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# Weak Bloch property and weight estimates for elliptic operators

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# ÉQUATIONS AUX DÉRIVÉES PARTIELLES

# WEAK BLOCH PROPERTY AND WEIGHT ESTIMATES FOR ELLIPTIC OPERATORS

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## 1. INTRODUCTION

The problem we are going to start with can be formulated as follows : let $A$ be a differential or pseudo-differential operator on a noncompact Riemannian manifold $M$ and let $A$ define a (unbounded) operator in the Hilbert space $L^2(M)$ ; let for some $\lambda \in \mathbf{C}$ we know a solution $\psi$ of the equation $A\psi = \lambda\psi$ satisfying some estimates at infinity ; when can we conclude that $\lambda$ is in the spectrum $\sigma(A)$ of the operator $A$ in $L^2(M)$ ? (The exact definition of $\sigma(A)$ will be given in Section 1).

An example of the sort is the well known Schnol theorem ([Sch], [C-F-K-S]) which (with some simplifying restrictions) states that if $A = -\Delta + q(x)$ is a Schrödinger operator in $L^2(\mathbf{R}^n)$ with the potential $q \in L^\infty_{\text{loc}}(\mathbf{R}^n)$ such that $q(x) \geq -C$ for all $x \in \mathbf{R}^n$ and there exists a non-trivial solution $\psi$ of the equation $A\psi = \lambda\psi$ such that for every $\varepsilon > 0$

$$\psi(x) = O(\exp(\varepsilon|x|)) \tag{0.1}$$

then $\lambda \in \sigma(A)$. Another Schnol theorem ([Sch]) also concerning the Schrödinger operator states that if the negative part $q_-(x) = \min(0, q(x))$ satisfies the estimate

$$q_-(x) = o(|x|^2)$$

then the existence of a non-trivial polynomially bounded solution (i.e. a solution $\psi$ such that $\psi(x) = 0((1 + |x|)^N)$ with some $N > 0$) for the equation $A\psi = \lambda\psi$ implies that $\lambda \in \sigma(A)$.

T. Kobayashi, K. One and T. Sunada ([K-O-S]) introduced.

**Definition 0.1.**— *An operator $A$ satisfies the* **weak Bloch property** *(WBP) if the following implication is true :*

$$\{ \text{ there exists a bounded } \quad \psi \not\equiv 0 \quad \text{such that} \quad A\psi = \lambda\psi\} \Longrightarrow \lambda \in \sigma(A)$$

So each of the mentionned Schnol theorems implies that the Schrödinger operator on $\mathbf{R}^n$ with a locally bounded and semi-bounded below potential satisfies WBP.

On the other hand the Laplacian $\Delta$ of the standard Riemannian metric on the hyperbolic space $\mathbf{H}^n$ does not satisfy WBP because $\Delta 1 = 0$ but $0 \notin \sigma(\Delta)$.

It is natural to investigate the following WBP-problem : **describe classes of manifolds and operators which satisfy WBP.**

It is easy to notice that the WBP-problem is closely connected with the problem of coincidence of spectra of an operator in spaces $L^p(M)$ for different $p$ : if all these spectra for $1 \leq p \leq \infty$ coincide then $WBP$ evidently holds because if $\sigma_p(A)$ means the spectrum of $A$ in $L^p(M)$ then the existence of a non-trivial bounded solution $\psi$ of $A\psi = \lambda\psi$ implies that $\lambda \in \sigma_\infty(A)$ so $\lambda \in \sigma_2(A) = \sigma(A)$. The problem of the coincidence of spectra was considered on discrete metric spaces in [S], where it was pointed that the coincidence follows from the exponential decay of the Green function off the diagonal provided the space has a subexponential growth of the number of points lying in a ball of the radius

$r$ as $r \to +\infty$. The exponential decay of the Green function off the diagonal was proved in [S] for some operators which where called pseudodifference operators, e.g. difference operators with a finite radius of action and bounded coefficients on disrete groups etc.

The same reasoning works also for continuous objects when the appropriate estimates of the Green function hold. Such estimates were obtained in [M-S] for uniformly elliptic operators on unimodular Lie groups and in [Kor 1,2] on general manifolds of bounded geometry. It follows (though it was not noted in [M-S] or [Kor 1,2]) that the spectra of corresponding operators in $L^p(M)$ coincide for all $p \in (1, +\infty)$ provided the volumes of balls of the radius $r$ grow subexponentially as $r \to +\infty$, and also that WBP is satisfied in this situation. The main ideas of this approach will be explained here in detail. The important point here is a use of some weight Sobolev spaces with exponential weights. In [K-O-S] the authors used an entirely different method which is quite close to the original Schnol method (see also [C-F-K-S]). The WBP was proved in [K-O-S] for the Schrödinger operators with periodic potentials on Riemannian manifolds $M$ with a subexponential growth of volumes of balls and with a discrete group of isometries $\Gamma$ such that the orbit space $M/\Gamma$ is compact.

Now let $M$ be a complete connected Riemannian manifold, $d(x,y)$ be the Riemannian distance between $x$ and $y$, $x, y \in M$. Let $A$ be a differential operator on $M$. Denote by $\sigma(A)$ its spectrum in $L^2(M)$.

**Definition 0.2.—**

*i) The operator $A$ satisfies the* **weak Schnol property** *(WSP) if the existence of a non-trivial solution $\psi$ of the equation $A\psi = \lambda\psi$ satisfying an estimate of the form*

$$|\psi(x)| = O(1 + d(x, x_0)^N) \tag{0.2}$$

*(with some $N > 0$ and a fixed $x_0$) implies that $\lambda \in \sigma(A)$.*

*ii) The operator $A$ satisfies the* **strong Schnol property** *(SSP) if the following implication is true : if there exists a non-trivial solution $\psi$ of the equation $A\psi = \lambda\psi$ such that for every $\varepsilon > 0$*

$$|\psi(x)| = O(\exp(\varepsilon d(x, x_0))) \tag{0.3}$$

*(with a fixed $x_0$) then $\lambda \in \sigma(A)$.*

Clearly (SSP) implies (WSP) and (WSP) implies (WBP). We shall prove that if $M$ is a manifold of bounded geometry with a subexponential growth of volumes of balls and $A$ is a uniformly elliptic operator with $C^\infty$-bounded coefficients on $M$ then $A$ satisfies (SSP) and even stronger property : if for every $\varepsilon > 0$ there exists a non-trivial solution $\psi_\varepsilon$ of $A\psi_\varepsilon = \lambda\psi_\varepsilon$ satisying (0.3) then $\lambda \in \sigma(A)$. So our result extends the result of [K-O-S] to a much more general situation.

The main ideas of this paper essentially contained in [S], [M-S] and [Kor 2] though the formulation of the corollaries of those results here is inspired by the beautiful paper [K-O-S] an is done here for the first time.

## 1. Preliminaries

In this section we shall fix notations and recall necessary definitions and facts. Let $M$ be a Riemannian manifold, $n = \dim M$. Denote by $T_xM$ the tangent space of $M$ at a point $x \in M$ and let $\exp_x : T_xM \to M$ be the usual exponential geodesic map : $\exp_x v = \gamma(1)$, where $\gamma(t)$ is the geodesic (with a canonical parameter which is proportional to the arc length) starting at $x$ with the initial speed $v \in T_xM$, i.e. $\gamma(0) = x, \dot\gamma(0) = v$. We shall always suppose that $M$ is complete or equivalently that $\exp_x$ is defined everywhere i.e. for every $x \in M$ and $v \in T_xM$ the corresponding geodesic $\gamma(t)$ can be defined for all $t \in \mathbf{R}$. The exponential map $\exp_x : T_xM \to M$ is a diffeomorphism of a ball $B_x(0,r) \subset T_xM$ of a radius $r > 0$ with the center 0 on a neighboorhood $\mathcal{U}_{x,r}$ of $x$ in $M$. Denoting by $r_x$ the supremum of possible radii of such balls we can define the injectivity radius of $M$ as $r_{inj} = \inf_{x \in M} r_x$. If $r_{inj} > C$ then taking $r \in (0, r_{inj})$ we see that $\exp_x : B_x(0,r) \to \mathcal{U}_{x,r}$ will be a diffeomorphism for every $x \in M$. Euclidean coordinates in $T_xM$ (associated with a orthonormal frame in $T_xM$) define coordinates on $\mathcal{U}_{x,r}$ (by means of $\exp_x$) which are called canonical.

**Definition 1.1.**— *(see e.g.* [C-G-T] or [R]). $M$ is called **a manifold of bounded geometry** if the following two conditions are satisfied :

a) $r_{inj} > 0$ ;

b) $|\nabla^k R| \le C_k$ , $k = 0, 1, 2, \cdots$ (i.e. every covariant derivative of the Riemann curvature tensor is bounded).

The property b) can be replaced by the following equivalent property which will be more convenient for the use in this paper

b') let us fix any $r \in (0, r_{inj})$ and let $\mathcal{U}_{x,r}, \mathcal{U}_{x',r}$ be two domains of canonical coordinates $y : \mathcal{U}_{x,r} \to \mathbf{R}^n, y' : \mathcal{U}_{x',r} \to \mathbf{R}^n$ such that $\mathcal{U}_{x,r} \cap \mathcal{U}_{x',r} \ne \emptyset$ : consider the vector function $y' \circ y^{-1} : y(\mathcal{U}_{x,r} \cap \mathcal{U}_{x',r}) \to \mathbf{R}^n$ ; then

$$|\partial_y^\alpha (y' \circ y^{-1})| \le C_{\alpha,r}$$

for every mutiindex $\alpha$.

Examples of manifolds of bounded geometry are Lie groups or more general homogeneous manifolds (with invariant metrics), covering manifolds of compact manifolds (with a Riemannian metric which is lifted from the base manifold), leaves of a foliation on a compact manifold (with a Riemannian metric which is induced by a Riemannian metric of the compact manifold).

Below in this paper we shall always use only canonical coordinates with a fixed $r \in (0, r_{inj})$. Then all the change of coordinate functions have bounded derivatives of all orders. This property allows to formulate a correct notion of $C^k$-boundedness ($k = 0, 1, 2, \cdots$) or $C^\infty$-boundedness for functions, vector fields, exterior forms and other tensor fields on $M$. Namely a function $f : M \to \mathbf{C}$ is called $C^k$-bounded if $f \in C^k(M)$ and $|\partial_y^\alpha f(y)| \le C_\alpha$ for every mutiindex $\alpha$ with $|\alpha| \le k$ and for any choice of canonical coordinates. A function $f : M \to \mathbf{C}$ is called $C^\infty$-bounded if $f \in C^\infty(M)$ and $f$ is $C^k$-bounded for every $k = 0, 1, 2, \cdots$. Let $C_b^k(M)$ be the space of all $C^k$-bounded complex-valued functions on $M$

(here $k = 0, 1, 2, \cdots$ or $k = \infty$). Of course $C^k$-boundedness of a function $f \in C^k(M)$ is equivalent to the estimate $|\nabla^k f(x)| \leq C$ but the formulation in local coordinates is sometimes more convenient.

Similarly a vector field, an exterior form on any general tensor field on $M$ is called $C^k$-bounded ($k = 0, 1, 2, \cdots$ or $k = \infty$) if all components of the field in any canonical coordunate system are $C^k$-bounded as $C^k$-functions of corresponding coordinates (with bounds depending only on the order of the differentiation but not on the chosen coordinate neighbourhood).

Let $A : C^\infty(M) \to C^\infty(M)$ be a differential operator of order $m$ with $C^\infty$-coefficients. We shall call it $C^\infty$-bounded if in any canonical coordinate system $A$ is written in the form

$$A = \sum_{|\alpha| \leq m} a_\alpha(y) \partial_y^\alpha \tag{1.1}$$

where the coefficients $a_\alpha$ are (complex-valued) functions satisfying the estimates
$|\partial_y^\beta a_\alpha(y)| \leq C_\beta$ for any multiindex $\beta$ (with a constant $C_\beta$ which does not depend on the chosen canonical neighbourhood). A $C^\infty$-bounded vector field defines a $C^\infty$-bounded differential operator of order 1.

Let $E$ be a complex vector bundle on $M$. We shall say that $E$ is a **bundle of bounded geometry** if it is supplied by an additional structure : trivializations of $E$ on every canonical coordinate neighbourhood $\mathcal{U}$ such that the corresponding matrix transition functions $g_{\mathcal{U}\mathcal{U}'}$ on all intersections $\mathcal{U} \cap \mathcal{U}'$ of such neighbourhoods are $C^\infty$-bounded i.e. all their derivatives $\partial_y^\alpha g_{\mathcal{U}\mathcal{U}'}(y)$ with respect to canonical coordinates are bounded with bounds $C_\alpha$ which do not depend on the chosen pair $\mathcal{U}, \mathcal{U}'$. Examples of vector bundles of bounded geometry are : trivial bundle $M \times \mathbf{C}$, complexified tangent and cotangent bundles $TM \otimes \mathbf{C}$ and $T^*M \otimes \mathbf{C}$, complexified exterior powers $\Lambda^\ell T^*M \otimes \mathbf{C}$ of the cotangent bundle ($C^\infty$-sections of $\Lambda^\ell T^*M \otimes \mathbf{C}$ are exterior complex-valued $\ell$-forms on $M$), complexified tensor bundles etc. The definition of $C^\infty$-bounded differential operator is easily generalized to the case of operators

$$A : C^\infty(M, E) \to C^\infty(M, F) \tag{1.2}$$

acting between spaces of $C^\infty$-sections of vector bundles of bounded geometry $E, F$ (the definition is the same as for scalar operators but with the use of the representation (1.1) in canonical coordinates). Examples of $C^\infty$-bounded differential operators in this more general context are the exterior differentiation de Rham operator $d : \Lambda^\ell(M) \to \Lambda^{\ell+1}(M)$ where $\Lambda^\ell(M) = C^\infty(M, \Lambda^\ell T^*M \otimes \mathbf{C})$, operators of covariant differentiation of tensors, Laplace-Beltrami operators on functions or forms etc.

If $E$ is a vector bundle of bounded geometry on $M$ then the notion of $C^\ell$-boundedness and the corresponding spaces $C_b^\ell(M, E)$ of $C^\ell$-bounded sections are also defined for $\ell = 0, 1, 2, \cdots$ or $\ell = \infty$. Also the space $L^p(M, E)$ of the sections with the integrable $p$-th power of a fiber norm ($1 \leq p < \infty$) is naturally defined.

In what follows we shall always suppose for the sake of simplicity that $M$ is connected. Then the Riemannian distance $d : M \times M \to [0, +\infty)$ is correctly defined ; namely $d(x, y)$ is the infimum of Riemannian lengths of all arcs connecting $x$ and $y$. Note that if $r > 0$ is small enough then the neighbourhoods $\mathcal{U}_{x,r}$ described before are balls $B(x, r)$ of the radius $r$ with the center $x$ with respect to this distance.

The following Lemma is essentially due to M. Gromov [G].

**Lemma 1.1.**— *For every $\varepsilon > 0$ there exists a countable covering of $M$ by balls of the radius $\varepsilon$ : $M = \cup B(x_i, \varepsilon)$ such that the covering of $M$ by the balls $B(x_i, 2\varepsilon)$ with the double radius and the same centers has a finite multiplicity.*

Here the multiplicity (or index in the terminology of [G]) of the covering by balls is the maximal number of the balls with non-empty intersection in this covering.

Lemma 1.1 implies the existence of "uniform" partition of unity which is subordinate to a covering by balls from Lemma 1.1. Let us choose $\varepsilon < r/2$ where $r \in (0, r_{inj})$ is fixed as before.

**Lemma 1.2.**— *For every $\varepsilon > 0$ there exists a partition of unity $1 = \Sigma_{i=1}^{\infty} \varphi_i$ on $M$ such that*

1) $$\varphi_i \geq 0, \varphi_i \in C_0^{\infty}(M), \mathrm{supp}\varphi_i \subset \mathrm{B}(\mathrm{x_i}, 2\varepsilon),$$

*where $\{x_i\}$ is the sequence of points from Lemma 1.1 ;*

2) $$|\partial_y^{\alpha} \varphi_i(y)| \leq C_{\alpha}$$

*for every multiindex $\alpha$ in canonical coordinates uniformly with respect to $i$ (i.e. with the constant $C_{\alpha}$ which does not depend on $i$).*

This Lemma is a useful tool to construct global objects on $M$ from their local prerequisites. One of the important examples is the uniform Sobolev spaces $W_p^s(M), s \in \mathbf{R}, 1 \leq p \leq \infty$ (see e.g. [R] in case $p = 2$). First introduce the Sobolev norm $\|.\|_{s,p}$ on $C_0^{\infty}(M)$ by the formula

$$\|u\|_{s,p}^p = \Sigma_{i=1}^{\infty} \|\varphi_i u\|_{s,p;B(x_i,2\varepsilon)}^p,$$

where $\|.\|_{s,p;B(x_i,2\varepsilon)}$ means the usual Sobolev norm of order $s$ in canonical coordinates on $B(x_i, 2\varepsilon)$. If $s \in \mathbf{Z}_+$ then the local Sobolev norm can be written for every open set $\Omega \subset \mathbf{R}^n$ as

$$\|v\|_{s,p;\Omega} = \left( \Sigma_{|\alpha| \leq s} \int_{\Omega} |\partial^{\alpha} v(y)|^p dy \right)^{1/p}, 1 \leq p < \infty ,$$

$$\|v\|_{s,\infty;\Omega} = \Sigma_{|\alpha| \leq s} \operatorname{ess\,sup}_{\Omega} |\partial^{\alpha} v(y)|$$

Also if we choose a system $X_1, \cdots, X_N$ of $C^{\infty}$-bounded vector fields on $M$ such that $X_1(x), \cdots, X_N(x)$ generate $T_x M$ for every $x \in M$ then we can introduce the following norm which is equivalent to (1.3)

$$(1.3') \qquad \|u\|_{s,p}^p = \sum_{k=0}^{s} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N} \int_M |X_{i_1} \cdots X_{i_k} u(x)|^p dx, 1 \leq p < \infty,$$

where $dx$ is the standard Riemannian density on $M$,

$$\|u\|_{s,\infty} = \sum_{k=0}^{s} \sum_{1\le i_1\le i_2\le\cdots\le i_k\le N} \operatorname{ess\,sup}_M |X_{i_1}\cdots X_{i_k}u(x)|.$$

Another equivalent norm for $s \in \mathbf{Z}_+$ is given by

$$\|u\|_{s,p}^p = \Sigma_{k=0}^s \int_M |\nabla^k u(x)|^p dx, 1 \le p < \infty,$$

$$\|u\|_{s,\infty} = \sum_{k=0}^{s} \operatorname{ess\,sup}_M |\nabla^k u(x)|$$

(here $|\cdot|$ is understood as the norm induced by the Riemannian metric on tensors).

Now we can introduce the uniform Sobolev space $W_p^s(M)$ as the completion of $C_0^\infty(M)$ with respect to the norm (1.3). The spaces $W_p^s(M)$ have the same properties as the corresponding spaces in the case $M = \mathbf{R}^n$. All of them are naturally included in the space of distributions $\mathcal{D}'(M)$. The space $W_2^s(M)$ has a natural Hilbert structure and will be also denoted $H^s(M)$. The usual embedding theorems are true, e.g. $W_p^0(M) =: L^p(M)$ if $1 \le p < \infty$ , $W_p^s(M) \subset C_b^k(M)$ if $s > k + n/p$. If $E$ is a vector bundle of bounded geometry then the Sobolev norms of sections and the corresponding Sobolev spaces of sections $W_p^s(M, E)$ are defined in the same way.

Denote $W_p^{-\infty}(M) = \cup_{s\in\mathbf{R}} W_p^s(M), W_p^\infty(M) = \cap_{s\in\mathbf{R}} W_p^s(M)$ and the similar meaning have the notations $W_p^{-\infty}(M, E), W_p^\infty(M, E)$.

Let $A$ be a differential operator of order $m$ acting as in (1.2) between spaces of sections of vector bundles of bounded geometry. The principal symbol of $A$ gives a family of linear maps

$$a_m(x, \xi) : E_x \to F_x$$

where $x \in M$ ,$(x, \xi) \in T_x^*M$ is a cotangent vector based at $x$, $E_x$ and $F_x$ are fibers of bundles $E$ and $F$ over $x$. Let us choose admissible trivializations of $E$ and $F$ over a neighbourhood of $x$. Then $a_m(x, \xi)$ becames a (complex) matrix. The operator $A$ is called elliptic if this matrix is invertible for every $(x, \xi)$ with $\xi \ne 0$. It is called **uniformly elliptic** if there exists $C > 0$ such that

$$|a_m^{-1}(x, \xi)| \le C|\xi|^{-m}, (x, \xi) \in T^*M, \xi \ne 0 \;. \tag{1.4}$$

Here $|\xi|$ is the length of $(x, \xi)$ with respect to the given Riemannian metric, $|a_m^{-1}(x, \xi)|$ is the operator norm of the matrix $a_{(x,\xi)}^{-1}$ in the above mentioned trivializations.

Let $A$ be a $C^\infty$-bounded differential operator of order $m$ on $M$. Then $A$ defines a bounded linear operator $A : W_p^s(M) \to W_p^{s-m}(M)$ for every $s \in \mathbf{R}, 1 \le p \le \infty$ (if $A$ acts as in (1.2) then it defines a bounded linear operator $A : W_p^s(M, E) \to W_p^{s-m}(M, F)$). Now we shall formulate regularity properties and a priori estimates which follow from uniform ellipticity.

**Lemma 1.3.**— *Let $A$ be a $C^\infty$-bounded uniformly elliptic differential operator acting as in (1.2) between spaces of sections of vector bundles of bounded geometry. Then for every $s, t \in \mathbf{R}, p \in (1, +\infty)$ there exists $C > 0$ such that*

$$\|u\|_{s,p} \leq C(\|Au\|_{s-m,p} + \|u\|_{t,p}), u \in C_0^\infty(M, E) \tag{1.5}$$

*Moreover if $u \in W_p^{-\infty}(M, E)$ and $Au \in W_p^{s-m}(M, F)$ then $u \in W_p^s(M, E)$.*

**Proof :** Let us choose the points $x_1, x_2, \cdots$ and $\varepsilon > 0$ as in Lemma 1.1. We have the usual local a priori estimate

$$\|u\|^p_{s,p;B(x_i,\varepsilon)} \leq C_1(\|Au\|_{s-m,p;B(x_i,2\varepsilon)} + \|u\|^p_{t,p;B(x_i,2\varepsilon)}) \tag{1.6}$$

with a constant $C_1$ which does not depend on $i$. Summing over all $i$ we evidently obtain an estimate which is equivalent to (1.5). The last statement also follows from the corresponding local regularity result and the estimate (1.6). □

Lemma 1.3 easily implies the coincidence of weak and strong extensions in $L^p, 1 < p < \infty$, for the corresponding operators. Namely let $A$ be an operator satisfying the conditions of Lemma 1.3. We can consider two unbounded operators generated by $A$ in $L^p(M, E), 1 \leq p \leq \infty$. Let ${}^sA$ be the closure of $A|_{C_0^\infty(M,E)}$ in $L^p(M, E)$ (such a closure is correctly defined because $A$ can be extended to a continuous linear operator from $\mathcal{D}'(M, E)$ to $\mathcal{D}'(M, F)$ and $L^p(M, E) \subset \mathcal{D}'(M, E)$, where the inclusion operator is continous). So the graph of ${}^sA$ is the closure of the set $\{(u, Au) | u \in C_0^\infty(M, E)\}$ in $L^p(M, E) \times L^p(M, F)$. Let ${}^wA$ be a weak extension of $A$ in $L^p(M, E)$ i.e. the domain of ${}^wA$ is

$$\mathcal{D}_p({}^wA) = \{u | u \in L^p(M, E), Au \in L^p(M, F)\}$$

where $A$ is applied in the sense of distributions and ${}^wAu = Au$ if $u \in \mathcal{D}_p({}^wA)$.

**Proposition 1.1.**— *If $A$ satisfies the conditions of Lemma 1.3 and $1 < p < \infty$ then ${}^wA = {}^sA$ and*

$$\mathcal{D}_p({}^sA) = \mathcal{D}_p({}^wA) = W_p^m(M, E)$$

**Proof :** It is clear that

$$W_p^m(M, E) \subset \mathcal{D}({}^sA) \subset \mathcal{D}({}^wA)$$

but Lemma 1.3. implies that $\mathcal{D}_p({}^wA) \subset W_p^m(M, E)$ so $\mathcal{D}_p({}^sA) = \mathcal{D}_p({}^wA) = W_p^m(M, E)$ q.e.d. □

**Corollary 1.1.**— *Let $A$ satisfies the conditions of Lemma 1.3 with $E = F$, $E$ has a hermitean $C^\infty$-bounded scalar product on fibers, $(\cdot\ ,\ \cdot)$ is the scalar product on $L^2(M, E)$ induced by the scalar product of fibers and the Riemannian density on $M$ and $A$ is formally self-adjoint with respect to the scalar product i.e.*

$$(Au, v) = (u, Av), u, v \in C_0^\infty(M, E)$$

*Then $A$ is essentially self-adjoint in $L^2(M, E)$.*

In the next sections we shall investigate spectra of $C^\infty$-bounded uniformly elliptic operators $A$ in $L^p(M,E)$ and use for them notations $\sigma_p({}^sA), \sigma_p({}^wA)$ and also the notation

$$\sigma_p(A) = \sigma_p({}^sA) = \sigma_p({}^wA), 1 < p < \infty ,$$

which is correct due to Proposition 1.1. We shall also write $\sigma(A)$ instead of $\sigma_2(A)$.

Clearly ${}^sA \neq^w A$ in $L^\infty(M,E)$ but the coincidence ${}^sA =^w A$ in $L^1(M,E)$ can be proved for the operators with positive principal symbols ([Kor 1,2]).

## 2. Weight estimates and decay of the green function

We begin with a construction which gives a substitute with natural smoothness properties for the distance $d = d(x,y)$ on a connected Riemannian manifold $M$ of bounded geometry. Such a substitute will be a function which we shall denote by $\tilde{d} = \tilde{d}(x,y)$. For the case of Lie groups it can be constructed as a convolution of $d(x,.)$ with a $C_0^\infty$-function ([M-S]). General case requires a more complicated procedure which we shall give now.

**Lemma 2.1.**— *(Yu.A. Kordyukov). There exists a function $\tilde{d} : M \times M \to [0,+\infty)$ satisfying the following conditions :*

*(i) there exists $\rho > 0$ such that*

$$|\tilde{d}(x,y) - d(x,y)| < \rho$$

*for every $x, y \in M$ ;*

*(ii) for every multiindex $\alpha$ with $|\alpha| > 0$ there exists a constant $C_\alpha > 0$ such that*

$$|\partial_y^\alpha \tilde{d}(x,y)| \leq C_\alpha, x, y \in M ,$$

*where the derivative $\partial_y^\alpha$ is taken with respect to canonical coordinates.*

*Moreover for every $\varepsilon > 0$ there exists a function $\tilde{d} : M \times M \to [0,\infty)$ satisfying (i) with $\rho < \varepsilon$.*

**Proof :** Let us choose a covering $M = \cup B(x_i, 2\varepsilon)$ and a partition of unity $1 = \Sigma\varphi_i$ described in Lemmas 1.1 and 1.2. We shall suppose that an orthonormal frame is choosen in every tangent space $T_{x_i}M, i = 1, 2, \cdots$, so $T_{x_i}M$ is identified with $\mathbf{R}^n$ and the exponential maps at the points $x_i$ can be considered as the maps $\exp_{x_i} : \mathbf{R}^n \to M$.

Let us choose a function $\theta_1 \in C_0^\infty(\mathbf{R}^n)$ such that $\theta_1 \geq 0, \mathrm{supp}\theta_1 \subset \{x||x| < 1\}$, $\int_{\mathbf{R}^n} \theta_1(x)dx = 1$ and define $\theta_\delta(x) = \delta^{-n}\theta_1(x/\delta)$ for any $\delta > 0$. Now choosing $\delta$ sufficiently small we can define

$$\tilde{d}(x,y) = \Sigma_{i=1}^\infty \varphi_i(y) \int_{\mathbf{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x, \exp_{x_i}(z))dz. \tag{2.1}$$

Subtracting the evident identity

$$d(x,y) = \Sigma_{i=1}^\infty \varphi_i(y) \int_{\mathbf{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x,y)dz$$

from (2.1) and using the triangle inequality we obtain the estimate

$$|\tilde{d}(x,y) - d(x,y)| \le \Sigma_{i=1}^{\infty}\varphi_i(y)\int_{\mathbf{R}^n}\theta_\delta(\exp_{x_i}^{-1}(y) - z)d(\exp_{x_i}(z),y),dz.$$

It follows from the bounded geometry conditions that there exists $C > 0$ such that $d(\exp_{x_i}(z),y) < C\delta$ if $y \in supp\varphi_i$ and $|\exp_{x_i}^{-1}(y) - z| < \delta$, so we obtain

$$|\tilde{d}(x,y) - d(x,y)| < C\delta$$

which proves (i) with small $\rho$ provided $\delta$ is chosen sufficiently small.

To prove (ii) let us consider first the case $|\alpha| = 1$.

Using the notation $\partial_j = \partial/\partial y_j$ in some canonical coordinates we obtain

$$\partial_j\tilde{d}(x,y) = \Sigma_{i=1}^{\infty}[\partial_j\varphi_i(y)]\int_{\mathbf{R}^n}\theta_\delta(\exp_{x_i}^{-1}(y) - z)d(x,\exp_{x_i}(z))dz+$$

$$(2.2)\qquad +\Sigma_{i=1}^{\infty}\varphi_i(y)\Sigma_{k=1}^{n}\int_{\mathbf{R}^n}b_{ijk}(y)[\frac{\partial}{\partial z_k}\theta_\delta(\exp_{x_i}^{-1}(y) - z)]d(x,\exp_{x_i}(z))dz$$

where $b_{ijk}$ are some functions (in the chosen canonical coordinates) which are $C^\infty$-bounded uniformly with respect to $i,j,k$ and the chosen coordinates. The same arguments as we used in proving (i) show that the first term in the right hand side of (2.2) is estimated by $C\delta$. To estimate the second term we can subtract from him a similar term which is obtained by changing $d(x,\exp_{x_i}(z))$ to $d(x,y)$ (this modified term evidently vanishes). Following then the reasoming used for the proof of (i) we obtain that the second term is estimated by a constant.

Further inductive reasoning shows that (ii) is true for every $\alpha$ q.e.d. □

Now we can introduce exponential weights $f_{\varepsilon,y} \in C^\infty(M)$ by

$$f_{\varepsilon,y}(x) = \exp(\varepsilon\tilde{d}(x,y)), x,y \in M\ ,$$

where $\varepsilon \in \mathbf{R}$ (usually $\varepsilon$ will be sufficiently small).

Let us introduce a weight Sobolev space

$$W_{p,\varepsilon}^s(M) = \{u | u \in \mathcal{D}'(M), f_{\varepsilon,y}u \in W_p^s(M)\}$$

where $s \in \mathbf{R}, p \in [1,+\infty]$ and $y$ is any fixed point in $M$. It is easy to check that

$$f_{\varepsilon,y_1}^{-1}f_{\varepsilon,y_2} \in C_b^\infty(M)$$

for any fixed points $y_1, y_2 \in M$. It follows that the space $W_{p,\varepsilon}^s(M)$ does not depend on the chosen point $y$. The space $W_{p,\varepsilon}^s(M)$ is a Banach space with the norm

$$(2.4)\qquad \|u\|_{s,p;\varepsilon,y} = \|f_{\varepsilon,y}u\|_{s,p}\ .$$

These norms obtained by use of different points $y$ are equivalent but the dependence on $y$ is sometimes essential.

Now we shall consider a $C^\infty$-bounded uniformly elliptic operator $A : C^\infty(M,E) \to C^\infty(M,E)$ where $E$ is a vector bundle of bounded geometry. Let us suppose that $\lambda \in \mathbf{C} \setminus \sigma_p(A)$ for some $p \in (1,+\infty)$. Then there is a bounded inverse operator

$$(A - \lambda I)^{-1} : L^p(M,E) \to L^p(M,E).$$

The L. Schwartz kernel of this inverse operator will be denoted $G = G(x,y)$ and will be called **the Green function** ($p$ and $\lambda$ are fixed). We are ready to prove estimates of decay of the Green function off the diagonal $\Delta = \{(x,x) | x \in M\} \subset M \times M$. Note that $G$ is a distributional section of the bundle $E \otimes E$ on $M \times M$ (the fiber of $E \otimes E$ over a point $(x,y) \in M \times M$ is $E_x \otimes E_y^*$, where $E_y^*$ is the dual linear space to $E_y$). We identify the density bundle over $M$ with a trivial bundle by use of the standard Riemannian density.

**Theorem 2.1.**— *Let $p \in (1,+\infty)$ and $\lambda \in \mathbf{C} \setminus \sigma_p(A)$ be fixed, $G = G(x,y)$ the Green function. Then $G \in C^\infty(M \times M \setminus \Delta)$ and there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every multiindices $\alpha, \beta$ there exists $C_{\alpha\beta\delta} > 0$ such that*

$$|\partial_x^\alpha \partial_y^\beta G(x,y)| \le C_{\alpha\beta\delta} \exp(-\varepsilon d(x,y)) \quad \text{if} \quad d(x,y) \ge \delta \ . \tag{2.5}$$

*Here the derivatives $\partial_x^\alpha$ and $\partial_y^\beta$ are taken with respect to canonical coordinates and absolute value in the left hand side is taken in the corresponding fibers.*

**Proof :** Without loss of generality we can suppose that $\lambda = 0$. For the sake of simplicity of notations we shall only consider scalar case i.e. the case of trivial $E = M \times \mathbf{C}$. Let us for every $\varepsilon \in \mathbf{R}$, $y \in M$ consider a differential operator $A_{\varepsilon,y} = F_{\varepsilon,y} A F_{\varepsilon,y}^{-1}$ where $F_{\varepsilon,y}$ is the multiplication operator $(F_{\varepsilon,y}u)(x) = f_{\varepsilon,y}(x)u(x)$ with $f_{\varepsilon,y}$ defined by (2.3). Choosing any $s \in \mathbf{R}$ we obtain a commutative diagram

$$\begin{array}{ccc} W_p^s(M) & \xrightarrow{A_{\varepsilon,y}} & W_p^{s-m}(M) \\ \uparrow {\scriptstyle F_{\varepsilon,y}} & & \uparrow {\scriptstyle F_{\varepsilon,y}} \\ W_{p,\varepsilon}^s(M) & \xrightarrow{A} & W_{p,\varepsilon}^{s-m}(M) \end{array} \tag{2.6}$$

where the vertical arrows are linear topological isomorphisms and even isometries if we use the norm (2.4) in $W_{p,\varepsilon}^s(M)$ and the corresponding norm in $W_{p,\varepsilon}^{s-m}(M)$. It follows from the properties of $\tilde{d}$ described in lemma 2.1 that

$$A_{\varepsilon,y} = A + \varepsilon B_{\varepsilon,y}, \tag{2.7}$$

where $\{B_{\varepsilon,y} | y \in M, |\varepsilon| < 1\}$ is a family of uniformly $C^\infty$-bounded differential operators of order $m-1$. It follows that the operator norm

$$\|A_{\varepsilon,y} - A : W_p^s(M) \to W_p^{s-m}(M)\|$$

tends to 0 as $\varepsilon \to 0$. The required invertibility of $A$ implies due to Proposition 1.1 that $A$ defines a linear topological isomorphism of Banach spaces

$$A : W_p^s(M) \to W_p^{s-m}(M),$$

so $A_{\varepsilon,y}$ in the diagram (2.6) also defines a linear topological isomorphism if $|\varepsilon| < \varepsilon_0$ where $\varepsilon_0 > 0$ is sufficiently small. Besides all norm estimates are uniform with respect to $y \in M$. Hence $A$ in the diagram is also uniformly topologically invertible if $|\varepsilon| < \varepsilon_0$.

Now notice that

$$G(x,y) = [A^{-1}\delta_y(\cdot)](x), \tag{2.8}$$

where $\delta_y$ is the standard Dirac $\delta$-measure on $M$ supported at $y \in M$. The Sobolev embedding theorem implies that if $s < -n/p$ then $\delta_y \in \cap_{\varepsilon \in \mathbf{R}} W^s_{p,\varepsilon}(M)$ and $\|\delta_y\|_{s,p;\varepsilon,y} \leq C_{s,p}$ uniformly over $y \in M$ and $\varepsilon$ with $|\varepsilon| < 1$. It follows from (2.8) that

$$\|G(\cdot,y)\|_{s+m,p;\varepsilon,y} \leq C_{s,p} \tag{2.9}$$

if $|\varepsilon| < \varepsilon_0$.

Now note that

$$A_x G(x,y) = 0 \quad \text{if} \quad x \neq y\ .$$

It follows from (2.9) and the uniform local a priori estimate like (1.6) that for every $\delta > 0, s \in \mathbf{R}, p \in (1, +\infty), y \in M$ and $x \in M$ with $d(x,y) > \delta$

$$\|G(\cdot,y)\|_{s,p,B(x,\delta/2)} \leq C_{s,p,\delta} \exp(-\varepsilon d(x,y)).$$

The Sobolev embedding theorem implies now that the required estimate (2.5) is satisfied if $\beta = 0$. Now the same reasoning can be applied with respect to $y$ because we can use the uniformly elliptic equation

$${}^tA_y G(x,y) = 0 \quad \text{if} \quad x \neq y$$

where ${}^tA$ is the formally transposed operator to $A$ defined by the equality

$$\langle Au, v\rangle = \langle u, {}^tAv\rangle, u, v \in C_0^\infty(M),$$

where

$$\langle f, g\rangle = \int_M f(x)g(x)dx,$$

$dx$ is the Riemannian density on $M$. This immediately leads to the estimates (2.5). □

We need also uniform local estimates of the Green function near the diagonal but the simplest way to obtain them is in a use of pseudo-differential operators. This will be done in the next Section.

## 3. Uniform properly supported pseudo-differential operators and structure of inverse operators

We shall introduce here classes of uniform properly supported pseudo-differential operators on a manifold $M$ of bounded geometry which coincide locally with well-known Hörmander classes $\psi^m$ and $\psi^m_{phg}$ ([H], vol.3). Such classes were introduced first on Lie groups in [M-S] and later in general case in [Kor 2]

**Definition 3.1.**— *$U\Psi^{-\infty}(M)$ is a class of all ;operators $R$ with a L. Schwartz kernel $K_R \in C^\infty(M \times M)$ satisfying the following conditions*

*(i) there exists $C_R > 0$ such that $K_R(x,y) = 0$ if $d(x,y) > C_R$ ;*

*(ii) $|\partial_x^\alpha \partial_y^\beta K_R(x,y)| \leq C_{\alpha\beta}, x, y \in M$, where the derivatives are taken in canonical coordinates.*

The class $U\Psi^{-\infty}(M)$ will serve as a class of negligible operators in our context. Notice that an operator $R \in U\Psi^{-\infty}(M)$ is not necessarily compact e.g. in $L^2(M)$.

In the next definition we fix $r \in (0, r_{inj})$ as was already done before.

**Definition 3.2.**— *$U\Psi^m(M)$ is a class of all operators $A : C_0^\infty(M) \to C_0^\infty(M)$ satisfying the following conditions :*

*(i) there exists $C_A > 0$ such that $K_A(x,y) = 0$ if $d(x,y) > C_A$ (here $K_A$ is the L. Schwartz kernel of $A$) ;*

*(ii) let $B(x_0, r)$ be a ball on $M$, then in canonical coordinates on $B(x_0,r)$ the operator*

$$A_{x_0} = A_{|C_0^\infty(B(x_0,r))} \begin{array}{l} : C_0^\infty(B(x_0,r)) \to C^\infty(B(x_0,r)) \\ \quad u \mapsto Au_{|B(x_0,r)} \end{array}$$

can be written as

$$A_{x_0} = a_{x_0}(x, D_x) + R_{x_0} \tag{3.1}$$

where $a_{x_0} \in S^n$ uniformly with respect to $x_0$, i.e.

$$|\partial_\xi^\alpha \partial_x^\beta a_{x_0}(x,\xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m-|\alpha|}$$

with $C_{\alpha\beta}$ which do not depend on $x_0$, and $R_{x_0}$ is an operator with a L. Schwartz kernel $K_{R_{x_0}} \in C^\infty(B(x_0,r) \times B(x_0,r))$ satisfying the following estimates

$$|\partial_x^\alpha \partial_y^\beta K_{R_{x_0}}(x,y)| \leq C'_{\alpha\beta}$$

with constants $C'_{\alpha\beta}$ which do not depend on $x_0$.

**Definition 3.3.**— *$U\Psi^m_{phg}(M)$ is a class of operators $A \in U\Psi^m(M)$ which have polyhomogeneous local symbols $a_{x_0}(x,\xi)$ with uniform estimates of homogeneous terms in local representations (3.1). More exactly it is required that there exist $a_{x_0,j} = a_{x_0,j}(x,\xi), j = 0, 1, 2, \cdots$, such that the following conditions are satisfied :*

(i) $a_{x_0,j}(x,\xi)$ *is defined when* $x \in B(x_0,r), \xi \neq 0$ *and is homogeneous of degree* $m-j$ *with respect to* $\xi$, *i.e.*

$$a_{x_0,j}(x,t\xi) = t^{m-j} a_{x_0,j}(x,\xi), x \in B(x_0,r), \xi \in \mathbf{R}^n \setminus 0, t > 0 \ ;$$

(ii) $a_{x_0,j} \in C^\infty$ *when* $\xi \neq 0$ *and* $|\partial_\xi^\alpha \partial_x^\alpha a_{x_0,j}(x,\xi)| \leq C_{\alpha\beta_j}$ *when* $x \in B(x_0,r)$ *and* $|\xi| = 1$ *with the constants* $C_{\alpha\beta j}$ *which do not depend on* $x_0$ ;

(iii) *let* $x \in C_0^\infty(\mathbf{R}^n), \chi(\xi) = 1$ *when* $\xi$ *is close to* 0, *and* $\chi$ *is fixed, then for every* $N, \alpha, \beta, x_0$

$$|\partial_\xi^\alpha \partial_x^\beta [a_{x_0}(x,\xi) - \Sigma_{j=0}^{N-1}(1-\chi(\xi)) a_{x_0,j}(x,\xi)]| \leq C_{\alpha\beta N}(1+|\xi|)^{m-N}$$

*with* $C_{\alpha\beta N}$ *which do not depend on* $x_0$.

So the classes $U\Psi^m, U\Psi^m_{phg}$ are just usual Hörmander classes of properly supported pseudo-differential operators but with appropriate uniformity conditions.

The classes $U\Psi^m, U\Psi^m_{phg}$ are defined for all $m \in \mathbf{R}$. The class $U\Psi^m_{phg}(M)$ can be defined also for $m \in \mathbf{C}$ as a class of operators $A \in U\Psi^{\mathrm{Re}\ m}(M)$ such that the conditions (i), (ii), (iii) of Definition 3.3 are satisfied if we replace $m$ by Re $m$ in (iii).

The usual algebraic and continuity properties are satisfied for the classes $U\Psi^m(M), U\Psi^m_{phg}(M)$.

In particular the following statements are easily checked :

(a) if $A_j \in U\Psi^{m_j}(M), j = 1,2$, then $A_1A_2 \in U\Psi^{m_1+m_2}(M)$ ; the same is true for the classes $U\Psi^m_{phg}(M)$ ;

(b) if $A \in U\Psi^m(M)$ (or $U\Psi^m_{phg}(M)$) then $A^* \in U\Psi^m(M)$ (resp. $U\Psi^{\bar{m}}_{phg}(M)$ where $\bar{m}$ is complex conjugate to $m$).

(c) if $A \in U\Psi^m(M)$ then $A$ defines for every $s \in \mathbf{R}$, $p \in (1,+\infty)$ a continuous linear operator

$$A : W_p^s(M) \to W_p^{s-m}(M)$$

**Proposition 3.1.**— *Let* $A$ *be a* $C^\infty$*-bounded uniformly elliptic differential operator of order* $m$ *on* $M$. *Then there exists* $B \in U\Psi^{-m}_{phg}(M)$ *such that* $I - AB, I - BA \in U\Psi^{-\infty}(M)$.

**Proof :** The operator $B$ with required properties is easily constructed by use of inform local parametrices $B_i$ for $A$ in the balls $B(x_i,\varepsilon)$ from Lemma 1.1 and then patching them up by the formula

$$B = \Sigma_i \Psi_i B_i \Phi_i \ ,$$

where $\Phi_i, \Psi_i$ are multiplication operators $\Phi_i u(x) = \varphi_i(x)u(x), \Psi_i u(x) = \psi_i(x)u(x), \varphi_i$ is taken from the partition of unity of Lemma 1.2, $\psi_i \in C_0^\infty(B(x_i,2\varepsilon))$ are chosen to be uniformly $C^\infty$-bounded and such that $\psi_i(x) = 1$ in a neighbourhood of supp $\varphi_i$ □

**Remark 3.1.** Chosing $\varepsilon > 0$ sufficiently small we can obtain the parametrix $B$ with a L. Schwartz kernel $K_B$ with

$$\mathrm{supp}\ K_B \subset \{(x,y) | d(x,y) < \varepsilon_1\}$$

Now we can describe the structure of the operator $(A - \lambda I)^{-1}$ in case $\lambda \notin \sigma_p(A)$ more precisely.

First note that all the definitions and statements of this Section can be easily generalized to operators acting in spaces of sections of vector bundles of bounded geometry on $M$. The corresponding classes of operators $A : C_0^\infty(M, E) \to C_0^\infty(M, F)$ will be denoted $U\Psi^{-\infty}(M; E, F), U\Psi^m(M; E, F), U\Psi^m_{phg}(M; E, F)$ or $U\Psi^{-\infty}(M; E)$ in case $E = F$.

**Theorem 3.1.**— *Let $A : C_0^\infty(M, E) \to C_0^\infty(M, F)$ be a uniformly elliptic $C^\infty$-bounded differential operator of order $m$. Let the closure of $A$ in $L^p(M, E)$ has an everywhere defined bounded inverse $A^{-1}$. Then there exist $\varepsilon > 0$ and a representation :*

$$A^{-1} = B + T \;, \tag{3.2}$$

*where $B \in U\Psi^{-m}_{phg}(M; F, E)$, $T$ has a L. Schwartz kernel $K_T \in C^\infty$ satisfying the following estimates*

$$|\partial_x^\alpha \partial_y^\beta K_T(x, y)| \le C_{\alpha\beta\delta} \exp(-\varepsilon d(x, y)) \quad \textit{if} \quad d(x, y) \ge \delta > 0 \;. \tag{3.3}$$

*Here the derivatives and the norm in the left-hand side are taken with respect to canonical and canonical trivializations of $E$ and $F$.*

**Proof :** For the sake of simplicity of notations we shall consider the case of trivial $E = F = M \times \mathbf{R}$. It follows from Proposition 3.1 that there exists $B \in U\Psi^{-m}_{phg}(M)$ such that

$$AB = I - R \;,$$

where $R \in U\Psi^{-\infty}(M)$. Multiplying by $A^{-1}$ from the left we obtain (3.2) with $T = A^{-1}R$. Now it is clear that

$$K_T(x, y) = [A^{-1} K_R(\cdot, y)](x). \tag{3.4}$$

Notice that $K_R(\cdot, y) \in C_0^\infty(M)$ and supp $K_R(\cdot, y) \subset B(y, r_c)$ for some $r_0 > 0$ which does not depend on $y$. Hence it follows from (3.4) and Theorem 2.1 that the estimates (3.3) are fulfilled if $d(x, y) \ge r_0 > 0$ arbitrarily small so the estimates (3.3) are proved outside $\delta$-neighbourhood of the diagonal for every $\delta > 0$.

It remains to prove (3.3) in the set

$$\{(x, y) | d(x, y) < \delta\}$$

where $\delta > 0$ can be choosen arbitrarily small. But then (3.3) reduces to the boundedness of all derivatives which follows from the Sobolev embedding theorem and the boundedness of the operator

$$A^{-1} : W_p^s(M) \to W_p^{s+m}(M)$$

for every $s \in \mathbf{R}$ which is due to the regularity properties (Lemma 1.3) and the closed graph theorem. □

Now we can prove estimates of the Green function near the diagonal.

**Theorem 3.2.**— *Let $A, p, \lambda$ satisfy the conditions of Theorem 2.1, $G$ be the Green function (the L. Schwartz kernel of $(A - \lambda I)^{-1}$). Then there exists $\varepsilon > 0$ such that*

$$|\partial_x^\alpha \partial_y^\beta G(x,y)| \leq C_{\alpha\beta} d(x,y)^{m-n-|\alpha|-|\beta|} \exp(-\varepsilon d(x,y)) \tag{3.5}$$

*provided $m < n$ ;*

$$|\partial_x^\alpha \partial_y^\beta G(x,y)| \leq C_{\alpha\beta}[1 + d(x,y)^{m-n-|\alpha|-|\beta|} |\log d(x,y)|] \exp(-\varepsilon d(x,y)) \tag{3.6}$$

*provided $m \geq n$.*

**Proof :** As usual we shall consider the scalar case. Due to Theorem 2.1 it is sufficient to prove (3.5) and (3.6) for $x, y \in M$ such that $d(x,y) \leq \delta$ with some fixed $\delta > 0$. Let us consider the representation (3.2). Clearly the L. Schwartz kernel $K_T$ satisfies the required estimates due to (3.3). Now we have to consider $K_B$ and to do this let us present $B$ locally in $B(x_0, r)$ in the form (3.1)

$$B_{x_0} = B_{x_0}(x, D_x) + R_{x_0}$$

where the L. Schwartz kernel of $R_{x_0}$ satisfies the required estimates and $b_{x_0} = b_{x_0}(x, \xi)$ is a polyhomogeneous symbol with uniform estimates. The L. Schwartz kernel of $b_{x_0}(x, D_x)$ in local canonical coordinates near $x_0$ is equal to

$$K_{x_0}(x,y) = F_{\xi \to x-y} b_{x_0}(x,\xi) = (2\pi)^{-n} \int b_{x_0}(x,\xi) e^{i\langle x-y,\xi\rangle} d\xi$$

so to prove the necessary estimates it is sufficient to use the well known properties of the Fourier transform of homogeneous functions or their appropriate distributional regularizations (see e.g. [H], vol. 1). □

## 4. Spectral properties of uniformly elliptic operators on manifolds of subexponential growth

**Definition 4.1.**— *Let $M$ be a manifold of bounded geometry. We shall say that $M$ is a* **manifold of subexponential growth** *if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\sup_{x \in M} \text{vol } B(x,r) \leq C_\varepsilon \exp(\varepsilon r), r > 0 \ , \tag{4.1}$$

where vol denotes the volume with respect to the Riemannian density.

The following Proposition will show the property of Green functions on subexponential manifolds which is most valuable for us.

**Proposition 4.1.**— *Let $M$ be a manifold of subexponential growth and $A, \lambda, p, G$ the same as in Theorem 3.2. Then there exists $C > 0$ such that*

$$\int_M |G(x,y)| dx \leq C, \int_M |G(x,y)| dy \leq C \tag{4.2}$$

*(the first estimate holds uniformly with respect to $y$ and the second uniformly with respect to $x$), where $dx, dy$ denote the Riemannian density on $M$.*

**Proof :** Using Theorem 3.2. we see that if we take the integrals in (4.2) over $B(y,1)$ and $B(x,1)$ respectively then they would be bounded as required. Therefore it is sufficient to estimate the integrals over $M \setminus B(y,1)$ and $M \setminus B(x,1)$. They are estimated as follows

$$\int_{M\setminus B(y,1)} |G(x,y)|dx \leq C_0 \int_M \exp(-\varepsilon d(x,y))dx = C_0 \int_0^\infty \exp(-\varepsilon r)d_r(\text{vol } B(y,r))$$

$$= C_0\varepsilon \int_0^\infty \exp(-\varepsilon r)\text{vol } B(y,r)dr \leq C_0\varepsilon C_{\varepsilon_1} \int_0^\infty \exp(-(\varepsilon - \varepsilon_1)r)dr \leq C < \infty$$

since we can take $\varepsilon_1 > 0$ arbitrarily small due to (4.1). □

**Corollary 4.1.**— *Let the conditions of Proposition 4.1 are satisfied. Then* $(A - \lambda I)^{-1}$ *can be extended to a bounded linear operator*

$$(A - \lambda I)^{-1} : L^q(M,E) \to L^q(M,E)$$

*for every* $q \in [1, +\infty]$.

**Proof :** The statement is true due to (4.2) and the well known Schur lemma (see e.g. [H], vol.3). □

Now we are ready to discuss the spectra in $L^p$ for different $p$. We shall use notations for spectra which were introduced in the end of Section 1. Namely the spectrum of the closure of $A$ in $L^p(M,E)$ will be denoted $\sigma_p(A)$ if $1 < p < \infty$ (remind that ${}^sA ={}^w A$ if $1 < p < \infty$) and we shall also use $\sigma_p({}^wA)$ in the extremal cases $p = 1$ and $p = \infty$. We shall also use the notation $\sigma(A) = \sigma_2(A)$.

**Theorem 4.1.**— *Let $M$ be a manifold of subexponential growth, $E$ a vector bundle of bounded geometry over $M$, $C_0^\infty(M,E) \to C_0^\infty(M,E)$ an uniformly elliptic $C^\infty$-bounded differential operator. Then $\sigma_p(A)$ does not depend on $p$ :*

$$\sigma_p(A) = \sigma(A), 1 < p < \infty. \tag{4.3}$$

*Furthermore*

$$\sigma_\infty({}^wA) \subset \sigma(A), \sigma(A). \tag{4.4}$$

**Proof :** As always let us consider the scalar case. We have to prove that if $\lambda \in \mathbf{C} \setminus \sigma_{p_0}(A)$ for some $p_0 \in (1,\infty)$ then $\lambda \notin \sigma_p({}^wA)$ for all $p \in [1,\infty]$ and $\lambda \notin \sigma_1({}^sA)$ (remind that ${}^wA ={}^s A$ in $L^p(M)$ if $p \in (1,\infty)$). For the sake of simplicity of notations suppose that $\lambda = 0$.

Let $G$ be an integral operator with the Green function $G(\cdot,\cdot)$ as the L. Schwartz kernel. Then $G$ can be extended to a linear bounded operator

$$G : L^p(M) \to L^p(M)$$

for every $p \in [1, \infty]$ due to Corollary 4.1. Let us introduce for any $\varepsilon > 0$ a space $W_\varepsilon$ which contains functions $\varphi \in C^\infty(M)$ such that

$$|\partial^\alpha \varphi(x)| = O(\exp(-\varepsilon d(x, x_0)))$$

for every multiindex $\alpha$ (with the derivative $\partial^\alpha$ in canonical coordinates) and a choosen fixed $x_0 \in M$ (the condition does not depend on $x_0$). The subexponentiality condition clearly implies that $W_\varepsilon \subset L^p(M)$ for all $\varepsilon > 0, p \in [1, \infty]$ and moreover

$$W_\varepsilon \subset \bigcap_{p \in [1,\infty]} \bigcap_{s \in \mathbf{R}} W_p^s(M), \ \varepsilon > 0 \ . \tag{4.5}$$

Now it follows from Theorem 3.1 that $G$ maps $C_0^\infty(M)$ into $W_\varepsilon$ with some $\varepsilon > 0$. Evidently $AG = GA = I$ on $C_0^\infty(M)$. Note that the first equality implies that $A_x G(x, y) = \delta_y(x)$ and the second implies that ${}^tA^tG = I$ on $C_0^\infty(M)$, hence ${}^tA_y G(x, y) = \delta_x(y)$. Another important algebraic corollary is that ${}^tG^tA = I$ on $C_0^\infty(M)$.

Now it is easy to check that $AG = I$ on $L^p(M)$ for every $p \in [1, \infty]$ if $A$ is applied in the sense of distributions. In fact if $u \in L^p(M), v \in C_0^\infty(M)$ then

$$\langle AGu, v\rangle = \langle Gu, {}^tAv\rangle = \langle u, {}^tG^tAv\rangle = \langle u, v\rangle \ ,$$

hence $AGu = u$. It follows that $Gu \in \mathcal{D}_p$. It follows that $Gu \in \mathcal{D}_p({}^wA)$ ; hence $A : \mathcal{D}_p({}^wA) \to L^p(M)$ is surjective.

Let us prove that $GA = I$ on $\mathcal{D}_p({}^wA), p \in [1, \infty]$. If $u \in \mathcal{D}_p({}^wA), v \in C_0^\infty(M)$ then

$$\langle G\,A\,u, v\rangle = \langle Au, {}^tGv\rangle$$

due to the Fubini theorem. Note that ${}^tGv \in W_\varepsilon$ for some $\varepsilon > 0$. So it is enough to prove that

$$\langle Au, \varphi\rangle = \langle u, {}^tA\varphi\rangle, u \in \mathcal{D}_p({}^wA), \varphi \in W_\varepsilon. \tag{4.6}$$

Let us define a cut-off function

$$\chi_N(x) = \Sigma_{i=1}^N \varphi_i(x)$$

where $\varphi_i$ are the functions from the partition of unity of Lemma 1.2. It is clear that $\chi_N \in C_0^\infty(M), 0 \le \chi_N \le 1$ and for every compact $K \subset M$ there exists $N$ such that $\chi_N = 1$ in a neighbourhood of $K$. Moreover $|\partial^\alpha \chi_N| \le C_\alpha$ in canonical coordinates uniformly with respect to $N$.

Now we can begin with the equality

$$\langle Au, \chi_N\varphi\rangle = \langle u, {}^tA(\chi_N\varphi)\rangle, u \in \mathcal{D}_p({}^wA), \varphi \in W_\varepsilon, \tag{4.7}$$

and try to take limit as $N \to \infty$ to obtain (4.8). Note that $(Au)\varphi \in L^1(M)$ due to (4.5), therefore $\lim_{N\to\infty}\langle Au, \chi_N\varphi\rangle = \langle Au, \varphi\rangle$ due to the dominated convergence theorem. The same reasoning can be applied to the right-hand side of (4.7) due to the estimates of derivatives of $\chi_N$, so we obtain (4.6).

We have proved that the operators $A : \mathcal{D}_p({}^wA) \to L^p(M)$ and $G : L^p(M) \to \mathcal{D}_p({}^wA)$ are mutually inverse. Therefore (4.3), (4.4) and hence Theorem 4.1 are proved. □

Theorem 4.1 immediately implies

**Corollary 4.2.**— *Let $M, A$ be as in Theorem 4.1. Then WBP holds i.e. if there exists $u \in L^\infty(M,E), u \not\equiv 0$, such that $Au = \lambda u$ in the sense of distributions then $\lambda \in \sigma(A)$.*

It is also easy to obtain SSP and a stronger result which was mentionned in Introduction.

**Theorem 4.2.**— *Let $M$ be as in Theorem 4.1, $\lambda \in \mathbf{C}$ and for every $\varepsilon > 0$ there exists a weak solution $\psi_\varepsilon \not\equiv 0$ of $A\psi_\varepsilon = \lambda\psi_\varepsilon$ satisfying*

$$|\psi_\varepsilon(x)| = O(\exp(\varepsilon d(x, x_0))) \tag{4.8}$$

*with a fixed $X_0$. Then $\lambda \in \sigma(A)$.*

**Proof :** Let us consider the scalar case and suppose that $\lambda = 0$. We should repeat arguments given in the proof of Theorem 4.1. Let us suppose that $0 \notin \sigma(A)$. Then we can construct the Green operator $G$.

Using the local a priori estimates it is easy to prove that (4.8) implies the same estimates for the derivatives of $\psi_\varepsilon$ :

$$|\partial^\alpha \psi_\varepsilon(x)| = O(\exp(\varepsilon d(x, x_0))) \tag{4.9}$$

for every mutiindex $\alpha$. If $\varepsilon$ is sufficiently small then (4.9) implies that $GA\psi_\varepsilon$ makes sense and equals $\psi_\varepsilon$ because for every $v \in C_0^\infty(M)$ we obtain due to Theorem 3.1 and the Fubini theorem

$$\langle GA\psi_\varepsilon, v\rangle = \langle A\psi_\varepsilon, {}^t Gv\rangle = \langle \psi_\varepsilon, {}^t A^t Gv\rangle = \langle \psi_\varepsilon, v\rangle$$

(the middle equality is obtained by a limit procedure with the same use of the cut-off functions $\chi_N$ as in the proof of Theorem 4.1). On the other hand $A\psi_\varepsilon = 0$ implies $GA\psi_\varepsilon = 0$, hence $\psi_\varepsilon = 0$, so we get a contradiction which proves the theorem. □

## 5. Generalizations and open questions

Most part of the results described earlier can be generalized to pseudo-differential operators. We shall mention some of the generalizations. Theorem 3.1 is true for uniformly elliptic pseudo-differential operators $A \in U\Psi^m_{phg}(M; E, F)$ if $m > 0$. Also if $A \in U\Psi^m(M; E, F)$ is uniformly elliptic in appropriate sense (see [M-S] for the case of Lie groups) then the statement of Theorem 3.1 is true with $B \in U\Psi^{-m}(M; F, E)$. So Theorem 3.2 is also true in the case $A \in U\Psi^m_{phg}(M; E, F)$ if $m > 0$ (the estimate (3.5) will be true when $m < n$ or $m - n \notin \mathbf{Z}$). All the results of Section 4 are then easily generalized to the case when $A \in U\Psi^m_{phg}(M; E), m > 0$.

In fact it is not necessary to consider only pseudo-differential operators which are properly supported. Everything is true e.g. for the operators like the right-hand side in (3.2) i.e. for the operators of the form $A = A_0 + T$, where $A_0 \in U\Psi^m_{phg}(M; E, F)$ and $T$ satisfies the same conditions as in the formulation of Theorem 3.1. Moreover the requirement of exponential decay of the kernel off the diagonal can also be relaxed if the volume of balls on $M$ grows even more slowly. The corresponding machinery was developped in

[M-S] for Lie groups and is perfectly suitable for general manifolds of bounded geometry so we omit the details.

Now we shall mention some open problems.

**1° Conjecture.** Corollary 4.2 holds not only for the manifolds of subexponential growth but also for all amenable manifolds i.e. manifolds $M$ of bounded geometry such that there exists a system of compacts $\{K_j\}_{j=1}^{\circ}$ in $M$ satisfying the following conditions :

(i) for every compact $K \subset M$ there exists $j$ such that $K \subset K_j$ ;

(ii) if $(K_j)_1 = \{|\text{dist}(x, K_j) \leq 1\}$ then

$$\lim_{j\to\infty} \frac{\text{vol}[(K_j)_1 \setminus K_j]}{\text{vol} K_j} = 0$$

where vol is the Riemannian volume on $M$.

In other words WBP holds for uniformly elliptic $C^\infty$-bounded operators on amenable manifolds.

Note that in a weaker form this conjecture was formulated in [K-O-S].

**2°** How to generalize Schnol results [Sch] about Schrödinger operators with unbounded potentials (see introduction) to more general operators and manifolds ? In particular how to describe general situations when (SSP) holds ?

## REFERENCES


[C-G-T] J. Cheeger, M. Gromov, M. Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator and the geometry of complete Riemannian manifolds. J. Diff. Geom. **17** (1982), 15-53.

[C-F-K-S] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon. Schrödinger operators. Springer, Berlin e.a., 1987.

[G] M. Gromov. Curvature, diameter and Betti numbers. Comment Math. Helvetici **56** (1981), 179-195.

[H] L. Hörmander. The analysis of linear partial differential operators. Berlin e.a., Springer-Verlag, vol. 1,2 (1983), vol. 3,4 (1985).

[K-O-S] T. Kobayashi, K. Ono, T. Sunada. Periodic Schrödinger opera tors on a manifold. Forum Math. **1** (1989), 69-79.

[Kor 1] Yu. A. Kordyukov. $L^p$-theory of elliptic differential operators with bounded coefficients. Vestnik Moskovskogo Universiteta, Ser. I Math. Mech. 1988, N°4, 98-100 (in Russian).

[Kor 2] Yu. A. Kordyukov. Elliptic operators on manifolds of bounded geometry. Thesis, Moscow State University, 1987 (in Russian)

[M-S] G.A. Meladze, M.A. Shubin. Properly supported uniform pseudo-differential operators on unimodular Lie groups. Trudy Sem. Petrovskogo **11** (1986), 74-97 ; Functional calculus of pseudo-differential operators on unimodular Lie groups. Trudy Sem. Petrovskogo **12** (1987), 164-200 (in Russian).

[Sch] E. Schnol. On the behaviour of the Schrödinger equation. Mat. Sbornik 42 (1957), 273-286 (in Russian).

[S] M.A. Shubin. Pseudodifference operators and their Green's functions. Math. USSR Izvestiya **26** (1986), N 3, 605-622.

[R] J. Roe. An index theorem on open manifolds. I. II. J. Diff. Geom. **27** (1988), 87-113, 115-136.

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