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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

A DIRECTIONAL COMPACTIFICATION OF THE COMPLEX FERMI SURFACE AND ISOSPECTRALITY

D. BÄTTIG

1. Introduction and Theorems :

The content of this report is joint work with H. Knörrer and E. Trubowitz (ETH-Zürich, Switzerland), [BKT].

We consider a lattice $\Gamma \subset \mathbf{R}^3$ of maximal rank and $L_{\mathbf{R}}^2(\mathbf{R}^3/\Gamma)$ the Hilbert-space of square-integrable real-valued functions on the torus $\mathbf{R}^3/\Gamma$. Let q be in $L_{\mathbf{R}}^2(\mathbf{R}^3/\Gamma)$.

For each $k \in \mathbf{R}^3$ the self-adjoint boundary value problem

$$\begin{aligned} (-\Delta + q(x))\psi(x) &= \lambda\psi(x) \\ \psi(x + \gamma) &= e^{i\langle k, \gamma \rangle} \psi(x) \quad \text{for all } \gamma \in \Gamma \end{aligned}$$

has discrete spectrum, customarily denoted by

$$E_1(k) \leq E_2(k) \leq E_3(k) \leq \dots$$

The eigenvalue $E_n(k)$, $n \geq 1$, defines a function of k called the n -th band function. It is continuous and periodic with respect to the lattice

$$\Gamma^\# := \{b \in \mathbf{R}^3 / \langle \gamma, b \rangle \in 2\pi\mathbf{Z} \quad \text{for all } \gamma \in \Gamma\}.$$

dual to Γ .

The physical Fermi surface for energy λ is the set

$$F_{\text{phys}, \lambda}(q) := \{k \in \mathbf{R}^3 / E_n(k) = \lambda \quad \text{for some } n \geq 1\}.$$

For example, if $q(x) = \text{constant}$, then $F_{\text{phys}, \lambda}(q)$ is the union of the spheres

$$\{k \in \mathbf{R}^3 / (k_1 + b_1)^2 + (k_2 + b_2)^2 + (k_3 + b_3)^2 = \lambda - \text{constant}\}$$

with $b = (b_1, b_2, b_3) \in \Gamma^\#$.

Theorem 1.— *If q is in $L_{\mathbf{R}}^2(\mathbf{R}^3/\Gamma)$ and if for a single λ in $\mathbf{R}$ one of the components of $F_{\text{phys}, \lambda}(q)$ is a sphere (not necessarily centered at a point of the dual lattice), then q is constant.*

Actually the same conclusion holds if $F_{\text{phys}, \lambda}(q)$ contains an algebraic component X , which fulfills certain assumptions, (see section 3). These assumptions are fulfilled if X is an ellipsoid.

To prove Theorem 1 we complexify the Fermi surface. The (lifted) complex Fermi surface is defined by $F_\lambda(q) := \{k \in \mathbf{C}^3 / \text{there exists a non trivial solution } \psi \text{ in } H_{\text{loc}}^2(\mathbf{R}^3) \text{ of } (-\Delta + q(x))\psi(x) = \lambda\psi(x) \text{ satisfying } \psi(x + \gamma) = e^{i\langle k, \gamma \rangle} \psi(x) \text{ for all } \gamma \in \Gamma\}$.

Clearly, the dual lattice $\Gamma^\#$ acts on $F_\lambda(q)$ by $k \mapsto k + b$, $b \in \Gamma^\#$. Furthermore we have $F_\lambda(q) \cap \mathbf{R}^3 = F_{\text{phys}, \lambda}(q)$.

It is easy to show, using regularized determinants (see [KT]), that $F_\lambda(q)$ is a complex analytic hypersurface in $\mathbf{C}^3$. The main purpose is to construct a directional compactification of $F_\lambda(q)$ in the sense of [KT]. The above theorem follows from the analysis of the points added at “infinity”.

To compactify $F_\lambda(q)$ we first embed $\mathbf{C}^3$ in a quadric Q lying in $\mathbf{P}^4$. For each affine line $g = \{c + tb/t \in \mathbf{R}^3\}$ in $\mathbf{R}^3$, where $b, c \in \Gamma^\#$ and b is primitive, we blow-up two distinguished points of $\mathbf{P}^4$ that lie on the quadric Q , to get, by using inverse limits, a space $\mathcal{M}$. Denote by $E_1(g)$ and $E_2(g)$ the corresponding exceptional divisors.

Theorem 2.— *The directional closure of $F_\lambda(q)$ in the space $\mathcal{M}$ intersects $E_1(g)$ and $E_2(g)$ along curves both of which are isomorphic to the one-dimensional Bloch-variety*

$$\mathcal{B}(q_g) \quad \text{where} \quad q_g(x) = \sum_{n=-\infty}^{\infty} \hat{q}(nb) e^{i\langle nb, x \rangle}, x \in g.$$

Here $\hat{q}(b)$ is the Fourier-coefficient $\int_{\mathbf{R}^3/\Gamma} q(x) e^{-i\langle b, x \rangle} dx$ ($b \in \Gamma^\sharp$, without loss of generality we assume that $\mathbf{R}^3/\Gamma$ has volume one). Recall that in [KT] the complex one dimensional Bloch-variety for $p(x) \in L^2(\mathbf{R}/|b|\mathbf{Z})$ is

$\mathcal{B}(p) = \{(k, \lambda) \in \mathbf{C} \times \mathbf{C} / \text{there is a non-trivial function } \psi \text{ in } H_{\text{loc}}^2(\mathbf{R}) \text{ satisfying } -\psi''(x) + p(x)\psi(x) = \lambda\psi(x) \text{ and } \psi(x + |b|n) = e^{ik|b|n}\psi(x) \text{ for all } n \in \mathbf{Z}\}.$

2. Sketch of the proof of Theorem 2

First we construct a compactification of $\mathbf{C}^3$, which serves as the ambient space for the directional compactification of $F_\lambda(q)$. This compactification will be independent of q . It's construction is motivated by considering the free Fermi-surface $F_\lambda(0)$. $F_\lambda(0)$ is the union of the quadrics

$$\{k \in \mathbf{C}^3 / (k_1 + b_1)^2 + (k_2 + b_2)^2 + (k_3 + b_3)^2 = \lambda\} \quad , b = (b_1, b_2, b_3) \in \Gamma^\sharp.$$

If we compactify $\mathbf{C}^3$ in the naive way to $\mathbf{P}^3$ or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ we would have to perform many blow-up's before the components of $F_\lambda(0)$ are in general position at infinity. Instead we embed $\mathbf{C}^3$ in the complex projective 3-dimensional nonsingular quadric

$$Q := \{(k_1, k_2, k_3, y, z) \in \mathbf{P}^4 / yz = k_1^2 + k_2^2 + k_3^2\}$$

by mapping (k_1, k_2, k_3) to $(k_1, k_2, k_3, k_1^2 + k_2^2 + k_3^2, 1)$.

The image of the embedding is the complement of

$$Q_\infty := \{(k_1, k_2, k_3, y, z) \in Q / z = 0\}.$$

The closures of the components of $F_\lambda(0)$ in Q are the intersections of Q with the hyperplanes H_b in $\mathbf{P}^4$ given by

$$y + 2\langle k, b \rangle + (b^2 - \lambda)z = 0 \quad , b \in \Gamma^\sharp.$$

If $b \neq b'$, then $H_b \cap H_{b'}$ is a plane in $\mathbf{P}^4$. It intersects Q_∞ in the set $D_{b,b'}$, consisting of two points, given by the equations

$$z = 0, k_1^2 + k_2^2 + k_3^2 = 0, \langle k, b - b' \rangle = 0 \quad , y + 2\langle k, b \rangle = 0.$$

One checks that $D_{b,b'}$ and $D_{b'',b'''}$ are disjoint if b, b', b'', b''' do not lie on a line and that $D_{b,b'} = D_{b'',b'''}$ if these four points of $\Gamma^\sharp$ are on a line. Thus we can denote the points $D_{b,b'}$ by $D(g)$, where g is the affine line through b and b' . The group $\Gamma^\sharp$ acts by translation on $\mathbf{C}^3$. This action extends to Q and it maps $D(g)$ to $D(c + g)$ for $c \in \Gamma^\sharp$.

If b and $b' \in \Gamma^\#$ are different points on the line $g = c_1 + \mathbf{R}c_2$ ($c_i \in \Gamma^\#$) then $Q \cap H_b$ and $Q \cap H_{b'}$ have different tangent planes in the points of $D(g)$. Therefore we can separate $Q \cap H_b$ and $Q \cap H_{b'}$ by blowing-up the points of $D(g)$. Precisely, for each line $g = c_1 + \mathbf{R}c_2$ ($c_i \in \Gamma^\#$), let $\mathcal{M}(g)$ be the space obtained from $\mathbf{P}^4$ by blowing-up the points of $D(g)$, $Q(g)$ the strict transform of Q in $\mathcal{M}(g)$ and $E_1(g), E_2(g)$ the two exceptional divisors over the two points of $D(g)$. As compactification $\mathcal{M}$ of $\mathbf{C}^3$ we take the inverse limit of all the spaces $\mathcal{M}(G)$, where G is a finite set of affine lines and $\mathcal{M}(G)$ is obtained from $\mathbf{P}^4$ by blowing-up the points of $\cup_{g \in G} D(g)$, defined by the natural maps $\mathcal{M}(G_1) \rightarrow \mathcal{M}(G_2)$ for $G_2 \subset G_1$.

Using the action of $\Gamma^\#$ we consider $\mathcal{M}(g)$ where g passes through the origin, and after rotating and scaling we further assume that $g = t(1, 0, 0)$.

Then

$$D(g) = \{(0, \pm i, 1, 0, 0) \in \mathbf{P}^4\}$$

Consider now the exceptional divisor $E_1 := E_1(g)$ lying above the point $(0, i, 1, 0, 0)$, the other divisor is treated similarly. Near this point we take coordinates $(\frac{k_1}{k_3}, \frac{k_2}{k_3} - i, \frac{y}{k_3}, \frac{z}{k_3})$. In $\mathcal{M}(g)$ we have coordinates (ℓ_1, ℓ_2, y', z) such that

$$\frac{k_1}{k_3} = z\ell_1, \frac{k_2}{k_3} - i = z\ell_2, \frac{y}{k_3} = zy', k_3 = \frac{1}{z}$$

For convenience we perform the change of variables

$$y' = -\mu + \ell_1^2 + \lambda$$

In these coordinates the blow-up map $\pi : \mathcal{M}(g) \rightarrow \mathbf{P}^4$ is

$$k_1 = \ell_1, k_2 = \ell_2 + \frac{i}{z}, y = -\mu + \ell_1^2 + \lambda, k_3 = \frac{1}{z}.$$

$Q(g)$ intersects E_1 in the plane $z = \ell_2 = 0$. The strict transform of the hyperplane $H_b, b \in \Gamma^\#$, does not meet E_1 if $b_2 \neq 0$ or $b_3 \neq 0$. Further, the strict transform of $H_{(b_1, 0, 0)}$ intersects E_1 in

$$(\ell_1 + b_1)^2 - \mu = 0$$

Remember that the strict transform of $Q \cap H_b$ is the closure of a component of the free Fermi-surface $F_\lambda(0)$, and that the one-dimensional Bloch-variety for potential zero is

$$\cup_{n \in \mathbf{Z}} \{(\ell, \mu) \in \mathbf{C} \times \mathbf{C} / (\ell + n)^2 - \mu = 0\}.$$

This shows that for $q \equiv 0$ the union of the closures of the components of $F_\lambda(0)$ meets $E_1 \cap Q(g)$ along a curve isomorphic to the one-dimensional Bloch-variety for potential zero. Observe however that the closure of $F_\lambda(0)$ in $Q(g)$ is bigger than the union of the closures of its components. This indicates that it is necessary for the general case to restrict the way one takes limits to E_1 , i.e. the **directional closure** in Theorem 2 is made precise by introducing a subset $\Sigma(g)$ of $\mathbf{C}^4$ such that the closure of $F_\lambda(q) \cap \Sigma(g)$ in $Q(g)$ intersects $E_1(g)$ and $E_2(g)$ along a curve each isomorphic to the Bloch-variety $\mathcal{B}(q_g)$.

An equation for $F_\lambda(q)$ outside of the free Fermi-surface $F_\lambda(0)$ is given by (see [KT]), assuming without loss of generality $\hat{q}(0) = 0$,

$$\det_2(-\Delta_k + q - \lambda \mathbf{1}) \circ (-\Delta_k - \lambda \mathbf{1})^{-1} = \det_2(\delta_{cb} + \frac{\hat{q}(c-b)}{(k+b)^2 - \lambda}) = 0.$$

This determinant can be computed by taking limits of finite principal minors. (It is not difficult to get an equation for $F_\lambda(q)$ on the whole $\mathbf{C}^3$, but to get the notations as small as possible we work with the above equation). In the coordinates (ℓ_1, ℓ_2, μ, z) of $\mathcal{M}(g)$ the entries of the matrix for $(-\Delta_k + q - \lambda) \circ (-\Delta_k - \lambda)^{-1}$ are

$$\delta_{cb} + \frac{\hat{q}(c-b)}{\frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]}$$

Block the matrix in the form

$$\begin{array}{c} c \in \mathbf{Z}(1,0,0) \{ \\ c \notin \mathbf{Z}(1,0,0) \{ \end{array} \left(\begin{array}{cc} \overbrace{A(\ell_1, \ell_2, \mu, z)}^{b \in \mathbf{Z}(1,0,0)} & \overbrace{B(\ell_1, \ell_2, \mu, z)}^{b \notin \mathbf{Z}(1,0,0)} \\ \dots\dots\dots & \dots\dots\dots \\ \overbrace{C(\ell_1, \ell_2, \mu, z)}^{b \in \mathbf{Z}(1,0,0)} & \overbrace{D(\ell_1, \ell_2, \mu, z)}^{b \notin \mathbf{Z}(1,0,0)} \end{array} \right) =: \mathcal{F}(\ell_1, \ell_2, \mu, z)$$

With this notation $A(\ell_1, \ell_2, \mu, z) = (\delta_{c_1 b_1} + \frac{\hat{q}(c_1 - b_1, 0, 0)}{(\ell_1 + b_1)^2 - \mu})_{c_1, b_1 \in \mathbf{Z}}$. This is the matrix whose determinant describes the Bloch-variety of the averaged potential q_g outside of $\mathcal{B}(0)$. Furthermore on $Q(g) \cap E_1 = \{z = \ell_2 = 0\}$ the matrix $B = 0$ and $D = \mathbf{1}$.

The square of the Hilbert-Schmidt norm of

$$\mathcal{F}(\ell_1, \ell_2, \mu, z) - \mathcal{F}(\ell_1, 0, \mu, 0)$$

is bounded by

$$\|q\|_2^2 \sum_{\substack{b \in \Gamma^\sharp \\ b \notin \mathbf{Z}(1,0,0)}} \frac{1}{|\frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^2}$$

Definition :

$$\begin{aligned} \Sigma(g) := & \{(\ell_1, \ell_2, \mu, z) \in \mathbf{C}^4 / \sum_{\substack{b \in \Gamma^\sharp \\ b \notin \mathbf{Z}(1,0,0)}} \frac{1}{|\frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^2} \\ & + \sum_{\substack{b \in \Gamma^\sharp \\ b \notin \mathbf{Z}(1,0,0)}} \frac{|\ell_1 + b_1|^2 + b_2^2}{|\frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]|^4} < |z|^{1/5}\} \end{aligned}$$

The restriction of $\det_2 \mathcal{F}$ to $\Sigma(g)$ is continous at $z = 0$:

$$\|\mathcal{F}(\ell_1, \ell_2, \mu, z) - \mathcal{F}(\ell_1, 0, \mu, 0)\|_{\text{Hilbert-Schmidt}}^2 = \mathcal{O}(\|q\|_2^2 |z|^{1/5})$$

Therefore we have :

$$\overline{F_\lambda(q) \cap \Sigma(g)} \cap (Q(g) \cap E_1) \subset \mathcal{B}(q_g). \quad (1)$$

To prove the converse we need information about the structure of $\Sigma(g)$ in the neighbourhood of any point of $Q(g) \cap E_1$:

Lemma 1.— For every point $p = (\ell_1^*, \ell_2^*, \mu^*, 0)$ of $E_1(g)$ and for all $A > 0$ there is a neighbourhood $\mathcal{U}$ of p in $\mathcal{M}(g)$ and an open set $Z \subset \mathbf{C}$ having 0 as a cluster point such that

$$T := \{(\ell_1, \ell_2, \mu, z) \in \mathcal{U} / z \in Z, |\ell_2 - \ell_2^*| \leq A|z|\} \subset \Sigma(g)$$

The proof of Lemma 1 is technical, very long and done by contradiction. One has to estimate the functions in the sums defining $\Sigma(g)$ outside of little discs centered at

$$z_b(\ell_1, \mu) := 2i(1 + \frac{(\ell_1 + b_1)^2 - \mu}{b_2^2 + b_3^2})^{-1}(-b_2 + ib_3)^{-1}$$

since

$$\begin{aligned} & \left| \frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2] \right|^2 = \\ & = (b_2^2 + b_3^2) \left| \frac{2i}{z} - \left(1 + \frac{(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2}{b_2^2 + b_3^2}\right)(-b_2 + ib_3) \right|^2. \end{aligned}$$

We do not know if $\Sigma(g)$ is path-connected, i.e. if Z is.

Let us fix now a smooth point $p = (\ell_1^*, 0, \mu^*, 0)$ of $Q(g) \cap E_1 \cap \mathcal{B}(g_g)$. For simplicity we assume that p doesn't lie on the free Bloch-variety $\mathcal{B}(0)$ in $Q(g) \cap E_1$. By Lemma 1 there is a neighbourhood $\mathcal{U}$ of p in $\mathcal{M}(g)$ and an open subset $Z \subset \mathbf{C}$ having 0 as a cluster point such that $T \subset \Sigma(g)$.

It is easy to see (using the definition of $\Sigma(g)$ and the fact that $\det_2$ is continuous in Hilbert-Schmidt norm) that we have

Lemma 2.— The restriction of the function

$$f(\ell_1, \ell_2, \mu, z) := \det_2 \mathcal{F}(\ell_1, \ell_2, \mu, z)$$

to $\bar{T}$ has the following properties :

i) $f(p) = 0$

ii) There is a constant C , such that

$$|f(\ell_1, \ell_2, \mu, z) - f(\ell_1, \ell_2, \mu, 0)| \leq C|z|^{1/5}$$

for all $z \in Z, (\ell_1, \ell_2, \mu, z) \in \mathcal{U}$

iii) For any $z \in \bar{Z}$ the mapping $f(., z)$ is differentiable and $(\ell_1, \ell_2, \mu, z) \mapsto (\nabla_{(\ell_1, \ell_2, \mu)} f)(\ell_1, \ell_2, \mu, z)$ is continuous on $\bar{T}$.

iv) $\frac{\partial f}{\partial \ell_1}(p)$ and $\frac{\partial f}{\partial \mu}(p)$ are not both equal to zero.

We apply this lemma as follows :

Since $Q(g)$ intersects E_1 transversally, we can choose (ℓ_1, μ, z) as local coordinates on $Q(g) \cap \mathcal{U} =: V$ near p (observe that there exists a $A > 0$ such that $|\ell_2| \leq A|z|$ for all points near p in $Q(g)$). Assume $\frac{\partial f}{\partial \ell_1}(p) \neq 0$ (the other case is treated similarly using $\frac{\partial \ell_2(\mu, z)}{\partial \mu}(p) = 0$) and consider the continuous mapping

$$F : V \subset \mathbf{R}^4 \times \overline{Z} \rightarrow \mathbf{R}^4$$

defined by

$$F(\ell_1, \mu, z) := (f(\ell_1, \ell_2(\mu, z), \mu, z), \mu - \mu^*) .$$

It is not difficult to apply the implicit function theorem to F , by imitating it's proof, to get a sequence $((\ell_1, \mu)_k, z_k)_{k \in \mathbf{N}}$ in $V \times Z$ with $z_k \neq 0$ converging to $((0, 0), 0)$ such that $F((\ell_1, \mu)_k, z_k) = 0$. Therefore p lies in the closure of the zero-set of f in $(Q(g)\text{-strict transform of } Q_\infty) \cap T$, hence in the closure of $F_\lambda(q) \cap \Sigma(g)$. From [Bo] one knows, that the equation defining the one-dimensional Bloch-variety $\mathcal{B}(q_g)$ is reduced. So the smooth points are dense in the zero-set of $f(\ell_1, 0, \mu, 0)$ and we get

$$\overline{F_\lambda(q) \cap \Sigma(g) \cap (Q(g) \cap E_1)} \supset \mathcal{B}(q_g) \quad (2)$$

(1) and (2) imply the Theorem 2.

3. Sketch of the proof of Theorem 1

First we claim :

Assume that q is a real potential and that $F_\lambda(q)$ contains an algebraic component X . If the closure $\overline{X}$ of X in Q contains of the curves $\{(k, Y, 0) \in Q_\infty / \langle k, c \rangle + y = 0\}$ with $c \in \Gamma^\sharp$, then q is constant.

Proof :

For $b \in \Gamma^\sharp - \{0\}$ let g_b be the line $\{c + tb/t \in \mathbf{R}\}$. Then $\overline{X}$ contains all the sets $D(g_b), b \in \Gamma^\sharp$. By Lemma 1 the closure of $X \cap \Sigma(g_b)$ in $Q(g_b)$ meets $E_1(g_b)$ and $E_2(g_b)$ along a (non-empty) algebraic curve, namely the intersection of the strict transform of $\overline{X}$ with $E_1(g_b)$ resp. $E_2(g_b)$. Hence by Theorem 2 the Bloch-varieties of all the averaged potentials $q_b, b \in \Gamma^\sharp$ each contain an algebraic component. As each q_b is real, Borg's Theorem [Bo] implies that q_b is constant. Therefore q is constant. $\square$

The assumption of the claim is fulfilled if $F_\lambda(q)$ contains a sphere around a point of $\Gamma^\sharp$. Assume that $F_\lambda(q)/\Gamma^\sharp$ is irreducible. Then, if X where any algebraic component of $F_\lambda(q)$, by Theorem 2 there would be an affine line g , such that $\overline{X \cap \Sigma(g)}$ intersects $E_i(g) (i = 1, 2)$ along a curve, and one would deduce the fact that q is constant as above.

Theorem 1' shows, under further assumptions on X , one does not need the irreducibility of $F_\lambda(q)/\Gamma^\sharp$ to conclude Theorem 1.

Theorem 1'.—

Let $q \in L_{\mathbf{R}}^2(\mathbf{R}^3/\Gamma)$. Assume that $F_\lambda(q)$ contains an algebraic component X whose closure $\overline{X} \subset Q$ is transversal to Q_∞ at almost every point of $\overline{X} \cap Q_\infty$. Then q is constant.

This is the case if for example X is a sphere or an ellipsoid.

For the proof of Theorem 1' it suffices to show that

$$\overline{X} \cap Q_\infty \subset \cup_{b \in \Gamma^\#} \{(k, y, 0) \in Q_\infty / \langle k, b \rangle + y = 0\}. \quad (*)$$

Let $\mathcal{D} := \{(\kappa_1, \kappa_2, \kappa_3, 1, 0) \in Q_\infty / \text{there are } M, \tau \geq 0 \text{ such that for all } b \in \Gamma^\# - \{0\} \text{ one has}$

$$|\langle \kappa, b \rangle| \geq M|b|^{-\tau}, |\langle \kappa, b \rangle + 1| \geq M|b|^{-\tau}\}.$$

Then one shows (by blowing up the point $p \in \mathbf{P}^4$ and using the methods to prove the Theorem 2).

Lemma 3.— *Let $q \in L_{\mathbf{R}}^2(\mathbf{R}^3/\Gamma)$ and $p = (\kappa, 1, 0) \in \mathcal{D}$. Then there is no algebraic component of $F_\lambda(q)$, whose closure passes through p and is transversal to Q_∞ in this point.*

If C is a component of $\overline{X} \cap Q_\infty$ which is not contained in $\cup_{b \in \Gamma^\#} \{(k, y, 0) \in Q_\infty / \langle k, b \rangle + y = 0\}$, then C meets $\{(k, y, 0) \in Q_\infty / y = 0\}$ in only finitely many points, i.e.

$$C' := \{(k, 1, 0) \in Q_\infty / (k, 1, 0) \in C\}$$

is an affine curve and by Lemma 3 $C' \cap \mathcal{D}$ consists of only finitely many points. One shows that this leads to a contradiction :

Let $\mathcal{D}_0$ be the set of points $(y_1, y_2, y_3) \in \mathbf{P}_2(\mathbf{R})$ which fulfil a diophantine estimate

$$|\langle y, b \rangle| \geq \frac{K}{|b|^\tau} \quad \text{for all } b \in \Gamma^\# - \{0\}$$

with some $K, \tau \geq 0$. Clearly a point $(k, 1, 0) \in Q_\infty$ with $k \neq 0$ lies in $\mathcal{D}$ if its imaginary part Imk represents a point of $\mathcal{D}_0$. Consider the map

$$\pi_0 : C' - \{(0; 1, 0)\} \rightarrow \mathbf{P}_2(\mathbf{R}), \quad (k; 1, 0) \mapsto Imk.$$

The image of π_0 intersects $\mathcal{D}_0$ in only finitely many points. On the other hand one easily verifies that $\mathbf{P}_2(\mathbf{R}) - \mathcal{D}_0$ has measure zero. Hence by Sard's theorem π_0 does not have maximal rank anywhere. From this one can conclude that C' is contained in a plane. Therefore it exists a $\gamma \in \mathbf{C}^3$ such that

$$C \subset \{(k, y, 0) \in Q_\infty / \langle k, \gamma \rangle + y = 0\}.$$

Since π_0 has rank ≤ 1 γ is either purely real or purely imaginary. We discuss here the case $\gamma \in \mathbf{R}^3$. We may now assume that

$$C' = \{(k, 1, 0) \in Q_\infty / \langle k, \gamma \rangle + 1 = 0\} = \{(k, 1, 0) \in \mathbf{P}^4 / k_1^2 + k_2^2 + k_3^2 = 0, \langle k, \gamma \rangle + 1 = 0\}.$$

We have to show : $\gamma \in \Gamma^\#$, i.e. (*) is true.

So let $\gamma \notin \Gamma^\#$. Consider for $k \in \mathbf{C}^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ $v(k)$, the unit vector in $\mathbf{R}^3$ such that $Rek, Imk, v(k)$ form an oriented orthogonal basis.

Put $\mathcal{D}_1 := \{v \in \mathbf{R}^3 / |v| = 1, v \neq \frac{b}{|b|} \text{ for all } b \in \Gamma^\# - \{0\} \text{ and there are only finitely many } b \in \Gamma^\# \text{ such that } |v - \frac{b}{|b|}| < \frac{1}{|b|^2}\}.$

It is easy to see that the complement of $\mathcal{D}_1$ in the unit sphere S^2 has Lebesgue measure zero. Further one shows

Lemma 4.— For any $k \in \mathbf{C}^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ and $v(k) \in \mathcal{D}_1$ there is a $K' > 0$ such that for all $b \in \Gamma^\# - \{0\}$

$$|\langle k, b \rangle| \geq K'|b|^{-2}.$$

But the map $C' \rightarrow S^2, (k, 1, 0) \mapsto v(k)$ has maximal rank almost everywhere. Therefore for all points $(k, 1, 0)$ outside a set of Lebesgue measure zero in C' there is a $K > 0$ such that $|\langle k, b \rangle| \geq K|b|^{-2}$ for all $b \in \Gamma^\# - \{0\}$.

Now the map

$$\pi : C' \rightarrow P := \{x \in \mathbf{R}^3 / \langle x, \gamma \rangle + 1 = 0\}, (k, 1, 0) \mapsto \text{Re } k$$

is surjective and submersive. Thus Theorem 1' follows immediately (since then $C' \cap \mathcal{D}$ consists of infinitely many points) from

Lemma 5.— The set of points $x \in P$ for which there is $K, \tau > 0$ such that $|\langle x, b \rangle + 1| \geq K|b|^{-\tau}$ has positive Lebesgue measure.

4. Appendix

It is possible to show that $F_\lambda(q)/\Gamma^\#$ for split potentials of the form $q(x) = p_1(x_1, x_2) + p_3(x_3)$ for a lattice $\Gamma = a_1\mathbf{Z} + a_2\mathbf{Z} + a_3\mathbf{Z}$ with $\langle a_1, a_3 \rangle = \langle a_2, a_3 \rangle = 0$ is always irreducible. One uses three facts :

- i) The Bloch-varieties $\mathcal{B}(p_1)$ and $\mathcal{B}(p_2)$ are irreducible (see [KT])
- ii) The map $\Phi : \mathcal{B}(p_1) \times \mathcal{B}(p_2) \rightarrow \mathcal{B}(p_1 + p_2)$ is surjective
- iii) Introducing

$$\begin{aligned} \pi_1^{(\lambda)} : \mathcal{B}(p_1) &\rightarrow \mathbf{C}, (k_1, k_2, \lambda_1) \rightarrow \lambda_1 - \frac{\lambda}{2} \\ \pi_2^{(\lambda)} : \mathcal{B}(p_2) &\rightarrow \mathbf{C}, (k_3, \lambda_2) \rightarrow \frac{\lambda}{2} - \lambda_2 \end{aligned}$$

the Fermi-surface $F_\lambda(q)$ is the fibered product

$$\mathcal{B}(p_1) \times_\lambda \mathcal{B}(p_2) = \{((k_1, k_2, \lambda_1), (k_3, \lambda_2)) \in \mathcal{B}(p_1) \times \mathcal{B}(p_2) / \pi_1^{(\lambda)}(k_1, k_2, \lambda_1) = \pi_2^{(\lambda)}(k_3, \lambda_2)\}$$

Therefore we have

Theorem 3.—

If $q \in L^2(\mathbf{R}^3/\Gamma)$ and the Fermi-surface $F_{\text{phys}, \lambda(q)}$ is the same as $F_{\text{phys}, \lambda(q')}$, where q' is a split potential of the above form, then q also splits.

Let us close this report by the remark that for the discrete periodic Schrödinger operator $F_\lambda(q)/\Gamma^\#$ is always irreducible (see [B]).

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