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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

EXISTENCE OF SOLUTIONS TO MICROHYPERBOLIC BOUNDARY VALUE PROBLEMS

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This is a joint work with K. Kataoka (Univ. of Tokyo)

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Résumé

L'objet de cet exposé est de donner des théorèmes d'existence des solutions pour les problèmes microhyperboliques mixtes. Les problèmes aux bords microhyperboliques ont été étudiés initialement par J. Sjöstrand [Sj 1]. Il a obtenu le théorème de type Holmgren pour ces problèmes. Récemment, K. Kataoka et l'auteur ont obtenu une généralisation du théorème de J. Sjöstrand et donné un théorème d'extendabilité des solutions. Ils ont appliqué l'analyse microlocale des faisceaux due à Kashiwara-Schapira [K-S 2] pour le software et utilisé les problèmes aux bords elliptiques due à K. Kataoka [Ka 2] et P. Schapira [Sc 2].

Cette méthode est appliquable aussi pour obtenir les théorèmes d'existence des solutions.

Introduction.

The object of this talk is to give theorems about the existence and the extendabiblity of solutions to microhyperbolic and hyperbolic mixed problems. The microhyperbolic boundary value problems were first studied by J. Sjöstrand [Sj 1]. He obtained theorems of Holmgren type about the uniqueness of the solutions. Moreover there are many results for hyperbolic mixed problems, but we don't give a review here.

We applied in this work the theory of microlocal analysis of sheaves due to M. Kashiwara-P. Schapira [K-S2] as a software and the microlocal analysis of elliptic boundary value problems developed by K. Kataoka [Ka 1,2] and P. Shapira [Sc 2,3] for a hardware. Especially, we give the statements of the theorems in the framework of mild microfunctions due to K. Kakataoka [Ka 1].

Several words about the philosophy of this work of course, there are several approach to the (micro)-hyperbolic problems in the interior domain. For example, we can list up J.M. Bony-P. Shapira [B-S], M. Kashiwara- T. Kawai [K-K], M. Kashiwara-P. Schapira [K-S1], and J.M. Trépreau [Tr] for a method reducing the problems to those in the complex domains. We should also mention the microlocal energy method of J. Sjöstrand [Sj 2] and the method of K. Kajitani and S. Wakabayashi [K-W1] based on the Bronstein's calculus. The strategy of this work is essentially the same as the first group who studied the extension of holomorphic solutions using the geometry of characteristic varieties and took the boundary values as hyperfunctions or microfunctions. In this paper, we reduce the boundary value problems to the interior problems for equations degenerate along the boundary and study the extension of $\mathcal{BO}$ solutions to the degenerate equations. (Here $\mathcal{BO}$ denotes the sheef of hyperfunctions with holomorphic parameters. See paragraph 7 for precise definition). For this purpose, we apply the microlocal analysis of elliptic boundary value problems due to K. Kataoka [K 1,2] and P. Schapira [Sc 2,3] and finally utilize the microlocal analysis of sheaves to deduce the results in the real domains from those in the complex domains.

1. Problems.

Let $(w, z) = (w, z_1, \ldots, z_n)$ be coordinates of $\mathbf{C} \times \mathbf{C}^n = \mathbf{C}^{n+1}$, $(t, x) = (t, x_1, \ldots, x_n)$ coordinates of $\mathbf{R} \times \mathbf{R}^n = \mathbf{R}^{n+1}$ with $t = \Re w$ and $x = \Re w, M =]-T, T[\times N$ an open subset of $\mathbf{R}^{n+1}$, and X a complex neighborhood of M in $\mathbf{C}^{n+1}$. Here N is an open subset of $\mathbf{R}^n$ with a complex neighborhood Y in $\mathbf{C}^n$. We set

$$M_+ = \{(t, x) \in M;\ t \geq 0\}$$

and identify N with the boundary ∂M_+ of M_+. We take a differential operator with analytic coefficients of order m :

$$P(t, x, D_t, D_x) = D_t^m + \Sigma_{k=0}^{m-1} A_k(t, x, D_x) D_t^k.$$

Here $(D_t, D_x) = (\partial_t, \partial_x)$ and A_k is of order $\leq m - k$. We consider the Dirichlet problem

$$(1) \qquad \begin{cases} Pu(t, x) = f(t, x) \\ D_t^k u\ |_{t \to +0} = g_j(x) \quad (0 \leq j \leq m_+ - 1) \end{cases}$$

and will make precise the classes of the functions $u, f, (g_j)$ in the statement of the theorems. One should remark that one can deal with general boundary value problems under the Lopatinskii condition. But to simplify the present talk, we restrict ourselves to the Dirichlet problems. (See [Ka - To] for the theorems about general boundary value problems).

2. Mild microfunctions.

To state the main theorems, one should first of all give a brief explanation about the theory of the sheaf $\overset{\circ}{\mathcal{C}}_{N|M_+}$ of mild microfunctions due to K. Kataoka [Ka 1,4]. We take a system of coordinates of T_M^*X, the conormal bundle of M in X : $(t, x; \sqrt{-1}(\tau dt + \eta dx))$ with $\tau \in \mathbf{R}$ and $\eta \in \mathbf{R}^n$ and take that of T_N^*Y : $(x; \sqrt{-1}\eta dx)$. Then $\underset{N|M_+}{\overset{\circ}{\mathcal{C}}}$ is a sheaf on T_N^*Y, and for $(\overset{\circ}{x}, A\overset{\circ}{\eta} dx) \in T_N^*Y \setminus N$, any $f(t, x) \in \underset{M|M_+}{\overset{\circ}{\mathcal{C}}}$ gives a microfunction defined on

$$\left\{(t, x; \sqrt{-1}(\tau, \eta)) \in \overset{\circ}{\underset{M}{T^*}} X\,; 0 < t < \delta,\ |x - \overset{\circ}{x}| < \delta\,, |\eta - \overset{\circ}{\eta}| < \delta\right\}$$

with $\delta > 0$ sufficiently small. Moreover, $f(t, x)$ has its trace $f(t0, x)$ on $\{t = 0\}$. More precisely, there exist the canonical morphisms

$$\begin{aligned} \text{trace} \quad &: \underset{N|M_+}{\overset{\circ}{\mathcal{C}}} \longrightarrow \mathcal{C}_N \\ &\quad f(t, x) \longmapsto f(t + 0, x) \\ \text{ext} \quad &: \underset{N|M_+}{\overset{\circ}{\mathcal{C}}} \longrightarrow \iota_*(\rho_M\ |_{T_M^*X \underset{M}{\times} N} \setminus \sqrt{-1}T_N^*M) \\ &\quad f(t, x) \longmapsto Y(t).f(t, x). \end{aligned}$$

Here $\iota : T^*_M X \underset{M}{\times} N \setminus \sqrt{1} T^*_N M \longrightarrow T^*_N Y$ is the natural projection, which is written by coordinates

$$(0, x; \sqrt{-1}(\tau dt + \eta dx)) \longmapsto (x; \sqrt{-1}\eta dx).$$

Thus it means that for $(\overset{\circ}{x}; \sqrt{-1}\overset{\circ}{\eta} dx) \in T^*_N Y \setminus N$ and $f(t,x) \in \overset{\circ}{\rho}_{NIM_+} |_{(\overset{\circ}{x}, \sqrt{-1}\overset{\circ}{\eta})}$, $ext(f) = Y(t).f(t,x)$ defines a microfunction on

$$\left\{(t, x; \sqrt{-1}(\tau, \eta)) \in \overset{\circ}{\underset{M}{T^*}} X; \mid t \mid < \delta, \mid x - \overset{\circ}{x} \mid < \delta, \mid \eta - \overset{\circ}{\eta} \mid < \delta, \tau \in \mathbf{R}\right\}$$

with $\delta > 0$ small enough. See [Ka 1, 4] for more details about mild microfunctions. Especially in [Ka 4], we can find an intuitive explanation using the expression of hyperfunctions as boundary values of holomorphic functions. Refer also to Schapira - Zampieri [S - Z] in which you can find the interpretation of $\overset{\circ}{\mathcal{C}}_{N|M_+}$ by use of the sheaf $\mathcal{C}_{\Omega|X}$.

3. Announcements of microlocal theorems.

First we give for the extendability

Theorem 1.—

Take a real C^1 function $\psi(t,x,\eta)$ defined in a neighborhood of $t = 0, x = \overset{\circ}{x}, \eta = \overset{\circ}{\eta}$ satisfying

$$(4) \qquad \psi(0, \overset{\circ}{x}, \overset{\circ}{\eta}) = 0 \quad , \quad d\psi(0, \overset{\circ}{x}, \overset{\circ}{\eta}) \wedge dt \neq 0,$$

(5) ψ is homogeneous of order 0 with respect to ξ. Set

$$\overset{\circ}{t}{}^* dt + \overset{\circ}{x}{}^* dx + \overset{\circ}{\eta}{}^* d\eta = d\psi(0, \overset{\circ}{x}, \overset{\circ}{\eta})$$

and assume the conditions (S1) and (S2) :

(S1) there exists a constant $\delta > 0$ such that

$$\sigma_m(P)(t, x + \sqrt{-1}\varepsilon \overset{\circ}{\eta}{}^*; \theta + \varepsilon \overset{\circ}{t}{}^*, \sqrt{-1}\eta + \varepsilon \overset{\circ}{x}{}^*) \neq 0$$

on

$$\{0 < \varepsilon < \delta, 0 \leq t < \delta, \mid x - \overset{\circ}{x} \mid < \delta, \mid \eta - \overset{\circ}{\eta} \mid < \delta, \theta \in \sqrt{-1}\mathbf{R}\},$$

(S2) the equation for θ

$$\sigma_m(P)(0, \overset{\circ}{x} + \sqrt{-1}\varepsilon \overset{\circ}{\eta}{}^*; \theta + \varepsilon \overset{\circ}{t}{}^*, \sqrt{-1}\overset{\circ}{\eta}{}^* + \varepsilon \overset{\circ}{x}{}^*) = 0$$

has $m - m_+$ roots on $\{\Re\theta > 0\}$ and m_+ roots on $\{\Re\theta < 0\}$ if $0 < \varepsilon < \delta$.

Let u be a $\overset{\circ}{\mathcal{C}}_{N|M_+}$ solution defined on

$$\{(x; \sqrt{-1}\eta) \in \overset{\circ}{\underset{N}{T^*}} Y; \psi(0, x, \eta) < 0, \mid x - \overset{\circ}{x} \mid < \delta, \mid \eta - \overset{\circ}{\eta} \mid < \delta\}$$

of the Dirichlet problem

$$(6)\qquad \begin{cases} P(t,x,D_t,D_x)u(t,x) = 0 \\ \quad D_t^j u(t+0,x) = 0 \qquad (0 \le j \le m_+ - 1) \end{cases}$$

We assume that $v = \mathrm{ext}(u)$ *give a microfunction solution of the equation* $Pv = 0$ *on*

$$\{(t,x;\sqrt{-1}(\tau dt + \eta dx)) \in \overset{\circ}{T^*_M}X;\ \psi(t,x,\eta) < 0,\ 0 < t < \delta,\ \tau \in \mathbf{R},\ |\ x - \overset{\circ}{x}\ | < \delta,\ |\ \eta - \overset{\circ}{\eta}\ | < \delta\}.$$

Then u *extends to the point* $(\overset{\circ}{x};\sqrt{-1}\overset{\circ}{\eta}dx) \in \overset{\circ}{T^*_N}Y$ *as a* $\underset{N|M_+}{\overset{\circ}{\mathcal{C}}}$ *solution to (6).*

Next we give a theorem of existence

Theorem 2.—

Let $\psi(t,x,\eta)$ *be a real* c' *function defined in a neighborhood of* $\{t = 0,\ x = \overset{\circ}{x},\ \eta = \overset{\circ}{\eta}\}$ *which satisfies the conditions (4), (5), (S1) and (S2). Let* $f(t,x)$ *be a section of* $\underset{N|M_+}{\overset{\circ}{\mathcal{C}}}$ *defined in a neighborhood of* $\rho_0 = (\overset{\circ}{x};\ \sqrt{-1}\overset{\circ}{\eta}) \in \overset{\circ}{T^*_N}Y$ *, and* $(g_j)_{0\le j\le m_+-1}$ *a section of* $\mathcal{C}_N^{m_+}$ *defined in a neighborhood of* ρ_0*. Suppose that*

$$\mathrm{supp}(f) \cup \mathrm{supp}(g_0) \cup \ldots \cup \mathrm{supp}(g_{m_+-1}) \subset \{(x;\sqrt{-1}\eta dx) \in \overset{\circ}{T^*_N}Y\ ;\ \psi(0,x,\eta) \ge 0\}$$

and that

$$\mathrm{supp}(\mathrm{ext}(f)) \subset \{(t,x;\sqrt{-1}(\tau dt + \xi dx)) \in \overset{\circ}{T^*_M}X; \psi(t,x,\eta) \ge 0\}.$$

Then there exists a unique section $u(t,x)$ *of* $\underset{N|M_+}{\overset{\circ}{\mathcal{C}}}$ *defined in a neighborhood of* $\rho_0 \in \overset{\circ}{T^*_N}Y$ *which satisfies*

$$(7)\qquad \begin{cases} \qquad Pu(t,x) = f(t,x) \\ D_t^j u(t+0,x) = g_j(x) \qquad (0 \le j \le m_+ - 1), \end{cases}$$

$$(8)\qquad \mathrm{supp}(u) \subset \{(x;\sqrt{-1}\eta dx) \in \overset{\circ}{T^*_N}Y\ ;\ \psi(0,x,\eta) \ge 0\},$$

and

$$(9)\qquad \mathrm{supp}(\mathrm{ext}(u)) \subset \{(t,x;\sqrt{-1}(\tau dt + \eta dx)) \in \overset{\circ}{T^*_M}X;\ \psi(t,x,\eta) \ge 0\}.$$

One should remark that in Theorem 2, supp(f) and supp(u) take for the supports as sections of $\underset{N|M_+}{\overset{\circ}{\mathcal{C}}}$.

Remark. i) The uniqueness in the theorems above is equivalent to the results of J. Sjöstrand [Sj 1]. However, our class of microhyperbolic operators at the boundary is a little wider than that of [Sj 1]. For example, one consider the Dirichlet problem

$$\begin{cases} P(t,x,D_t,D_x)u = \{D_t^2 - (\sqrt{-1}t + x_1)D_{x_2}^2\}u = 0 \\ \qquad u(t+0,x) = 0 \end{cases}$$

with $\psi(t,x,\eta) = \eta$, at $\rho_1 = (\overset{\circ}{x} = 0, \overset{\circ}{\eta} = (0,1))$. Then it satisfies the conditions in Theorem 1. But [Sj 1] is not applicable for the problem. Recall that our conditions on microhyperbolicity concerns only with the subset $\{t \geq 0\}$.

4. Announcement of local theorems.

We can deduce at the same time theorems about hyperfunction solutions.

Theorem 3.— *We set $\overset{\circ}{x}{}^* = (1,0,\dots,0) \in \mathbf{R}^n$ and suppose the following conditions (H1) and (H2) :*

(H1) $\sigma_m(P)$ is hyperbolic with respect to x_1, i.e.

$$\sigma_m(P)(t,x;\sqrt{-1}\tau,\ \sqrt{-1}\eta + \overset{\circ}{x}{}^*) \neq 0$$

on $\{0 \leq t < \delta, \mid x - \overset{\circ}{x} \mid < \delta,\ \tau \in \mathbf{R},\ \eta \in \mathbf{R}^n\}$ for some $\delta > 0$ small enough,

(H2) the equation

$$H(\theta) = \sigma_m(P)(0,\overset{\circ}{x},\theta,\overset{\circ}{x}{}^*) = 0$$

has $m - m_+$ positive roots and m_+ negative roots. Let u be a hyperfunction solution to the Dirichlet problems

$$(10) \qquad \begin{cases} \qquad Pu(t,x) = 0 \\ D_t^j u(t+0,x) = 0 \quad (0 \leq j \leq m_+ - 1) \end{cases}$$

defined on

$$\{(t,x) \in M;\ 0 < t < \delta,\ \mid x - \overset{\circ}{x} \mid < \delta,\ ,x_1 < \ \overset{\circ}{x}_1\}.$$

Then u extends uniquely to

$$\{0 < t < \delta',\ \mid x - \overset{\circ}{x} \mid < \delta\}$$

as a hyperfunction solution to (10) for some $\delta' > 0$ small enough.

For the existence, we give

Theorem 4.— *We assume the same conditions as in Theorem 3. Let $f(t,x)$ be a hyperfunction defined on $\{(t,x) \in M; 0 < t < \delta, \mid x - \overset{\circ}{x} \mid < \delta\}$, and for $0 \leq j \leq m_+ - 1$, $g_j(x)$ a hyperfunction defined on $\{x \in N; \mid x - \overset{\circ}{x} \mid < \delta\}$. We assume that $f(t,x)$ is mild on $\{t = 0\}$ and that*

$$\mathrm{supp}(f) \subset \{x_1 \geq \overset{\circ}{x}_1\}, \ \mathrm{supp}(g_j) \subset \{x_1 \geq \overset{\circ}{x}_1\}(0 \leq j \leq m_+ - 1).$$

Then there exists a hyperfunction solution $u(t,x)$ to the problem

$$(11) \qquad \begin{cases} Pu(t,x) = f(t,x) \\ D_t^j u(t+0,x) = g_j(x) \quad (0 \leq j \leq m_+ - 1) \\ \mathrm{supp}(u) \subset \{x_1 \geq \overset{\circ}{x}_1\} \end{cases}$$

on

$$\{0 < t < \delta', \mid x - \overset{\circ}{x} \mid < \delta'\} \quad \text{for some} \quad \delta' > 0.$$

Remark. Recently, Kajitani - Wakabayaski obtained theorems analous to Theorem 3 and Theorem 4 in Gevrey classes. See [K - W 2]. Of course, there should be many results to quote here about hyperbolic mixed problems. Refer to the references of [K - W 2].

5. Microlocal Analysis of sheaves due to Kashiwara - Schapira.

5.1 We recall several notions concerning Microlocal Analysis of sheaves due to Kashiwara-Schapira [K-S 2]. Let X be a C^∞ manifold, and N a closed submanifold of X. Then $D(X)$ denotes the derived category of the category of sheaves of abelian groups on X, and $D^b(X)$ the full sub-category of $D(X)$ consisting of complexes with bounded cohomologies. For $F \in 0b(D^b(x))$, $SS(F)$ express the microsupport of F, which is a closed conic subset in T^*X. Explicitly, for $p_0 \in T^*X, p_0 \notin SS(F)$ if and only if the following condition is satisfied :

(MS) There exists an open neighborhood U of p_0 such that for any $x_1 \in X$ and any real C^∞ function φ defined in an open neighborhood of x_1 with $\varphi(x_1) = 0$, $d\varphi(x_1) \in U$, we have

$$\mathbf{R}\Gamma_{\{\varphi(x) \geq 0\}}(F)_{x_1} = 0.$$

5.2 Consider a complex manifold X and a coherent $\mathcal{D}_X$ module $\mathcal{M}$. If we put $F = \mathbf{R}\underset{\mathcal{D}_X}{\mathrm{Hom}}(\mathcal{M}, \mathcal{O}_X)$, we have

$$(12) \qquad SS(F) = ch(\mathcal{M}).$$

Here $ch(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. This identity is a corollary of the theorem of Cauchy-Kowalevsky. See theorem 10.1.1 of [K-S2] for details about (12).

5.3 We treat again the general case in 5.1. For $F \in Ob(D^+(X))$, $\mu_N(F)$ denotes the microlocalization of F along N, which is an object of $D^b(T_N^*X)$. Then we have

$$(13) \qquad SS(\mu_N(F)) \subset C_{T_N^*X}(SS(F)).$$

Here $C_{T_N^*X}(.)$ is the normal cone along T_N^*X, which is a closed conic set of $T_{T_N^*X}T^*X$ and is identified with a subset in $T^*T_N^*X$ through the Hamiltonian isomorphism multiplied by (-1) :

$$(-H) : TT^*X \xrightarrow{\sim} T^*T^*X.$$

5.4 Next we take a real analytic manifold M with a complexification X. Then we have

$$\mathcal{C}_M = \mu_M(\mathcal{O}_X)[\dim M] \otimes or_M$$

where $\mathcal{C}_M$ is the sheaf of Sato's microfunctions, and or_M denotes the orientation sheaf on M. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$ module on X, we have

$$\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) = \mu_M(\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_x))[\dim M] \otimes or_M.$$

Thus one can deduce by (12) and (13)

$$SS(\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)) \subset C_{T_M^*X}(ch(\mathcal{M})),$$

from which we can show the results for microhyperbolic systems due to [K-S1]. See paragraph 10.5 of [K-S2] for further details.

6. The idea of proof.

6.1 First of all, we remark that the boundary value problems

$$\begin{cases} P(t,x)u(t,x) = f(t,x) & (t > 0) \\ D_t^j u(t+0,x) = g_j(x) & (0 \leq j \leq m_+ - 1) \end{cases}$$

can be reduced to the interior problem pf the form

$$\begin{cases} t^{m-m_+}P(t,x,D_t,D_x)\tilde{u}(t,x) = \tilde{f}(t,x) \\ \operatorname{supp}(\tilde{u}) \subset \{t \geq 0\}. \end{cases}$$

By this reason, we set $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X t^{m-m_+}P$ where $\mathcal{D}_X$ is the sheaf of differentiel operators on X.

6.2 We first study in the complex domain, more precisely in the partially complex domain : we put

$$\tilde{M} = X \cap (\mathbf{R}_t^n \times \mathbf{C}_z^n)$$

in X and

$$\tilde{M}_+ = \{(t,z) \in \tilde{M}\,;\, t \geq 0\}.$$

Then $\mathcal{BO}$ expresses the sheaf of hyperfunctions with holomorphic parameters z :

$$\mathcal{BO} = \mathcal{H}^1_{\tilde{M}}(\mathcal{O}_x).$$

6.3 To show microlocal theorems, we estimate the microsupport of $\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tilde{M}_+}(\mathcal{BO}))$ by using the microlocal theory of elliptic boundary value problems due to [Ka 1, 2] and [Sc 2, 3]. For $SS(\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tilde{M}_+}(\mathcal{BO})))$, we give

Theorem 5.— *Take a point* $\rho_0 = (\overset{\circ}{t}, \overset{\circ}{z}; \overset{\circ}{\tau} dt + \Re(\overset{\circ}{\zeta}.dz)) \in \overset{\circ}{T^*} \tilde{M}$ *with* $\overset{\circ}{\zeta} \neq 0$ *and suppose the following condition (A 1) :*
A.1 the equation

$$H(\theta) = \sigma_m(P)(\overset{\circ}{t}, \overset{\circ}{z}; \overset{\circ}{\tau} + \theta, \overset{\circ}{\zeta}) = 0$$

has $m - m_+$ *roots on* $\{\Re\theta > 0\}$ *and* m_+ *roots on* $\{\Re\theta < 0\}$. *Then we have*

$$\rho_0 \notin SS(\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tilde{M}_+}(\mathcal{BO})).$$

We shall give the spirit of the proof of Theorem 5 in the following section 7 and admit it for the moment.

6.3. Next we apply the formula (13) and obtain an estimate of $SS[\mathbf{R}\operatorname{Hom}(\mathcal{M}, \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO})))]$. In order to be precise, we take a coordinate of $T^*_M\tilde{M}$ as $(t, x; \eta dx)$ and that of $T^*T^*_M\tilde{M}$ as $(t, x, \eta\,;\, t^*dt + x^*dx + \eta^*d\eta)$. Then by Theorem 5 and the formula (13), one can show.

Proposition 6.— *If a point* $\rho_1 = (0, \overset{\circ}{x}, \overset{\circ}{\eta}; \overset{\circ}{t^*}dt + \overset{\circ}{x^*}dx + \overset{\circ}{\eta^*}d\eta) \in T^*T^*_M\tilde{M}$ *satisfies the conditions (S 1) and (S 2) in Theorem 1, then we have*

$$\rho_1 \notin SS[\mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO})))].$$

The theorem in paragraph 3 follows from Proposition 6 since $\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n]$ is almost the same as the sheaf of mild microfunctions. We will sketch it in several lines. We identify

$$T^*_M\tilde{M} \simeq]-T, T[\times T^*_N Y\ ,$$

and $T^*_N Y$ with $\{t = 0\} \times T^*_N Y$. One can construct the sheaf $\overset{\circ}{\mathcal{C}}_{\mathbf{R}_+\times N}$ on $T^*_M\tilde{M}$ which satisfies

$$\overset{\circ}{\mathcal{C}}_{\mathbf{R}_+\times N} = \begin{cases} 0 & \text{on } \{t<0\} \\ \overset{\circ}{\mathcal{C}}_{N|M_+} & \text{on } \{t=0\} \\ \iota!(\mathcal{C}_M \mid \overset{\circ}{T^*_M} X \setminus \overset{\circ}{T^*_M} \tilde{M}) & \text{on } \{t>0\}. \end{cases}$$

Here $\iota : \overset{\circ}{T^*_M} X \setminus \overset{\circ}{T^*_M} \tilde{M} \longrightarrow \overset{\circ}{T^*_M} \tilde{M}$ a natural projection. Moreover one can associate $\overset{\circ}{\mathcal{C}}_{\mathbf{R}_+\times N}$ with $\mu_M(\Gamma_{\tilde{M}_1}(\mathcal{BO}))[n]$ in the following.

Proposition 7.—
i) $\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n]$ *is concentrated in degree* 0.
ii) There exist two morphisms

$$\alpha : \overset{\circ}{\mathcal{C}}_{\mathbf{R}_+\times N} \overset{Y(t)\cdot}{\longrightarrow} \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n]$$

and

$$\beta : \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n] \longrightarrow \hat{\iota}_*(\mathcal{C}_{M_+\setminus X}\ |_{T^*_{M_+}X \setminus T^*_{\tilde{M}_+}X})$$

which satisfy for any section f of $\overset{\circ}{\mathcal{C}}_{\mathbf{R}_+\times N}$

$$\beta\circ\alpha=\begin{cases} f & \text{on } \{t>0\}\\ ext(f) & \text{on } \{t=0\}.\end{cases}$$

Here one set $\hat{\imath}: T^*_{M_*}X\setminus T^*_{\tilde{M}_+}X\longrightarrow T^*_M\tilde{M}$.

By proposition 6 and proposition 7, one can immediately prove Theorem 1 and Theorem 2. Before ending this section, we remark that in a sense Theorem 5 corresponds to the work of A. Martinez [M].

7. Spirit of the proof of Theorem 5.

We take a coordinate of $\tilde{M}$ as (t,z) with $z=x+\sqrt{-1}y$ and set $y'=(y_2,\dots,y_n)$. We take a C^ω function $\varphi(t,x,y')$ defined in a neighborhood of $(0,\overset{\circ}{z}=\overset{\circ}{x}+\sqrt{-1}\overset{\circ}{y})$ with $\varphi(0,\overset{\circ}{x},\overset{\circ}{y}{}')=0$ and put

$$F=\{(t,z)\in\tilde{M};\ -y_1+\varphi(t,x,y')\geq 0\},$$

$$\Omega=\tilde{M}\setminus F,\ L=\bar{\Omega}\setminus\Omega,$$

and $j:\Omega\hookrightarrow\tilde{M}$. We assume that $\bar{\Omega}$ is complex and $q_0=d\varphi-dy_1$ is sufficiently near ρ_0 in Theorem 5. Then it suffices to show that

$$\mathbf{R}\Gamma_F(\mathbf{R}\underset{\mathcal{D}_X}{\mathcal{H}om}(\mathcal{M},\Gamma_{\tilde{M}_+}(\mathcal{B}_{\mathcal{O}})))\mid_{(\overset{\circ}{t},\overset{\circ}{z})}=0. \tag{14}$$

To see this, we will reduce the problem to that on L. We remark that $\mathcal{BO}$ can be identified with

$$\{u\in\mathcal{B}_{\tilde{M}};\ \bar{\partial}_z u=0\}.$$

By this identification and the results about non-characteristic boundary value problems due to [Sc 1] and [Ko - Ka], construct the injective morphism

$$bv: j_*j^{-1}\mathcal{BO}\mid_L\longrightarrow\underset{\mathcal{D}_Z}{\mathcal{H}om}(\mathcal{N}_Z,\beta_L).$$

Here Z denotes a complexification of L and $\mathcal{N}_Z$ denotes the tangential system for $\bar{\partial}_Z u=0$ on L which is a coherent $\mathcal{D}_Z$ module. On the other hand, we have

$$\mathbf{R}j_*j^{-1}\mathcal{BO}=j_*j^{-1}\mathcal{BO}.$$

and the distinguished triangle

$$\to\mathbf{R}\underset{\mathcal{D}_X}{\mathcal{H}om}(\mathcal{M},\mathbf{R}\Gamma_F(\Gamma_{\tilde{M}_+}(\mathcal{BO})))\mid_L\to\mathbf{R}\underset{\mathcal{D}_X}{\mathcal{H}om}(\mathcal{M},\Gamma_{\tilde{M}_+}(\mathcal{BO}))\mid_L\to\mathbf{R}\underset{\mathcal{D}_X}{\mathcal{H}om}(\mathcal{M},j_*j^{-1}\mathcal{BO})\mid_L^{+1}\to$$

Next we put

$$\mathcal{J}_0=bv(\Gamma_{\tilde{M}_+}(\mathcal{BO})\mid_L)\quad\text{and}\quad\mathcal{J}=bv(j_*j^{-1}\Gamma_{\tilde{M}_+}(\mathcal{BO})\mid_L).$$

Then the above triangle implies that it suffices to show

$$\mathcal{J}/\mathcal{J}_0 \xrightarrow{t^{m-m}+\tilde{P}(t,x,y',D_t,D_x,D_{y'})} \mathcal{J}/\mathcal{J}_0 \tag{15}$$

is isomorphic since bv is a injective sheaf homomorphism. Here $\tilde{P}$ is the tangential operator of P on L. But we can show this by using results of [Sc 2] and [Ka 1] concerning elliptic boundary value problems. Moreover we have to characterize $\mathcal{J}$ and $\mathcal{J}_0$ by microlocal conditions in order to apply [Sc 1] and [Ka 1]. Precisely we take a coordinate of L as (t,x,y') and T_L^*Z as $(t,x,y';\ A(\tau dt+\xi dx+\eta' dy))$ and put

$$\Sigma = \{(t,x,y';\sqrt{-1}(\tau dt+\xi dx+\eta' dy)) \in T_L^*Z;\ \xi=0,\ \eta=0\},$$

which is a regular involutory submanifold of T_L^*Z. By this coordinate, we can show that

$$\mathcal{J} = \{u \in \Gamma_{\{t\geq 0\}}(\mathcal{B}_L);\ u \quad \text{satisfies} \quad \mathcal{N}_Z,\ (\mu),\ (2-\mu)\}$$

and

$$\mathcal{J}_0 = \{u \in \Gamma_{\{t\geq 0\}}(\mathcal{B}_L);\ u \quad \text{satisfies} \quad \mathcal{N}_Z,\ SS(u) \subset \{(t,x,y';\ \tau dt)\}, SS_\Sigma^2(u)=\phi\}$$

where the conditions (μ) and $(2-\mu)$ are given by

$$(\mu) \qquad SS(\mu) \subset \left\{(t,x,y';\tau,\xi,\eta');\xi,>0,\right.$$

$$\left.\xi_j = \frac{-(\partial\varphi/\alpha y_j)+(\partial\varphi/\partial x_1)(\partial\varphi/\partial x_j)}{1+(\partial\varphi/\partial x_j)^2}\xi_1\ (2\leq j\leq n)\right\},$$

and

$$(2\mu)\quad SS_\Sigma^2(u) \subset \left\{(t,x,y';\sqrt{-1}\tau dt;\sqrt{-1}\lambda dx_1 + \sum_{j=2}^{n}\frac{-(\partial\varphi/\partial y_j)+(\partial\varphi/\partial x_1)(\partial\varphi/\partial x_j)}{1+(\partial\varphi/\partial x_1)^2}dx_j\right.$$

$$\left.+\sum_{j=2}^{n}\frac{-(\partial\varphi/\partial x_j)+(\partial\varphi/\partial x_1)(\partial\varphi/\partial\varphi_j)}{1+(\partial\varphi/\partial x_1)^2}dy_j;\ \lambda>0\right\}.$$

Here $SS_\Sigma^2(u)$ is the 2-singular spectrum of u along Σ and we take a partial complexification $\tilde{\Sigma}$ of Σ and a coordinate of $T_\Sigma^*\tilde{\Sigma}$ as $(t,x,y';\sqrt{-1}\tau dt;\sqrt{-1}(x^*dx+y'^*dy'))$ with $x^* \in \mathbf{R}^n$ and $y'^* \in \mathbf{R}^{n-2}$. Refer to Kashiwara-Laurent [K-L] for $SS_\Sigma^2(u)$. We end here this note by remarking that we utilize a result in [To] to characterize $\mathcal{J}_0$ and $\mathcal{J}$ (See also [To-u]).

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