcal{P}_h = -h^2\Delta + V(x)$, where

Δ is the Laplace-Beltrami operator on M, associated with a real analytic Riemannian metric and $V(x)$ is a real analytic potential on M bounded from below.

We suppose that there exists a real analytic exact symplectic diffeomorphism $\chi_1 : \mathbf{T}^n \times D \longrightarrow U \subset T^*(M)$, where D is a domain in $\mathbf{R}^n$ such that the Hamiltonian $H(\varphi, I) := (P_0 \circ \chi_1)(\varphi, I)$ admits a $G^{\tau'+2}$-BNF given by (BF) around a family of invariant tori with frequencies in Ω_κ. The map χ_1 provides "action-angle" coordinates for the "completely integrable part" of P_0 and it can be constructed by the Liouville-Arnold theorem. For example, if we take $M = \mathbf{R}^n$, $\mathcal{P}_h = -h^2\Delta + V(x)$, and suppose that V has a nondegenerate minimum $V(0)$ and that there are no resonances of order ≤ 4, then Corollary 1.3 holds. We set $\chi = \chi_1 \circ \chi_0$, where χ_0 is given by (BF). Then $P_0(\chi(\varphi, I)) = K_0(I) + R_0(\varphi, I)$. Staring from the $G^{\tau'+2}$-BNF of the Hamitonian P_0, we are going to obtain a QBNF of the operator $\mathcal{P}_h$ in suitable Gevrey classes of pseudodifferential operators.

Let Λ be the union of the invariant tori $\Lambda_\omega = \chi(\mathbf{T}^n \times \{I(\omega)\})$ of P_0 with frequencies $\omega \in \Omega_\kappa$, where $\Omega \ni \omega \to I(\omega) \in D$ is the inverse to the frequency map $D \ni I \to \omega(I) \in \Omega$. The Maslov class of Λ_ω, $\omega \in \Omega_\kappa$, can be identified with an element ϑ of $H^1(\mathbf{T}^n; \mathbf{Z}) = \mathbf{Z}^n$ via the symplectic map χ. Notice that $\vartheta = (2, \ldots, 2)$ in the case when V has a nondegenerate minimum $E_0 = V(0)$. As in [3] we consider the flat Hermitian line bundle $\mathbf{L}$ over $\mathbf{T}^n$ which is associated to the class ϑ. The sections f in $\mathbf{L}$ can be identified canonically with functions $\widetilde{f} : \mathbf{R}^n \to \mathbf{C}$ so that $\widetilde{f}(x + 2\pi p) = e^{i\frac{\pi}{2}\langle\vartheta, p\rangle}\widetilde{f}(x)$ for each $x \in \mathbf{R}^n$ and $p \in \mathbf{Z}^n$. It is easy to see that an orthonormal basis of $L^2(\mathbf{T}^n; \mathbf{L})$ is given by e_m, $m \in \mathbf{Z}^n$, where

$$\widetilde{e}_m(x) = \exp\left(i\langle m + \vartheta/4, x\rangle\right).$$

Set $\nu = \tau + n + 1$ and fix τ' such that $\tau + n - 1 > \tau' > \max(\tau, 5/2)$. Then fix μ_0 such that $\nu > \mu_0 > \tau' + 2$, choose $\sigma > 1$ sufficiently close to 1 such that $\nu > \mu_0 > \sigma(\tau' + 1) + 1$, and set $\varrho = \sigma\nu$. Thus ϱ could be any number bigger than ν and sufficiently close to ν. Set $\ell = (\sigma, \mu_0, \varrho)$ and consider the class of Gevrey symbols $S_\ell(\mathbf{A}^n)$, $\mathbf{A}^n = \mathbf{T}^n \times D$, defined as follows: First we introduce a class of formal Gevrey symbols $FS_\ell(\mathbf{A}^n)$. Consider a sequence of smooth functions $p_j \in C_0^\infty(\mathbf{A}^n)$, $j \in \mathbf{Z}_+$ such that $\operatorname{supp} p_j$ is contained in a fixed compact subset of $\mathbf{A}^n$. We say that $\sum_{j=0}^\infty p_j(\varphi, I)\, h^j$ is a formal Gevrey symbol in $FS_\ell(\mathbf{A}^n)$ if there exists a positive constant C such that p_j satisfies the estimates

$$\sup_{\mathbf{A}^n} |\partial_\varphi^\beta \partial_I^\alpha p_j(\varphi, I)| \leq C^{j+|\alpha|+|\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, j!^{\,\varrho}$$

for any α, β and j. The function $p(\varphi, I; h)$, $(\varphi, I) \in \mathbf{A}^n$, is called a realization in $\mathbf{A}^n$ of the formal symbol given above, if for each $0 < h \leq h_0$ it is smooth with respect to (φ, I), p is compactly supported in $I \in D$ uniformly with respect to (φ, h), and if there exists a positive constant C_1 such that

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha (p(\varphi, I, h) - \sum_{j=0}^{N} p_j(\varphi, I) h^j)| \ \leq \ h^{N+1}\, C_1^{N+|\alpha|+|\beta|+2}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, (N+1)!^{\,\varrho}$$

for any multi-indices α, β and $N \in \mathbf{Z}_+$, where $\mathbf{Q} = \mathbf{A}^n \times (0, h_0]$. For example, one can take

$$p(\varphi, I, h) \ = \ \sum_{j \leq \eta h^{-1/\varrho}} p_j(\varphi, I)\, h^j \ ,$$

where $0 < \eta \leq \eta_0$ and $\eta_0 > 0$ depends only on the constant C and the dimension n. We denote by $S_\ell(\mathbf{A}^n)$ the corresponding class of symbols. Moreover, $g \in S_\ell^{-\infty}(\mathbf{A}^n)$ if

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha\, g(\varphi, I; h)| \ \leq \ h^N\, C^{N+|\alpha+\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, N!^{\,\varrho}$$

for $0 < h \leq h_0$, $\forall N \in \mathbf{Z}_+$, and any multi-indices $\alpha, \beta \in \mathbf{Z}_+^N$, or equivalently

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha\, g(\varphi, I; h)| \ \leq \ C_1^{|\alpha+\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, \exp(-ch^{-1/\varrho})$$

for some $C_1,\ c > 0$, and any $h \in (0, h_0]$, $\alpha,\ \beta\ \in \mathbf{Z}_+^n$.

To each symbol $p \in S_\ell(\mathbf{A}^n)$ we associate an h-pseudodifferential operator $P_h : C^\infty(\mathbf{T}^n, \mathbf{L}) \to C^\infty(\mathbf{T}^n, \mathbf{L})$ by

$$\widetilde{\mathcal{P}_h u}(x) \ = \ (2\pi\, h)^{-n} \int_{\mathbf{R}^{2n}} e^{i\langle x-y, \xi\rangle/h}\, p(x, \xi, h)\, \tilde{u}(y) d\xi dy, \ u \in C^\infty(\mathbf{T}^n, \mathbf{L}).$$

It is well defined modulo $\exp(-ch^{-1/\varrho})$. Indeed, for any $p \in S_\ell^{-\infty}$ we have

$$||\mathcal{P}_h u||_{L^2} \ \leq \ C \exp(-ch^{-1/\varrho}) ||u||_{L^2}\ , \ u \in L^2(\mathbf{T}^n, \mathbf{L}),$$

with some positive constants c and C.

Theorem 2.1 *Suppose that there exists a real analytic exact symplectic map $\chi_1 : \mathbf{T}^n \times D \to U \subset T^*(M)$ such that the Hamiltonian $H(\varphi, I) \ = \ P_0(\chi_1(\varphi, I))$, $(\varphi, I) \in \mathbf{T}^n \times D$, satisfies (BF) for $\mu \ = \ \tau' + 2$. Then there exist a family of uniformly bounded h-Fourier integral operators $U_h : L^2(\mathbf{T}^n; \mathbf{L}) \to L^2(M)$, $0 < h \leq h_0$, associated with the canonical relation* graph (χ) *such that the following holds:*

(i) $U_h^* U_h - \mathrm{Id}$ *is a pseudodifferential operator with a symbol in the Gevrey class* $S_\ell(\mathbf{T}^n \times D)$ *which belongs to* $S_\ell^{-\infty}$ *on* $\mathbf{T}^n \times Y$*, where* Y *is a subdomain of* D *containing* E_κ*,*

(ii) $\mathcal{P}_h \circ U_h \;=\; U_h \circ \mathcal{P}_h^0$*, and the full symbol* $p^0(\varphi, I, h)$ *of* $\mathcal{P}_h^0$ *has the form* $p^0(\varphi, I, h) \;=\; K^0(I, h) \;+\; R^0(\varphi, I, h)$*, where the symbols*

$$K^0(I,h) \;=\; \sum_{0 \le j \le \eta h^{-1/\varrho}} K_j(I) h^j \quad \textit{and} \quad R^0(\varphi, I, h) \;=\; \sum_{0 \le j \le \eta h^{-1/\varrho}} R_j(\varphi, I) h^j$$

belong to the Gevrey class $S_\ell(\mathbf{A}^n)$*,* $\eta > 0$ *is a constant,* K^0 *is real valued, and* R^0 *is equal to zero to infinite order on the Cantor set* $\mathbf{T}^n \times E_\kappa$*.*

The idea of the proof of Theorem 2.1 is as follows: Conjugating $\mathcal{P}_h$ with suitable Fourier integral operator with a Gevrey symbol we obtain a pseudodifferential operator $\widetilde{\mathcal{P}}_h$ with principal symbol $K_0 + R_0$, subprincipal symbol 0, and full symbol $p(\varphi, I; h)$ in $S_{\widetilde{\ell}}(\mathbf{T}^n \times D)$, $\widetilde{\ell} = (\sigma, \mu_0, \sigma + \mu_0 - 1)$. Then we transform $\widetilde{\mathcal{P}}_h$ to a normal form $\mathcal{P}_h^0$ conjugating it with an elliptic pseudodifferential operator A_h with a symbol $a(\varphi, I, h)$ in $S_\ell(\mathbf{A}^n)$. Denote by $p \circ a$ the symbol of $\widetilde{\mathcal{P}}_h A_h$. The main technical part in the proof is the following:

Theorem 2.2 *There exist symbols* a *and* p^0 *in* $S_\ell(\mathbf{T}^n \times D)$*,* $\ell = (\sigma, \mu_0, \varrho)$*, given by*

$$a(\varphi, I, h) \;\sim\; \sum_{j=0}^{\infty} a_j(\varphi, I) h^j \;,\; p^0(\varphi, I, h) \;\sim\; \sum_{j=0}^{\infty} p_j^0(\varphi, I) h^j \;,$$

where $a_0 = 1$*,* $p_0^0(\varphi, I) = K_0(I) + R_0(\varphi, I)$*,* $p_1^0 = 0$*, the functions* $p_j^0(\varphi, I) - p_j^0(0, I)$*,* $j \ge 0$*, are flat at* $\mathbf{T}^n \times E_\kappa$*, and* $p \circ a - a \circ p^0 \in S_\ell^{-\infty}(\mathbf{T}^n \times Y)$*.*

2.2 Quasimodes with exponentially small errors

We define a G^ϱ (Gevrey) quasimode $\mathcal{Q}$ of $\mathcal{P}_h$ as follows:

$$\mathcal{Q} = \{(u_m(\cdot, h), \lambda_m(h)) : m \in \mathcal{M}_h\},$$

where $u_m(\cdot, h) \in C_0^\infty(M)$ has a support in a fixed bounded domain independent of h, $\lambda_m(h)$ are real valued functions of $h \in (0, h_0]$, $\mathcal{M}_h$ is a finite index set for each fixed h, and

$$(i) \quad ||\mathcal{P}_h u_m \;-\; \lambda_m(h) u_m||_{L^2} \;\le\; C\, e^{-c/h^{1/\varrho}} \;,\; m \in \mathcal{M}_h \,,$$

$$(ii) \quad |\langle u_m, u_l \rangle_{L^2} \;-\; \delta_{m,l}| \;\le\; C\, e^{-c/h^{1/\varrho}} \;,\; m, l \in \mathcal{M}_h \,,$$

for $0 < h \le h_0$. Here C and c are positive constants, and $\delta_{m,l}$ is the Kronecker index.

We define the G^ϱ micro-support $MS^\varrho(\mathcal{Q}) \subset T^*(M)$ of $\mathcal{Q}$ as follows: $(x_0,\xi_0) \notin MS^\varrho(\mathcal{Q})$ if there exist compact neighborhoods U of x_0 and V of ξ_0 in a given local chart such that for any G^ϱ function v with support in U

$$\int e^{-i\langle x,\xi\rangle/h}\, v(x)u_m(x,h)dx \;=\; O\left(e^{-c/h^{1/\varrho}}\right)\;,\; \text{as } h \searrow 0\,,$$

uniformly with respect to $m \in \mathcal{M}_h$ and $\xi \in V$.

As a consequence of Theorem 2.1 we obtain a G^ϱ - quasimode $\mathcal{Q}$ of $\mathcal{P}_h$ with an index set

$$\mathcal{M}_h \;=\; \{m \in \mathbf{Z}^n :\; |E_\kappa \;-\; h(m+\vartheta/4)| \;\le\; h^\varepsilon\}$$

where $\varepsilon = \varepsilon(\mu_0) \in (0,1)$. It is easy to see that

$$\begin{aligned}\#\{m \in \mathcal{M}_h\} \;&=\; \frac{1}{(2\pi h)^n}\,\mathrm{Vol}\,(\mathbf{T}^n \times E_\kappa)(1+o(1)) \\ &=\; \frac{1}{(2\pi h)^n}\,\mathrm{Vol}\,(\Lambda)(1+o(1))\,,\;\; h\searrow 0\,,\end{aligned} \tag{2.2}$$

where $\mathrm{Vol}\,(\Lambda)$ stands for the Lebesgue measure of the union Λ of the invariant tori in $T^*(M)$.

Corollary 2.3 *Let $u_m(x,h) = U_h(e_m)(x)$, and $\lambda_m(h) = K^0(h(m+\frac{1}{4}\vartheta),h)$, for $m \in \mathcal{M}_h$. Then*

$$\mathcal{Q} \;=\; \{(u_m(x,h),\lambda_m(h)) :\; m \in \mathcal{M}_h\}$$

is a G^ϱ-quasimode of $\mathcal{P}_h$. Moreover, $MS^\varrho(\mathcal{Q}) \;=\; \Lambda$.

Notice that $\varrho > \tau + n + 1$ could be any number $> 2n$ choosing $\tau > n-1$ sufficiently small. To prove Corollary 2.2 we write $P_h^0 = K_h^0 + R_h^0$, where the symbols of K_h^0 and R_h^0 are $K^0(I,h)$ and $R^0(\varphi,I,h)$ respectively. It is easy to see that

$$P_h^0(e_m)(\varphi) \;=\; (\lambda_m(h) \;+\; R^0(\varphi, h(m+\vartheta/4),h))\, e_m(\varphi)$$

for any $m \in \mathcal{M}_h$. On the other hand,

$$|D_\varphi^\beta D_I^\alpha R^0(\varphi,I,h)| \;\le\; C^{|\alpha|+|\beta|+1}\,\beta!^{\,\sigma}\alpha!^{\,\mu_0}\;,\;\forall\,(\varphi,I,h) \in \mathbf{T}^n \times Y \times (0,h_0]\,,$$

where Y is a fixed compact neighborhood of E_κ in D. Then there exist two positive constants C_1 and c depending only on the constant C such that for every $\alpha,\beta \in \mathbf{Z}_+^n$ the following estimate holds

$$|\partial_\varphi^\beta \partial_I^\alpha R^0(\varphi,I,h)| \;\le\; C_1^{|\alpha|+|\beta|+1}\,\beta!^{\,\sigma}\alpha!^{\,\mu_0}\,\exp\left(-c\,|E_\kappa - I|^{-\frac{1}{\mu_0-1}}\right)\,,$$

for each $(\varphi, I, h) \in \mathbf{T}^n \times Y \times (0, h_0]$, $I \notin E_\kappa$. Using the inequality $\mu_0 < \nu < \varrho$, and choosing appropriately ε we prove that $\mathcal{Q}$ satisfies (i). On the other hand (ii) follows directly from the definition of the index set $\mathcal{M}_h$, the orthogonality of e_m, and (i) in Theorem 2.1. □

The construction of quasimodes with polynomially small error terms (C^∞ quasimodes) is well known (see [3], [4], and the references there).

2.3 Applications to resonances.

Consider a selfadjoint second order differential operator in $\mathbf{R}^n$

$$\mathcal{P}_h = \sum_{|\alpha|+j\leq 2} a_\alpha(x)(hD)^\alpha h^j .$$

As in [13] we impose the following hypothesis:

(H_1) The coefficients $a_\alpha(x)$ are real analytic and they can be extended holomorphically to

$$\{r\omega : \omega \in \mathbf{C}^n,\ \mathrm{dist}(\omega, \mathbf{S}^n) < \varepsilon,\ r \in \mathbf{C},\ |r| > R,\ \arg r \in [-\varepsilon, \theta_0 - \varepsilon]\}$$

for some $\varepsilon > 0$ and $\theta_0 > 0$ and the coefficients of $-h^2\Delta - \mathcal{P}_h$ tend to zero as $|x| \to \infty$ in that set uniformly with respect to h.

(H_2) For some $C > 0$ we have

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C\,|\xi|^2,\ (x,\xi) \in T^*(\mathbf{R}^n) .$$

Then the resonances $\mathrm{Res}\,\mathcal{P}_h$ of $\mathcal{P}_h$ close to the real axis can be defined in a conic neighborhood Γ of the positive half axis in the lower half plain by the method of complex scaling (see [10] and [11]). They coincide in Γ with the poles of the meromorphic continuation of the resolvent

$$(\mathcal{P}_h - z)^{-1} : L^2_{\mathrm{comp}}(\mathbf{R}^n) \to H^2_{\mathrm{loc}}(\mathbf{R}^n),\ \mathrm{Im}\, z > 0 .$$

Thang and Zworski [13] obtained lower bounds of the number of resonances $\mathrm{Res}\,\mathcal{P}_h$ of $\mathcal{P}_h$ close to the real axis for any $h \in (0, h_0]$, provided that there exists a quasimode $\mathcal{Q}$ for $\mathcal{P}_h$. Stefanov [12] obtained sharp lower bounds, he showed that for each $h \in (0, h_0]$ the number of resonances of $\mathcal{P}_h$ close to the real axis is not less than the cardinality of the index set $\mathcal{M}_h$ of the quasimode $\mathcal{Q}$. Fix $\varrho > 2n$ and set

$$N_h = \#\{\lambda \in \mathrm{Res}\,\mathcal{P}_h : \mathrm{Re}\,\lambda \in [E_0, E],\ 0 < -\mathrm{Im}\,\lambda \leq h^{-n-2} e^{-c/h^{1/\varrho}}\},$$

where the resonances are counted with multiplicities, $c > 0$ is the constant in the definition of $\mathcal{Q}$, and $E_0 < \inf(P_0(\Lambda))$ and $E > \sup(P_0(\Lambda))$. Combining Corollary 2.3 with Theorem 1.1 in [12] (which holds also for non-compactly supported perturbations of $-h^2\Delta$ satisfying (H_1) and (H_2)), and using (2.2), we obtain the following:

Theorem 2.4 *Suppose that $\mathcal{P}_h$ satisfies (H_1), (H_2), and the assumptions of Theorem 2.1. Then*

$$N_h \;\geq\; \frac{1}{(2\pi h)^n}\,\mathrm{Vol}\,(\Lambda)(1+o(1))\,,\;\; h \searrow 0\,.$$

On the other hand, it is known from Burq [1] that there exists $\varepsilon > 0$ and $C > 0$ such that there are no resonances of $\mathcal{P}_h$, $0 < h \leq h_0$, in

$$\{\lambda \in \mathbf{C} \;:\; \mathrm{Re}\,\lambda \in [E_0, E],\, 0 \;<\; -\mathrm{Im}\,\lambda \;\leq\; \varepsilon e^{-C/h}\,\}\,.$$

The details of the proof of the results above will appear in [6] and [7]. Quasimodes with exponentially small errors associated with a broken elliptic ray in an analytic manifold with boundary is constructed in [2]. Similar results could be obtained also for Gevrey smooth Hamiltonians but the proof will be technically more complicated.

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