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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

SPECTRUM DISTRIBUTION FUNCTION AND VARIATIONAL PRINCIPLE FOR AUTOMORPHIC OPERATORS ON HYPERBOLIC SPACE

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INTRODUCTION.

Let $\mathbf{H}^n$ be the n-dimensional hyperbolic space (Lobachevsky space) which can be modelled as a half-space $\{x = (\tilde{x}, x_n) | \tilde{x} \in \mathbf{R}^{n-1}, x_n > 0\}$ with the Riemannian metric $ds^2 = x_n^{-2}(d\tilde{x}^2 + dx_n^2)$. Let Γ be a discrete group of isometries of $\mathbf{H}^n$ such that vol $\mathbf{H}^n/\Gamma < \infty$. In this paper we study some aspects of spectral theory of elliptic (pseudodifferential) operators on $\mathbf{H}^n$ which are automorphic with respect to Γ i.e. commute with the shift operators given by the transformations from Γ in function spaces on $\mathbf{H}^n$. We also consider more general situations with a general Riemannian manifold M instead of $\mathbf{H}^n$.

Similar operators on $\mathbf{R}^n$ are called periodic (with respect to a lattice $\Gamma \subset \mathbf{R}^n$ which acts on $\mathbf{R}^n$ by translations). The spectrum of self-adjoint periodic operators on $\mathbf{R}^n$ has a well-known band structure [E], [S]. There is also an important spectral invariant of such operators which is called the integrated density of states (IDS) and naturally arises in the quantum theory of solids. It is defined for self-adjoint semibounded below elliptic periodic differential operators as a limit distribution function

$$N(\lambda) = \lim_{V \to \infty} |V|^{-1} N_V(\lambda) \tag{0.1}$$

where $N_V(\lambda)$ is the usual eigenvalue distribution function of the operator in a bounded open domain $V \subset \mathbf{R}^n$ with Dirichlet boundary conditions, $|V|$ is the Euclidean volume of $V, V \to \infty$ means that V blows up in a sufficiently regular way (e.g. homothetically) filling the whole space $\mathbf{R}^n$ in this limit process. IDS can be defined by (0.1) not only for periodic operators but also for general almost periodic or random self-adjoint elliptic operators (see e.g. [Sh 4], [G]).

For an almost periodic operator A it was proved in [Sh 2] that his IDS can be written as

$$N(\lambda) = tr_B E_\lambda \tag{0.2}$$

where E_λ is the spectral projection of some operator which is closely connected with A, and tr_B is the trace on a II_∞-factor introduced by L.A. Coburn, R.D. Moyer and I.M. Singer [C-M-S] to build an index theory of almost periodic elliptic operators. In particular (0.2) implies a variational principle for $N(\lambda)$ (see [Sh 1]) and allows a description of the spectrum $\sigma(A)$ as the set of all points of increasing of $N(\lambda)$:

$$\sigma(A) = \{\lambda : N(\lambda + \varepsilon) - N(\lambda - \varepsilon) > 0 \quad \text{for every} \quad \varepsilon > 0\} \tag{0.3}$$

There are also some results on asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$ (see [Sh 4] and references there). The most unusual one tells that if $A = -\Delta + q(x)$ with almost periodic or random q, then

$$N(\lambda) = c\lambda^{n/2}(1 + 0(\lambda^{-1})) \tag{0.4}$$

(Note that the estimate of remainder is remarkably better here then the best possible Hörmander estimate in the usual Weyl formula for the eigenvalue distribution function of a second order operator on n-dimensional compact manifold). In [Sh 1] (0.4) was proved with help of (0.2) and the variational principle but later a very simple proof was found [Sh 4]. The latter uses directly (0.1) and the fact that for $A = -\Delta$ IDS is explicitly known.

Now let M be the universal covering space of a compact manifold X and $\Gamma = \pi_1(X)$ acts on M by deck transformations. More generally M can be a Galua covering space of a compact manifold X. M.F. Atiyah [A] introduced then a von Neumann algebra (NA) $\mathcal{A}$ of Γ-periodic operators and a natural trace tr_Γ on $\mathcal{A}$. He used them to build up real-valued index theory of elliptic Γ-periodic operators on M.

Generalising a bit in this paper we consider the case when M is an n-dimensional Riemannian manifold and Γ a discrete group of isometries of M such that vol $M/\Gamma < \infty$ (we do not suppose M/Γ to be compact and allow that transformations $\gamma \in \Gamma$ have fixed points). Using the same Atiyah trace tr_Γ we introduce the spectrum distribution function $N_\Gamma(\lambda)$ by the formula which is similar to (0.2) ; so (0.3) is also true. We do not know whether some kind of interpretation of $N_\Gamma(\lambda)$ as IDS (i.e. a formula like (0.1)) generally exists. It probably does not when the area of the sphere of the radius R increases at the same rate as the volume of the ball of radius R as $R \to \infty$ (e.g. this is the case in $\mathbf{H}^n$). Then we use NA technique to prove a variational principle which is similar to the well known Courant principle in the form of the Glazman lemma. This can be done also without using NA's (see [B-S] where the case of random elliptic operators was treated) but that would be more complicated.

The variational principle allows us to apply perturbation technique to obtain the asymptotic results like (0.4) for operators $A = (-\Delta)^{m/2} + Q_{m-r}$, where Δ is the Laplacian on $\mathbf{H}^n$, Q_{m-r} a Γ-periodic pseudodifferential operator of order $m - r$. More exactly we prove that in this case

$$N_\Gamma(\lambda) = \sum_{j=0}^{k} c_j \lambda^{(n-2j)/m} + 0(\lambda^{(n-r)/m}) \tag{0.5}$$

where $k = [(r-1)/2], c_j$ are constants depending only on n (actually they coincide with the Minakshisundaram - Pleijel coefficients of $-\Delta$). The proof goes along the same lines as that of (0.4) in [Sh 1] (i.e. it is based on the variational principle and not on formulas like (0.1) as in [Sh 4]).

Note finally that S.P. Novikov and M.A. Shubin [N-S] introduced some invariants of a compact manifold X with nontrivial fundamental group as exponents characterizing asymptotic behaviour of $N_\Gamma(\lambda)$ as $\lambda \to +0$ for the Laplacians on exterior p-forms on the universal covering space M of X. In Sect.5 we show that the variational principle easily implies that these invariants do not depend on the chosen Riemannian metric on X.

1. VON NEUMANN ALGEBRA OF Γ-INVARIANT OPERATORS.

1.1. Preliminaries on von Neumann algebras.

We start by reminding the necessary preliminaries on von Neumann algebras ([D], [T]). Let $\mathcal{H}$ be a Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators in $\mathcal{H}$. A subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is called *von Neumann algebra* (NA) if

a) $I \in \mathcal{A}$ (I is the identity operator in $\mathcal{H}$) ;

b) $A \in \mathcal{A}$ implies $A^* \in \mathcal{A}$ (i.e. $\mathcal{A}$ is a $*$-algebra) ;

c) $\mathcal{A}$ is closed with respect to the weak operator topology.

For $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ denote by $\mathcal{M}'$ the commutant of $\mathcal{M}$, i.e.

$$\mathcal{M}' = \{B | B \in \mathcal{L}(\mathcal{H}), BM = MB \quad \text{for every} \quad M \in \mathcal{M}\}.$$

A $*$-algebra $\mathcal{A}$ is NA iff $\mathcal{A}'' = \mathcal{A}$ where $\mathcal{A}'' = (\mathcal{A}')'$. Let a NA $\mathcal{A}$ be given. Let A be an unbounded operator in $\mathcal{H}$. We write $A\eta\mathcal{A}$ and say that A is *affiliated* with $\mathcal{A}$ iff $BA \subset AB$ for every $B \in \mathcal{A}'$. If A is closed and densely defined operator in $\mathcal{H}$, $A = US$ its polar decomposition and $A\eta\mathcal{A}$ then $U \in \mathcal{A}$ and $S\eta\mathcal{A}$. If A is a self-adjoint operator in $\mathcal{H}$, $A\eta\mathcal{A}$ and $f = f(\lambda)$ is a Borel function on the spectrum $\sigma(A)$ of A, then $f(A)\eta\mathcal{A}$. In particular $E_\lambda \in \mathcal{A}$ for all spectral projections E_λ of A.

Denote $\mathcal{A}^+ = \{A \in \mathcal{A} | A \geq 0\}$. A *trace* on $\mathcal{A}$ is an additive positively homogeneous (of order 1) nonnegative functional $f : \mathcal{A}^+ \to [0, +\infty]$ satisfying $f(CC^*) = f(C^*C)$ for every $C \in \mathcal{A}$. Such a trace can be extended from $J^+ = \{A \in \mathcal{A}^+ | f(A) < +\infty\}$ to a linear functional on an ideal $J \subset \mathcal{A}$ which is a linear hull of J^+. Operators $A \in \mathcal{J}$ are called f-class operators.

We will always deal with the traces satisfying some additional conditions which are called faithfullness, normality and semifiniteness. A trace f on $\mathcal{A}^+$ is called *faithful* iff $A \in \mathcal{A}^+$ and $f(A) = 0$ imply $A = 0$. It is called *normal* iff for every increasing directed set $\{A_\alpha\} \subset \mathcal{A}^+$ the existence of $\sup\limits_\alpha A_\alpha = A \in \mathcal{A}^+$ implies that $f(A) = \sup\limits_\alpha f(A_\alpha)$. *Semifiniteness* of f means that if $A \in \mathcal{A}^+$ then $f(A) = \sup f(B)$ with the supremum over all $B \in \mathcal{A}^+$ such that $B \leq A$ and $f(B) < +\infty$.

1.2 Γ-invariant operators and Γ-trace.

Let M be a complete connected Riemannian manifold, Γ a discrete group of isometries of M. Let $\mathcal{F}$ be a fundamental domain of Γ. We suppose that vol $\mathcal{F} < \infty$. Remark that some elements $\gamma \in \Gamma$ may have fixed points and M/Γ is not supposed to be compact. But we suppose that Γ acts effectively i.e. that there is no $\gamma \in \Gamma \setminus \{e\}$ such that γ defines the identity transformation of M. We denote γx the result of action of $\gamma \in \Gamma$ on a point $x \in M$. Define $L^2(M)$ with respect to the usual Riemannian measure and a shift operator $L^2(M) \to L^2(M)$ by the formula $(L_\gamma u)(x) = u(\gamma^{-1}x)$, so L_γ is an unitary operator. An operator A in $L^2(M)$ is called a Γ-*invariant operator* iff $AL_\gamma = L_\gamma A$ for every $\gamma \in \Gamma$.

By $\mathcal{A}_\Gamma$ denote the algebra of all bounded linear Γ-invariant operators. Then $\mathcal{A}_\Gamma$ is a NA and $\mathcal{A}'_\Gamma$ is the weak closure of the linear hull of $\{L_\gamma | \gamma \in \Gamma\}$. It is easy to see that if A is a self-adjoint operator in $L^2(M)$ and A is Γ-invariant then $A\eta\mathcal{A}_\Gamma$.

Define an action of Γ on $M \times M$ by $\gamma(x,y) = (\gamma x, \gamma y)$. Then $\mathcal{F} \times M$ or $M \times \mathcal{F}$ are fundamental domains of this action.

Let us introduce a class ΓHS of Γ-*Hilbert-Schmidt operators* in $L^2(M)$. By definition $A \in \Gamma\text{HS}$ means that $A \in \mathcal{A}_\Gamma$ and for the L. Schwartz kernel K_A of A we have $K_A \in L^2_{loc}(M \times M)$ and $K_A \in L^2(\mathcal{F} \times M)$. Note that $K_A(\gamma x, \gamma y) = K_A(x,y)$ for almost all $(x,y) \in M \times M$ so $K_A \in L^2(\mathcal{F} \times M)$ iff $K_A \in L^2(M \times \mathcal{F})$. Let χ be such a function on M that

$$\sum_{\gamma \in \Gamma} |L_\gamma \chi(x)|^2 \leq C \tag{1.1}$$

for almost all $x \in M$. It is easy to see that $A \in \Gamma\text{HS}$ iff χA is a usual Hilbert-Schmidt operator for every χ satisfying (1.1). So ΓHS is a bilateral *-ideal.

Now we can introduce Γ-*trace class* ΓTR as a bilateral ideal spanned by $A_1 A_2$ with $A_1, A_2 \in \Gamma\text{HS}$. If σ_1 and σ_2 are measurable functions satisfying (1.1) and $A \in \Gamma\text{TR}$ then $\sigma_1 A \sigma_2$ is a usual trace-class operator. Suppose in addition that

$$\sum_{\gamma \in \Gamma} L_\gamma(\sigma_1 \sigma_2) = 1. \tag{1.2}$$

It is not difficult to see (cf. [A] that $tr \sigma_1 A \sigma_2$ does not depend on the choice of σ_1 and σ_2 satisfying (1.1) and (1.2). So we can define Γ-*trace* of A as

$$tr_\Gamma A = tr \sigma_1 A \sigma_2 \tag{1.3}$$

Let $\{e_i, i = 1, 2, \cdots\}$ be an orthonormal base in $L^2(\mathcal{F})$. Set $e_i(x) \equiv 0$ for $x \notin \mathcal{F}$, then $\{L_\gamma e_i | \gamma \in \Gamma, i \in \mathbf{N}\}$ is an orthonormal base in $L^2(M)$. Taking $\sigma_1 = \sigma_2 = \chi_\mathcal{F}$ where $\chi_\mathcal{F}$ is a characteristic function of $\mathcal{F}$ we obtain

$$tr_\Gamma A = \sum_{i=1}^{\infty} (A e_i, e_i)$$

Note that there exists a function $\sigma \in C^\infty(M)$ such that $\Sigma_{\gamma \in G} L_\gamma(\sigma) \equiv 1$. It may be essentially constructed in the same way as in [A]. Some difference is caused by the existence of fixed points. We leave details to the reader. For any $\psi \in L^1_{loc}(M) \cap L^1(\mathcal{F})$ we have

$$\int_M \sigma(x) \psi(x) d\mu = \int_\mathcal{F} \psi(x) d\mu \tag{1.4}$$

where $d\mu$ is the Riemannian density induced by a given Γ-invariant Riemannian metric on M. Obviously we can take $\sigma_1 = \sigma_2 = \sqrt{\sigma}$ in (1.3). We also can suppose that σ is positive on $\bar{\mathcal{F}}$. So $A \in \mathcal{A}_\Gamma$ belongs to ΓHS iff σA is a Hilbert-Schmidt operator.

With these preliminaries it is not hard to prove the following properties of the Γ-trace.

Proposition 1.1.—

i) Let $A \in \mathcal{A}_\Gamma^+$, then $A \in \Gamma TR$ iff $\sqrt{A} \in \Gamma HS$.

ii) Let $A \in \mathcal{A}_\Gamma, S \in \Gamma TR$, then $tr\ SA = tr\ AS$.

iii) Let $S \in \Gamma TR$, $A_j \in \mathcal{A}_\Gamma, A_j \to A$ strongly. Then $tr_\Gamma SA_j \to tr_\Gamma A$.

iv) (Normality of Γ-trace). Let $\{A_\alpha\}$ be an increasing directed set of operators from $\mathcal{A}_\Gamma^+$ and suppose that there exists $\sup_\alpha A_\alpha = A \in \mathcal{A}_\Gamma^+$. Then $tr_\Gamma A = \sup_\alpha tr_\Gamma A_\alpha$

Proofs of the statements of this proposition are the same as in [A] or follow from analogous statements for usual trace.

2. UNIFORM PSEUDODIFFERENTIAL OPERATORS.

Here we introduce appropriate classes of uniform pseudodifferential operators (ΨDO) and corresponding uniform Sobolev spaces on M. For the sake of simplicity we suppose that discrete group Γ acts on M without fixed points and $X = M/\Gamma$ is a compact manifold. For an unimodular Lie group similar classes of ΨDO were studied in [M-S 1] and for general manifolds of bounded geometry in [Ro].

2.1. Classes of uniform ΨDO.

Let $\mathcal{U}$ be a Γ-invariant covering of M obtained by lifting of finite covering of X by small balls with coordinates chosen on them. Then $\mathcal{U}$ has finite multiplicity, and we may suppose that $\gamma U \cap U = \emptyset$ for any $U \in \mathcal{U}$ and $\gamma \in \Gamma, \gamma \neq e$.

For a domain $V \subset \mathbf{R}^n$ we denote by $S^m(V)$ a usual class of functions $a(x,\xi) \in C^\infty(V \times \mathbf{R}^n)$ such that for any $K \subset\subset V$ and any multiindices α and β there exists a constant $C_{\alpha\beta K}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha\beta K}(1+|\xi|)^{m-|\alpha|} \tag{2.1}$$

Now $a(x, D_x)$ denotes ΨDO on V with the symbol $a(x,\xi)$.

Definition 2.1.— *An operator $A : C_0^\infty(M) \to C^\infty(M)$ is called properly supported uniform ΨDO of order m iff the following conditions are satisfied :*

i) Let $K_A(x,y)$ be the L. Schwartz kernel of A. Then there exists a constant C_A such that $K_A(x,y) = 0$ when $d(x,y) > C_A$ where $d(x,y)$ denotes the distance between x and y, induced by a given Riemannian metric.

ii) $K_A \in C^\infty(M \times M \backslash \Delta_M)$ where $\Delta_M \subset M \times M$ is the diagonal, and for every $U_1, U_2 \in \mathcal{U}$, every $K_1 \subset\subset U_1, K_2 \subset\subset U_2, K_1 \cap K_2 = \emptyset$, and every multiindices α, β there exists a constant $C = C_{\alpha\beta K_1 K_2}$ such that C does not change if we replace K_1, K_2, U_1, U_2, by γK_1, $\gamma K_2, \gamma U_1, \gamma U_2$ and
$$|\partial_x^\alpha \partial_y^\beta K_A(x,y)| \leq C_{\alpha\beta K_1 K_2}, x \in K_1, y \in K_2\ ;$$

iii) Let $A_U = A_{|U} : C_0^\infty(U) \to C^\infty(U)$ be a restriction of A on $U \in \mathcal{U}$. Then $A_U = a_U(x, D_x) + R_U$, where $a_U \in S^m(U), R_U$ is a smoothing operator such that
$$|\partial_x^\alpha \partial_y^\beta K_{R_U}(x,y)| \leq C_{\alpha\beta K} \tag{2.2}$$

when $x, y \in K \subset\subset U$. The constants $C_{\alpha\beta K}$ in (2.1) for a_U and in (2.2) for K_{R_U} do not change if we replace K and U by γK and γU for any $\gamma \in \Gamma$.
Denote this class of operators by $U\Psi^m(M)$.
Now we shall define uniformly elliptic operators.

Definition 2.2.— *ΨDO $A \in U\Psi^m(M)$ is called uniformly elliptic iff there exist constants $C > 0$ and $R > 0$ which do not depend on $U \in \mathcal{U}$ such that*

$$|a_U(x,\xi)| \geq C|\xi|^m$$

when $x \in U, |\xi| > R$. Denote this class of uniformly elliptic operators by $EU\Psi^m(M)$.

We shall also need classical or polyhomogeneous (in the sence of [Sh 3] or [Hö]) ΨDO. In this case for each U we have

$$a_U(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j,U}(x,\xi)\psi(\xi)$$

where $m \in \mathbf{C}, a_{m-j,U}$ are positively homogeneous of order $m-j$ in ξ, $\psi \in C^\infty(\mathbf{R}^n), \psi \equiv 0$ when $|\xi| < 1/2$ and $\psi \equiv 1$ when $|\xi| > 1$. Denote also

$$r_{N,U}(x,\xi) = a_U(x,\xi) - \sum_{j=0}^{N-1} \psi(\xi)a_{m-j,U}(x,\xi).$$

Obviously $a_{m-j,U} \in S^{Re m-j}(U), r_{N,U} \in S^{Re m-N}(U)$. ΨDO A is called uniform classical ΨDO if for $\psi a_{m-j,U}, r_{N,U}$ one can choose constants in (2.1) uniformly for all $U \in \mathcal{U}$. Denote this class by $CU\Psi^m(M)$. Denote also $ECU\Psi^m(M) = CU\Psi^m(M) \cap EU\Psi^{Re m}(M)$.

A rootine technique allows us to show that $U\Psi^\infty(M) = \bigcup_{m\in\mathbf{R}} U\Psi^m(M)$ is an involutive algebra. For $A \in EU\Psi^m(M)$ there exists a *parametrix* $B \in EU\Psi^{-m}(M)$ i.e. an inverse to A modulo $U\Psi^{-\infty}(M) = \cap_{m\in\mathbf{R}} U\Psi^m(M)$. If $A \in ECU\Psi^m(M)$ then B may be chosen from $ECU\Psi^{-m}(M)$. We will write D instead of Ψ for classes of differential operators and Γ instead of U for classes of Γ-invariant operators.

2.2. Uniform Sobolev spaces.

Let $\Lambda_s \in E\Gamma\Psi^s(M)$ be an operator with a positive principal symbol. We may suppose that Λ_{-s} is a parametrix for Λ_s so that $\Lambda_{-s}\Lambda_s = I + R_s$ where $R_s \in U\Psi^{-\infty}(M)$. Define $H^{-\infty}(M)$ as a set of $u \in \mathcal{D}'(M)$ such that there exist $P_1, \ldots, P_N \in \Gamma D^\infty(M)$ and $u_1, \ldots, u_N \in L^2(M)$ such that $u = P_1u_1 + \ldots + P_Nu_N$. Now we define uniform Sobolev spaces $H^s(M)$.

Definition 2.3.— $H^s(M) = \{u \in H^{-\infty}(M) | \Lambda_s u \in L^2(M)\}$. *Define also $H^\infty(M) = \cap_{s\in\mathbf{R}} H^s(M)$. The following proposition is a trivial corollary of developped technique.*

Proposition 2.1.—

i) Let $A \in U\Psi^m(M), u \in H^s(M)$, then $Au \in H^{s-m}(M)$.

ii) (Elliptic regularity.) Let $u \in H^{-\infty}(M), A \in EU\Psi^m(M), Au \in H^s(M)$, then $u \in H^{s+m}(M)$.

Let us introduce an inner product in $H^s(M)$ in a usual way. Namely, let $Q_1, \dots, Q_N$ be generators of left $C^\infty(X)$-module $\Gamma D^p(M), p \geq s$. For $u, v \in H^s(M)$ set

$$(u,v)_s = (\Lambda_s u, \Lambda_s v) + \sum_{j=1}^{N} (Q_j R_s u, Q_j R_s v) \tag{2.3}$$

where $(\cdot,\cdot)$ is the usual L^2-inner product. The inner product (2.3) turns $H^s(M)$ into a Hilbert space, and the Sobolev norm defined by (2.3) is locally equivalent to usual $H^s(\mathbf{R}^n)$ norm. An operator $A \in U\Psi^m(M)$ extends to a continuous operator $A : H^s(M) \to H^{s-m}(M)$ for each $s \in \mathbf{R}$. Such simple facts as the density of $C_0^\infty(M)$ in every Sobolev space $H^s(M)$ and the duality of $H^s(M)$ and $H^{-s}(M)$ with respect to L^2-inner product are also true (for the proofs see e.g. [Sh 3] and [M-S 1]). There is also a uniform Sobolev embedding theorem : if $s > k + n/2$ with an integer $k \geq 0$ then $H^s(M) \subset C_b^k(M)$ where $C_b^k(M)$ is the space of all $u \in C^k(M)$ such that Lu is bounded for every $L \in \Gamma D^k(M)$. From Proposition 2.1 easily follows

Proposition 2.2.— *Let $A \in EU\Psi^m(M)$, be a formally self-adjoint operator, $m > 0$. Consider A as an unbounded operator in $L^2(M)$ with the domain $D(A) = C_0^\infty(M)$. Then it is essentialy self-adjoint, and $D(\bar{A}) = H^m(M)$.*

2.3 Complex powers of uniform ΨDO.

Let $A \in ECU\Psi^m(M), m > 0$. Suppose that there exists an $\varepsilon > 0$ not depending on U such that

$$|a_{m,U}(x,\xi) - \lambda| > \varepsilon \tag{2.4}$$

when $x \in U, |\psi| > 1, \lambda \in \Lambda_\delta = \{\mu | |arg\mu - \pi| < \delta\}$. In this case one can construct complex powers A^z of A by the scheme of Seeley [S1]. The uniform structure of M allows us to obtain estimates of $(A-\lambda)^{-1}$ which are similar to ones of [S1], so one can define $A^z, \mathrm{Re}z < 0$, by a standard contour integral. All the local properties of the complex powers and their L. Schwartz kernels are the same as for compacts manifolds. In addition it is easy to see that $A^z = A^{(z)} + R_z$ where $A^{(z)} \in ECU\Psi^{mz}(M)$ and R_z is a smoothing operator in the uniform Sobolev scale (i.e. $R_z : H^s(M) \to H^t(M)$ is a bounded operator for all $s,t \in \mathbf{R}$). (Remark that complex powers of uniform elliptic operators on a unimodular Lie group were studied in [M-S 2] and the case of Γ-invariant ΨDO was considered in [E-E]).

Remark. All the results of this section are easily generalized on arbitrary manifolds with a bounded geometry (see e.g. [Ro]). We use the action of Γ just for the sake of simplicity of definitions (we do not use the manifolds without free action of a discrete group in applications below).

2.4. Operators in spaces of sections of vector bundles.

We shall need a generalization of the situation which has been described. Let M be as before and E, F are some complex vector Γ-bundles on M. This means that Γ acts on E and F and this action agrees with the action of Γ on M e.g. every element $\gamma \in \Gamma$ defines a linear map $\gamma_{*x} : E_x \to E_{\gamma x}$ for every $x \in M$. We always suppose that the bundles E, F have Hermitian inner product on their fibers. All structures are supposed to be C^∞. Then the usual Hilbert spaces $L^2(M, E)$ and $L^2(M, F)$ of square-integrable sections are defined. The shift operators L_γ are then defined on $C_0^\infty(M, E), C^\infty(M, F)$ and $L^2(M, E)$ (and also on $C_0^\infty(M, F), C^\infty(M, F)$, and $L^2(M, F)$). Now we can define Γ-invariant operators $A : C_0^\infty(M, E) \to C^\infty(M, F)$ or $A : L^2(M, E) \to L^2(M, F)$ as operators which commute with all L_γ i.e. $AL_\gamma = L_\gamma A$ for every $\gamma \in \Gamma$. Note that here L_γ in the left side of the equality acts in $C_0^\infty(M, E)$ or $L^2(M, E)$ and that in the right side acts in $C^\infty(M, F)$ or $L^2(M, F)$. All the definitions and results of this and previous sections are easily generalized to this case. Let us note that the kernel of an operator $A : C_0^\infty(M, E) \to C^\infty(M, F)$ will be considered as a distribution section of the vector bundle $E \otimes F$ over $M \times M$ with the fiber $E_x \otimes F_y$ over the point $(x, y) \in M \times M$. This is possible because we identify functions and densities with the help of the canonical Riemannian density on M and F_y and F_y^* with the help of Hermitian structure on F. The NA of all bounded linear Γ-invariant operators in $L^2(M, E)$ will be denoted by $\mathcal{A}_\Gamma(E)$. The main example of this situation naturally arises when Γ acts on M as before and E, F are some tensor bundles or their natural subbundles like $\Lambda^p T^* M$. The exterior differentiation de Rham operators $d_p : \Lambda^p(M) \to \Lambda^{p+1}(M)$ as well as the Laplacians $\Delta_p = -(d_p^* d_p + d_{p+1} d_{p+1}^*) : \Lambda^p(M) \to \Lambda^p(M)$ are Γ-invariant operators. The invariance condition for an operator $A : L^2(M, E) \to L^2(M, F)$ with a continuous kernel K_A can be written as follows :

$$K_A(\gamma x, \gamma y) = \gamma_{*x} K_A(x, y) \gamma_{*y}^{-1} \quad \text{for every} \quad x, y \in M.$$

In particular,

$$K_A(\gamma x, \gamma x) = \gamma_{*x} K_A(x, x) \gamma_{*x}^{-1} \quad \text{for every} \quad x \in M$$

and

$$tr\ K_A(\gamma x, \gamma x) = tr K_A(x, x)$$

i.e. $x \mapsto tr\ K_A(x, x)$ is a Γ-invariant scalar continuous function. If in addition $A \in \Gamma TR$ then

$$tr_\Gamma A = \int_{\mathcal{F}} tr\ K_A(x, x) d\mu. \tag{2.5}$$

The class of operators $A : C_0^\infty(M, E) \to C^\infty(M, F)$, naturally generalizing $U\Psi^m(M)$, will be denoted $U\Psi^m(M; E, F)$. We shall write $U\Psi^m(M, E)$ instead of $U\Psi^m(M; E, E)$. Analogous notations will be used for other classes of ΨDO. The natural uniform Sobolev space of sections of a vector Γ-bundle E will be denoted by $H^s(M, E)$ (if E is the trivial bundle with the fiber $\mathbf{C}^1$ then $H^s(M, E) = H^s(M)$).

3. SPECTRUM DISTRIBUTION FUNCTION AND VARIATIONAL PRINCIPLE.

3.1. Preliminaries

In this section we change notations a little. We return to the situation which is described in 1.2 i.e. we suppose that M is a complete connected Riemannian manifold and Γ is a discrete group of isometries of M such that $\mathrm{vol} M/\Gamma < \infty$. Moreover we shall suppose for the sake of simplicity that there is another discrete group of isometries of M which is denoted by Γ_1 and acts on M without fixed points so that M/Γ_1 is a compact manifold. Instead of the existence of such a group Γ_1 we could just suppose that M has a bounded geometry because all we need is that the technical tools described in Section 2 could be applied. So we can construct classes of uniform pseudodifferential operators $U\Psi^m(M)$, uniform Sobolev spaces $H^s(M)$ and combine results of the two previous sections.

An example of particular importance is $M = \mathbf{H}^n$. Since $\mathbf{H}^n$ is a symmetric space we can find a discrete group Γ_1 satisfying the above mentioned conditions ([B]). Consider the Laplacian Δ on $\mathbf{H}^n$, which we identify with the self-adjoint operator in $L^2(\mathbf{H}^n)$ defined as the closure of Δ from the initial domain $C_0^\infty(\mathbf{H}^n)$. In this way Δ becames a strictly negative self-adjoint operator (strictly negative means here that the spectrum of Δ is on the negative real half-line and is separated from 0). So following the results of Section 2 we can define complex powers $(-\Delta)^z, z \in \mathbf{C}$. We shall study Γ-invariant operators of the form $H = (-\Delta)^{m/2} + H_1$ where $m > 0, H_1 = H_1^{(1)} + H_1^{(2)}, H_1^{(1)} \in U\Psi^{m-r}(\mathbf{H}^n), r > 0, H_1^{(2)}$ is a smoothing operator in the uniform Sobolev scale. If H_1 is formally self-adjoint (i.e. symmetric an $C_0^\infty(\mathbf{H}^n)$) then the closure of H (which we will denote by the same letter) is a self-adjoint operator in $L^2(\mathbf{H}^n)$. An important example is the Schrödinger operator

$$H_0 = -\Delta + q \tag{3.1}$$

where q is a real-valued C^∞-function with bounded derivatives of every order (this means that $|Lq| \le C_2$ for every Γ_1-invariant differential operator L with C^∞-coefficients).

3.2. Spectrum distribution function.

Let E be a hermitian vector Γ-bundle on M, A be a $\Psi DO, A : C_0^\infty(M,E) \to C^\infty(M,E)$ and $A = A_1 + R_1$ where $A_1 \in U\Psi^m(M), R_1$ is a smoothing operator in the scale $H^s(M,E)$ i.e. R_1 defines a bounded operator $R_1 : H^s(M,E) \to H^t(M,E)$ for every $s,t \in \mathbf{R}$. Let us suppose that $m > 0$ and A_1 has a strictly positive principal symbol a_m so that $a_m(x,\xi) : E_x \to E_x$ is a positive hermitian map for every $x \in M$ and every $(x,\xi) \in T_x^*M$. Suppose that A is formally self-adjoint i.e. symmetric on $C_0^\infty(M,E)$. Then its closure (also denoted A) is a self-adjoint operator in $L^2(M,E)$ which is semibounded below. Let us consider an open, closed or semiclosed interval $\Lambda \subset \mathbf{R}$ and denote by E_Λ the corresponding spectral projection of A. If $\Lambda = (-\infty, \lambda]$ we shall write E_λ instead of E_Λ.

Let Λ be a bounded interval. Then one can show (cf.[F-S]) that E_Λ is a smoothing operator in the uniform Sobolev scale. If $e_\Lambda(x,y)$ denotes the L. Schwartz Kernel of E_Λ then $e_\Lambda(x,x) \ge 0$ for every $x \in M$. It is easy to see that

$$|(e_\Lambda(x,y)f,g)|^2 \le (e_\Lambda(y,y)f,f)(e_\Lambda(x,x)g,g)$$

for every $f \in E_y, g \in E_x$. It follows that

$$\|e_\Lambda(x,y)\|^2 \leq \|e_\Lambda(x,x)\| \|e_\Lambda(y,y)\|$$

for every $x, y \in M$. Since A is semibounded below all these statements are fulfilled for E_λ too.

Note that $e_\Lambda(\cdot, y)f = E_\Lambda(\delta(\cdot, y)f)$ for every $y \in M$ and $f \in E_y$. Using the results of Sect.2 we get

$$\|e_\Lambda(\cdot,y)\|_{L^2(M,E)} \leq \|E_\Lambda\|_{-\frac{n}{2}-1,0} \|\delta(\cdot - y)\|_{H^{-\frac{n}{2}-1}(M,E)} \leq C, \tag{3.2}$$

where we used the notation $\|A\|_{s,t} = \|A : H^s(M,E) \to H^t(M,E)\|$. Due to (3.2) it is obvious that $E_\Lambda \in \Gamma HS$ and since $E_\Lambda = E_\Lambda^2$ we obtain that $E_\Lambda \in \Gamma TR$. Now (with the notations of Sect. 1.2)

$$tr_\Gamma E_\Lambda = tr\sqrt{\sigma}E_\Lambda\sqrt{\sigma} = \int_M \sigma(x) tr e_\Lambda(x,x) d\mu = \int_{\mathcal{F}} tr\ e_\Lambda(x,x) d\mu$$

because of (1.4).

Now we can define the spectrum distribution function $N_\Gamma(\lambda)$.

Definition 3.1.— $N_\Gamma(\lambda) = tr_\Gamma E_\lambda$.

Note that $N_\Gamma(\lambda)$ is semicontinuous from the right (i.e. $N_\Gamma(\lambda+0) = N_\Gamma(\lambda)$), $N_\Gamma(\lambda) \equiv 0$ when $\lambda \leq \inf \sigma(A)$, and $\sigma(A)$ coincides with the set of points of increasing of N_Γ, i.e. (0.3) is fulfilled for N_Γ. We will also write $N_{\Gamma,A}$ instead of N_Γ when it is necessary to specify the operator A for which N_Γ is constructed.

3.3 The variational principle.

We shall formulate now the variational principle which is similar to the well known Glazman lemma. For the sake of simplicity we shall write $\mathcal{A}_\Gamma$ instead of $\mathcal{A}_\Gamma(E)$. Let $\mathrm{Proj}(\mathcal{A}_\Gamma)$ be the set of all orthogonal projections in $\mathcal{A}_\Gamma$. Let A be any semibounded below self-adjoint Γ-invariant operator. Then we can define $N_{\Gamma,A}(\lambda) = tr_\Gamma E_\lambda$ (it may happen that $N_{\Gamma,A}(\lambda) = \infty$ when $\lambda \geq \lambda_0$).

Theorem 3.1.— *Let $A\eta\mathcal{A}_\Gamma$ be self-adjoint and semibounded below. Then $N_{\Gamma,A}(\lambda) = \sup tr_\Gamma P$ where supremum is taken over $P \in \mathrm{Proj}(\mathcal{A}_\Gamma)$ such that $\mathrm{Im}P \subset D(A)$ and $P(A-\lambda)P \leq 0$ (here $D(A)$ is the domain of A).*

Remark. This theorem was proved in [Sh 1] for I or II factors. The algebra $\mathcal{A}_\Gamma$ is not generally a factor but the proof for $\mathcal{A}_\Gamma$ is the same. We will give a sketch of it for the sake of completeness.

Proof. Since E_λ satisfy the conditions on P in the theorem the inequality $N_{\Gamma,A}(\lambda) \leq \sup tr_\Gamma P$ is evident. To prove the inverse inequality note that if P satisfies the conditions then $\mathrm{Im}P \cap \mathrm{Im}(I - E_\lambda) = 0$ since $A - \lambda I > 0$ on $\mathrm{Im}(I - E_\lambda)$. So $\mathrm{Im}P \cap \mathrm{Ker}E_\lambda = 0$. This implies that E_λ gives a monomorphism of $\mathrm{Im}P$ to $\mathrm{Im}E_\lambda$ so it should be expected that $tr_\Gamma P \leq tr_\Gamma E$. For the rigorous proof one should consider the polar decomposition $E_\lambda P =$

UT where $T = \sqrt{PE_\lambda P}$, U is the partially isometric operator mapping isometrically ImT to Im$E_\lambda P$ and vanishing on $(\mathrm{Im}T)^\perp$. Since ImT = ImP, Im$E_\lambda P \subset$ ImE_λ, U is an isometric monomorphism of ImP onto a subspace in ImE_λ. We evidently have $U^*U = P, UU^* = F$ and $F \in \mathrm{Proj}(\mathcal{A}_\Gamma), F \leq E_\lambda$. So $tr_\Gamma P = tr_\Gamma(U^*U) = tr_\Gamma(UU^*) = tr_\Gamma F \leq tr_\Gamma E_\lambda$. □

Corollary 3.1.— *If both A_1, A_2 satisfy conditions of Theorem 3.1. and $A_1 \leq A_2$ then $N_{\Gamma,A_1}(\lambda) \geq N_{\Gamma,A_2}(\lambda)$ for every $\lambda \in \mathbf{R}$.*

4. ASYMPTOTICS OF THE SPECTRUM DISTRIBUTION FUNCTION.

4.1 Zeta function, teta function and rough asymptotics of $N_\Gamma(\lambda)$ as $\lambda \to +\infty$.

Suppose that $A \in ECU\Psi^m(M, E), m > 0$, and A is a self-adjoint positive Γ-invariant operator satisfying the matrix analogue of (2.4), i.e. $\|(a_{m,U}(x,\xi)-\lambda)^{-1}\| \leq C$ when $x \in U$, $|\xi| > 1, \lambda \in \Lambda_\delta$. It is easy to see that in this case all the constructions of sect.3.2 are valid. On the other hand one can construct complex powers A^z of A. The usual arguments connected with uniform Sobolev spaces (see 3.2) show that $A^z \in \Gamma TR$ when Re$z < -n/m$ and in this case

$$(4.1) \qquad tr_\Gamma A^z = \int_{\mathcal{F}} tr\ A_z(x,x) d\mu$$

where $A_z(x,y)$ is the L. Schwartz kernel of A^z. This fact allows us to give the following

Definition 4.1.— $\zeta_\Gamma(z) = tr_\Gamma A^z$, Re$z < -n/m$. *In view of (4.1) and standard local properties of $A_z(x,y)$ we obtain that ζ_Γ can be extended to a meromorphic function on* $\mathbf{C}$ *with simple poles in $z_k = (k-n)/m, k = 0, 1, \ldots$. Residues in these poles are given by the well known Seeley formulas ([S1]).*

We can also define the exponent e^{-zA} of A by the formula

$$(4.2) \qquad e^{-zA} = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s = c} z^{-s} A^{-s} \Gamma(s) ds$$

where $c > 0$, Re$z > 0$. Let $P_z(x,y)$ be the L. Schwartz kernel of e^{-zA}. Then it is easy to prove that $e^{-zA} \in \Gamma TR$ (see 3.2) and

$$(4.3) \qquad tr_\Gamma e^{-zA} = \int_{\mathcal{F}} tr\ P_z(x,x) d\mu$$

Definition 4.2.— $\theta_\Gamma(z) = tr_\Gamma e^{-zA}$, Re$z > 0$.

Taking Γ-trace in (4.2) we obtain a usual connection between ζ_Γ and θ_Γ. So (see e.g. [D-G]) there exists Minakshisundaram-Pleijel expansion for $\theta_\Gamma(z) (z \to 0)$ and the coefficients of it (which are called the Minakshisundaram-Pleijel coefficients of A) are related with the residues of ζ_Γ by standard formulas. Note also that if $A \geq \lambda_0 > 0$ then

$$(4.4) \qquad \zeta_\Gamma(z) = \int_0^\infty \lambda^z dN_{\Gamma,A}(\lambda), \mathrm{Re}\, z < -n/m.$$

There are two important corollaries from this formula.

First, if we know the residue $\beta_1 = \mathrm{res}_{z=-n/m}\zeta_\Gamma(z)$ then applying Ikehara tauberian theorem we obtain

$$N_{\Gamma,A}(\lambda) \sim -(m/n)\beta_1\lambda^{n/m}(\lambda \to +\infty) \tag{4.5}$$

For free actions of Γ with a compact factor M/Γ this formula was proved in [E-E]. Second, using (4.4) one can obtain expressions of coefficients of asymptotic expansion of $N_{\Gamma,A}(\lambda)$ through Minakshisundaram-Pleijel coefficients, if such an expansion exists. We will use this fact below for the Laplace-Beltrami operator.

4.2 Precise spectral asymptotics for Schrödinger type operators.

Here we will obtain for the spectrum distribution function of operator H on $\mathbf{H}^n$ much more precise asymptotic formula than (4.5) by using the variational principle. First of all we need the following

Lemma 4.1.— *Let H be as described in 3.1. Then there exists a constant $C > 0$ such that $\pm H_1 \le C(-\Delta)^{(m-r)/2}$.*

Proof. It is sufficient to prove the statement for H_1. We shall prove that for $u \in C_0^\infty(\mathbf{H}^n)$

$$(H_1u, u) \le C((-\Delta)^{\frac{m-r}{2}}u, u)$$

Since $(-\Delta)^{\frac{m-r}{4}} : L^2(\mathbf{H}^n) \to H^{\frac{-m-r}{2}}(\mathbf{H}^n)$ is an isomorphism, it is sufficient to prove that

$$((-\Delta)^{\frac{r-m}{4}}H_1(-\Delta)^{\frac{r-m}{4}}v, v) \le C(v, v), v \in H^\infty(\mathbf{H}^n) \tag{4.6}$$

But the operator $B = (-\Delta)^{\frac{r-m}{4}}H_1(-\Delta)^{\frac{r-m}{4}}$ is bounded in every $H^s(\mathbf{H}^n)$ so (4.6) is actually true. □

Now we shall formulate the main theorem.

Theorem 4.1.— *Let $N_\Gamma(\lambda)$ be the spectrum distribution function of H, where H is described in 3.1. Then*

$$N_\Gamma(\lambda) = \sum_{j=0}^{k} c_j\lambda^{\frac{n-2j}{m}} + 0(\lambda^{\frac{n-r}{m}})$$

when $\lambda \to +\infty$. Here $k = [(r-1)/2]$, and c_j do not depend on H_1 (they depend on j and n only).

Proof. Consider the functions $f_\pm(\lambda) = \lambda^{m/2} \pm C\lambda^{(m-r)/2}$ with C from Lemma 4.1. Then Lemma 4.1 and Theorem 3.1 imply that

$$N_{\Gamma,f_+(-\Delta)}(\lambda) \le N_\Gamma(\lambda) \le N_{\Gamma,f_-(-\Delta)}(\lambda). \tag{4.7}$$

Denote by $N_0(\lambda)$ the spectrum distribution function for $-\Delta$, then $N_{\Gamma,f_\pm(-\Delta)}(f_\pm(\lambda)) = N_0(\lambda)$. For $N_0(\lambda)$ the complete asymptotic expansion is known [Ef]

$$N_0(\lambda) \sim \sum_{j=0}^{\infty} c_j\lambda^{\frac{n}{2}-j} + d_0 \ln\lambda \tag{4.8}$$

and d_0 actually vanishes because of the regularity of $\zeta_\Gamma(z)$ near $z = 0$. We will study the case of $f_+(\lambda)$, another case is treated similarly. Let $\mu = f_+(\lambda)$, then $\lambda = \mu^{2/m}(1 + 0(\mu^{-r/m}))$. Substituting this expression to (4.8) we obtain

$$N_{\Gamma, f_+(-\Delta)}(\mu) \sim \sum_{j=0}^{\infty} c_j \mu^{\frac{n-2j}{m}} (1 + 0(\mu^{-r/m}))^{\frac{n}{2}-j} =$$

$$= \sum_{j=0}^{k} c_j \mu^{\frac{n-2j}{m}} + 0(\mu^{\frac{n-r}{m}})$$

where $k = [(r-1)/2]$ is the maximal integer which is less then $r/2$. The statement of the theorem follows now from (4.7). □

For the Schrödinger operator (3.1) with a Γ-periodic potential q Theorem 4.1 gives $N_\Gamma(\lambda) = c_0 \lambda^{n/2}(1 + 0(\lambda^{-1}))$. The estimate of remainder is remarkably better here then the usual Hörmander one.

As was mentionned above the coefficients c_j are related with the Minakshisundaram-Pleijel coefficients of $-\Delta$. In particular $c_j = 0$ when n is even and $j > n/2$. Denote by $P_t^{(n)}(r)$ the heat kernel on $\mathbf{H}^n$, where $r = d(x,y)$. The explicit formulas for $P_t^{(n)}(r)$ are known for $n = 2,3$ and there is a recurrent formula connecting $P_t^{(n+2)}(r)$ and $P_t^{(n)}(r)$ (see [D-G-M]). Using these formulas one can obtain more explicit formulas for $P_t^{(n)}(0)$ (and consequently for the Minakshisundaram-Pleijel coefficients) for odd n. Namely

$$P_t^{(2\ell+1)}(0) = (-1)^{\ell+1}(2t)^{-3/2}(2\pi)^{-(2\ell+1)/2} e^{-\ell^2 t} \sum_{m=0}^{\ell-1} \omega_m^{(\ell)} t^{-m}$$

where

$$\omega_m^{(\ell)} = (-1)^m \frac{2^{\ell-1-m}}{m!} \sum_{\substack{m_1=m-1 \\ m_1 \geq 0}}^{\ell-2} \cdots \sum_{\substack{m_{\ell-2}=m_{\ell-3}-1 \\ m_{\ell-2} \geq 0}}^{1} (m_1+1)\ldots(m_{\ell-2}+1)\kappa_{1-m_{\ell-2}} \cdots \kappa_{m_1-m+1}$$

and $\kappa_k = 2(1-2^{2k-1})|B_{2\ell}|/(2\ell)!, \ell > 0, \kappa_0 = 1, B_{2\ell}$ are Bernoulli numbers. In particular $P_t^{(2\ell+1)}(0) = (4\pi t)^{-\frac{2\ell+1}{2}}(1 + 0(t))$ and we obtain $c_0 = (2\pi)^{-n} n^{-1} S_{n-1} \mathrm{vol}\mathcal{F}$ where S_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbf{R}^n, n = 2\ell+1$, the formula, which agrees with (4.5).

5. TOPOLOGICAL APPLICATION : A NEW INVARIANT OF NON SIMPLY CONNECTED MANIFOLDS.

The results of this section belong to S.P. Novikov and M.A. Shubin. We suppose here that M is a Riemannian manifold with a free action of a discrete group of isometries such that $X = M/\Gamma$ is compact. It is well known that the spectrum of the Laplacian on M is connected with the properties of Γ, hence with topological properties of X. As an example remind a result of R. Brooks ([Br]) telling that for simply connected M (i.e. when M is the universal covering space for X) $0 \in \sigma(\Delta)$ iff Γ is amenable. Also Laplacians Δ_p on exterior p-forms on M are important. For example in [N-S1] von Neumann real Betti numbers $\bar{b}_p$ first introduced by M.F. Atiyah ([A]) were used to improve the Morse inequalities for manifolds which are not simply connected. These Betti numbers can be identified with von Neumann dimensions of $\mathrm{Ker}\Delta_p$ on L^2-forms on M and are homotopy invariants of X. Note that they are easily expressed in terms of the spectrum distribution function $N_p(\lambda)$ of $(-\Delta_p)$:

$$\bar{b}_p = N_p(+0) = \lim_{\lambda \to +0} N_p(\lambda) \tag{5.1}$$

Note that if $M = \mathbf{H}^n$ or M is a strictly pseudoconvex domain in $\mathbf{C}^k$ then $\bar{b}_p = 0$ when $p \neq \dim M/2$. But the condition $\bar{b}_p = 0$ does not imply that $0 \notin \sigma(-\Delta_p)$. It was remarked in [N-S1] and [N-S2] that if $0 \in \sigma(-\Delta_p)$ then new and deep topological phenomena appear : invariants lying between the von Neumann Reidemeister torsion and homology. An example of such an invariant is the exponent arising in the power asymptotics of $N_p(\lambda)$ as $\lambda \to +0$ or in the power asymptotics of $\theta_p(t) = tr_\Gamma e^{t\Delta_p}$ as $t \to +\infty$ ([N-S2]). Now we shall prove with help of the variational principle that these exponents do not depend on the Riemannian metric an X and introduce a more general invariant.

Proposition 5.1.— *([N-S2]). Let N_p, N_p' be spectrum distribution functions corresponding to the Laplacians $(-\Delta_p), (-\Delta_p')$, constructed by means of Γ-invariant Riemannnian metrics g, g' on M. Then there exists $C > 0$ such that*

$$N_p(C^{-1}\lambda) \leq N_p'(\lambda) \leq N_p(C\lambda) \quad \textit{for every} \quad \lambda \in \mathbf{R} \tag{5.2}$$

Proof. It is clear that there exists $C > 0$ such that $C^{-1}\Delta_0 \leq \Delta_0' \leq C\Delta_0$. Now $N_0(C^{-1}\lambda), N_0(C\lambda)$ are the spectrum distribution functions of the operators $C\Delta_0, C^{-1}\Delta_0$, so if $p = 0$ then (5.2) immediately follows from Corollary 3.1. It is not so easy when $p > 0$ because as was pointed out by M. Hilsum and G. Scandalis it is not always true that $\Delta_p' \leq C\Delta_p$ (e.g. take the case when $\bar{b}_p \neq 0$ and there exists a L^2-form ω such that $\Delta_p\omega = 0$ but $\Delta_p'\omega \neq 0$). But the arguments given in [Hi], Sect.5, show that there exists a bounded invertible Γ-invariant operator B (with a bounded inverse operator) such that

$$\Delta_p' \leq CB^*\Delta_p B$$

with a positive constant C. So (5.2) immediately follows. □

Denote by $\mathcal{N}$ the class of all (non strictly) increasing functions on $\mathbf{R}$ vanishing on the open negative half-line i.e. $\mathcal{N}$ contains all increasing functions $N(\lambda)$ such that $N(\lambda) = 0$

when $\lambda < 0$. For two such functions $N_1(\lambda), N_2(\lambda)$ we shall write $N_1 \sim N_2$ iff there exists a constant $C > 0$ such that

$$N_1(C^{-1}\lambda) \le N_2(\lambda) \le N_1(C\lambda) \quad \text{for every} \quad \lambda \in \mathbf{R}.$$

Let $\tilde{\mathcal{N}} = \mathcal{N}/\sim$ be the set of all equivalence classes. Then (5.2) means that the equivalence class $\tilde{N}_p = N_p \mathrm{mod} \sim$ of the function N_p does not depend on the choice of a Γ-invariant Riemannian metric on M (or in other words it does not depend on the choice of a Riemannian metric on X). Let us try to extract number invariants from the equivalence class of a function $N \in \mathcal{N}$. To do so it is natural to look at asymptotics of $N(\lambda)$ as $\lambda \to +\infty$ or $\lambda \to +0$. Now when $\lambda \to +\infty$ then we have Weyl asymptotics like (4.5) with $m = 2$ for every function $N_p(\lambda)$. It is easy to see that the equivalence class of such a function contains no information concerning the asymptotic behaviour as $\lambda \to +\infty$ exept of the exponent in the power asymptotics i.e. the only number we can extract from the asymptotic behaviour of $N_p(\lambda)$ as $\lambda \to +\infty$ is the dimension n. The asymptotic behaviour as $\lambda \to +0$ is much more interesting. First of all the limit $N(+0)$ does not depend on the choice of a function $N(\lambda)$ in a given equivalence class. But we know already that $\bar{b}_p = N_p(+0)$ are homotopy invariants. Let us write that $f(\lambda) \asymp \lambda^\alpha$ as $\lambda \to +0$ iff there exists $c > 0$ such that $c\lambda^\alpha \le f(\lambda) \le c^{-1}\lambda^\alpha$ when $\lambda \in (0, \lambda_0)$, where λ_0 is any fixed positive number. It is clear that if $N(\lambda) - N(+0) \asymp \lambda^\alpha$ with some $\alpha > 0$ then the same is true for any function $N_1 \in \mathcal{N}$ such that $N_1 \sim N$. So we have

Corollary 5.1.— *Suppose that*

$$N_p(\lambda) - \bar{b}_p \asymp \lambda^{\alpha_p} \quad \text{as} \quad \lambda \to +0 \tag{5.2}$$

for a Γ-invariant Riemannian metric on M. Then the same it is true for every such metric with the same α_p.

So α_p is an invariant of smooth structure on X. It can also be defined in terms of corresponding θ-function θ_Γ. More generally for every $N \in \mathcal{N}$ define the Laplace transform of the corresponding measure

$$\theta(t) = \int e^{-\lambda t} dN(\lambda).$$

Suppose that $\theta(t)$ is finite for every $t > 0$ which is certainly the case for the functions $N_p(\lambda)$. Then it is a (non strictly) decreasing function on $\mathbf{R}_+ = \{t : t > 0\}$. It is easily seen that if $N_1 \sim N_2$ and θ_1, θ_2 are the Laplace transforms of N_1, N_2 then $\theta_1 \sim \theta_2$ in the same sense i.e. there exists $C > 0$ such that

$$\theta_1(C\, t) \le \theta_2(t) \le \theta_1(C^{-1}t) \quad \text{for every} \quad t > 0.$$

So we have proved

Proposition 5.2.— *([N-S2]) Let $\theta_p(t) = tr_\Gamma e^{t\Delta_p}, \theta'_p(t) = tr_\Gamma e^{t\Delta'_p}$ where Δ_p, Δ'_p are as in Proposition 5.1.*
Then

$$\theta_p(Ct) \le \theta'_p(t) \le \theta_p(C^{-1}t) \quad \textit{for every} \quad t > 0. \tag{5.3}$$

The number invariants that we discussed before can be usually desribed in terms of θ_p as well. For example the Weyl asymptotic like (4.5) for $N_p(\lambda)$ can be deduced with help of tauberian Karamata theorem from the asymptotic

$$\theta_p(t) \sim ct^{-n/2}, t \to +0.$$

We also have

$$\bar{b}_p = \lim_{t \to +\infty} \theta_p(t).$$

If we now suppose that

$$\theta_p(t) - \bar{b}_p \asymp t^{-\alpha_p} \quad \text{when} \quad t \geq 1, \tag{5.4}$$

then the same is true for θ'_p, so α_p in (5.4) does not depend on the Riemannian metric on X.

The connection between (5.2) and (5.4) is clear from the following

Proposition 5.3.— *Estimates (5.2) and (5.4) are equivalent i.e. (5.2) holds if and only if (5.4) holds.*

Proof. We shall omit the subscript p in the notations $\alpha_p, \bar{b}_p, N_p, \theta_p$. Integrating by parts gives

$$\theta(t) = t \int e^{-\lambda t} N(\lambda) d\lambda$$

Subtracting $\bar{b}H(\lambda)$ from $N(\lambda)$ (with $\bar{b} = N(+0)$ and $H(\lambda)$ the Heaviside function) we may assume that $\bar{b} = 0$. Taking into account that

$$t \int_0^\infty e^{-\lambda t} \lambda^\alpha d\lambda = \Gamma(\alpha + 1) t^{-\alpha}$$

we easily obtain that (5.2) implies (5.4).

In proving the inverse implication we shall also suppose that $\bar{b} = N(+0) = 0$, so $\lim_{t \to +\infty} \theta(t) = 0$. Note first that

$$\theta(t) \geq \int_0^{\lambda+0} e^{-\lambda t} dN(\lambda) \geq e^{-\lambda t} N(\lambda)$$

for $t \geq 1$ and for every $\lambda \geq 0$, so if (5.4) is true then

$$N(\lambda) \leq e^{\lambda t} \theta(t) \leq C e^{\lambda t} t^{-\alpha}, t \geq 1.$$

Taking $t = \lambda^{-1}$ we obtain

$$N(\lambda) \leq C_1 \lambda^\alpha, \lambda \leq 1.$$

On the other hand we have

$$\theta(t) = \int_0^{+\infty} e^{-\lambda t} dN(\lambda) = \int_0^{\lambda-0} e^{-\lambda t} dN(\lambda) + \int_{\lambda-0}^{\infty} e^{-\lambda t} dN(\lambda) \leq$$

$$\leq N(\lambda) + e^{-\lambda t/2} \int_{\lambda-0}^{\infty} e^{-\lambda t/2} dN(\lambda) \leq N(\lambda) + e^{-\lambda t/2} \int_{0}^{\infty} e^{-\lambda t/2} dN(\lambda)$$

$$= N(\lambda) + e^{-\lambda t/2}\theta(t/2),$$

so if (5.4) is true then

$$N(\lambda) \geq ct^{-\alpha} - Ce^{-\lambda t/2}t^{-\alpha},$$

with some positive constants c, C for every $\lambda \geq 0$ and for every $t \geq 1$. Taking $t = M/\lambda$ with $M > 0$ sufficiently large we obtain $N(\lambda) \geq C_2\lambda^\alpha$, $\lambda \leq 1$, with some positive C_2, q.e.d. □

Note that if $M = \mathbf{R}^n$ then $\alpha_p = n/2$ for all $p = 0, 1, \ldots, n$. But for $M = \mathbf{H}^3$ we have $\alpha_1 = \alpha_2 = 1/2$ ([V]) so geometry of M really influences α_p.

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