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REGULARITY OF THE $\overline{\partial}$-NEUMANN PROBLEM

by

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Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a smooth boundary $b\,\Omega$. If $z_1,\dots,z_n$ are coordinates in $\mathbb{C}^n$ we set $x_j = \mathrm{Re}(z_j), y_i = \mathrm{Im}(z_j)$ and, when u is a differentiable function, we define :

$$(1)\qquad \begin{aligned} u_{z_j} &= \frac{1}{2}\,(u_{x_j} - \sqrt{-1}\,u_{y_j}) \\ u_{\bar{z}_j} &= \frac{1}{2}\,(u_{x_j} + \sqrt{-1}\,u_{y_j})\ . \end{aligned}$$

Given $\alpha_1,\dots,\alpha_n$ on Ω we will study the system

$$(2)\qquad u_{\bar{z}_j} = \alpha_j \text{ for } j = 1,\dots,n.$$

We will assume that the α_j satisfy the compatibility conditions

$$(3)\qquad \alpha_{j\bar{z}_k} = \alpha_{k\bar{z}_j}\ .$$

We shall also assume that $b\,\Omega$ is pseudo-convex, that is if r is a real C^∞ function defined in a neighborhood of $b\,\Omega$ such that $dr \neq 0$, $r < 0$ in Ω and $r > 0$ outside of $\overline{\Omega}$, if $P \in b\,\Omega$ and if $(\zeta_1,\dots,\zeta_n) \in \mathbb{C}^n$ satisfy

$$(4)\qquad \sum r_{z_i}(P)\,\zeta_i = 0$$

then we have

$$(5)\qquad \sum r_{z_i\bar{z}_j}(P)\,\zeta_i\,\bar{\zeta}_j \geq 0.$$

Definition : If $u \in L_2(\Omega)$ we define the singular support of u, denoted by s.s.(u), to be the subset of $\overline{\Omega}$ defined as follows. The point $P \notin$ s.s.(u) if and only if there exists a neighborhood U of P such that the restriction of u to $U \cap \overline{\Omega}$ is in $C^\infty(U \cap \overline{\Omega})$.

This lecture will be concerned with the following two problems.

Problem I. : Given $\alpha_j \in L_2(\Omega)$, $j = 1,\dots,n$ satisfying (3) does

there exist a $u \in L_2(\Omega)$ such that

$$\text{(5)} \qquad s.s.(u) \subset \bigcup_j s.s.(\alpha_j) \quad ?$$

Problem II. : Let $\mathcal{H}(\Omega)$ denote the subspace of $L_2(\Omega)$ consisting of holomorphic functions. Let u be the unique solution of (2) such that $u \perp \mathcal{H}(\Omega)$. Does u satisfy (5) ?

Observe that the system (2) is elliptic thus $\Omega \cap s.s.(u)$ is contained in the singular support of the α_j. Hence problem I is only interesting for points in $b\,\Omega$. It is easy to construct a holomorphic function h which is singular at $b\,\Omega$, then if u satisfies (2) so does u + h, so that (5) cannot hold for all solutions of (2). In case the right side of (5) is empty we have the following result.

Theorem : If Ω is pseudo-convex and if $\alpha_j \in C^\infty(\overline{\Omega})$, $j = 1,\ldots,n$ satisfying (3), then there exists $u \in C^\infty(\overline{\Omega})$ which satisfies (2).

It is at present unknown whether, under the hypotheses of the above theorem, the solution of (2) which is orthogonal to $\mathcal{H}(\Omega)$ is also in $C^\infty(\overline{\Omega})$. One of the reasons for the solution orthogonal to $\mathcal{H}(\Omega)$ is because the Bergman projection $B : L_2(\Omega) \longrightarrow \mathcal{H}(\Omega)$ can be expressed in terms of it. Namely, if $f \in L_2(\Omega)$, if the $f_{z_j} \in L_2(\Omega)$ and if $u \perp \mathcal{H}(\Omega)$ satisfies $u_{z_j} = f_{z_j}$ for $j = 1,\ldots,n$ then clearly Bf, the orthogonal projection of f in $\mathcal{H}(\Omega)$, is given by :

$$\text{(6)} \qquad Bf = f - u.$$

This (5) implies $s.s.B(f) \subset s.s.(f)$. By a result of Bell and Ligocka regularity properties of B imply boundary regularity of biholomorphic maps.

It is known that the solution of Problem I depends on the geometry of Ω. The following results of Catlin show this .

Theorem (Catlin) : If Ω is pseudo-convex and if V is a complex-analytic curve contained in $b\,\Omega$ then there exists α_j satisfying (3) such that (5) is not satisfied for any solution of (2).

Theorem (Catlin) : There exists a pseudo-convex domain $\Omega \subset \mathbb{C}^3$ such that b Ω does not contain any complex curves and such that there exist α_j as above, so that (5) does not hold.

The $\overline{\partial}$-Neumann problem enables one to express the solution of II in terms of a boundary value problem. In particular we have the notion of subellipticity of this problem defined as follows.

Definition : The $\overline{\partial}$-Neumann problem on Ω is subelliptic at $x_o \in$ b Ω if there exists a neighborhood U of x_o and constants $\varepsilon > 0$, $c > 0$ such that for all n-tuples $(\varphi_1,\ldots,\varphi_n)$ with $\varphi_j \in C_o^\infty (U \cap \overline{\Omega})$ and

$$\sum r_{z_j} \varphi_j = 0 \text{ on b } \Omega \tag{7}$$

we have

$$\sum \|\varphi_j\|_\varepsilon^2 \leq C\Big(\sum_1^n \|\varphi_{jz_k} - \varphi_{kz_j}\|^2 + \Big\|\sum_1^n \varphi_j\, z_j\Big\|^2\Big) , \tag{8}$$

where $\| \|_\varepsilon$ denotes the ε-Sobolev-norm.

The quadratic from defined by the left side of (8) is connected with the solution u of II as follows. If $(\alpha_1,\ldots,\alpha_n)$ satisfies (3) and $(\varphi_1,\ldots,\varphi_n)$ with $\varphi_j \in C^\infty(\overline{\Omega})$ satisfies (7) and

$$\sum_{j,k} (\varphi_{j\bar{z}_k} - \varphi_{k\bar{z}_j}, \psi_{j\bar{z}_k} - \psi_{k\bar{z}_j}) + (\sum_j \varphi_{jz_j}, \sum_k \psi_{k\bar{z}_k}) = \sum (\alpha_j, \psi_j) , \tag{9}$$

for all $(\psi_1,\ldots,\psi_n)$ with $\psi_j \in C^\infty(\overline{\Omega})$ and $\sum r_{z_j} \psi_j = 0$ on b Ω. Then

$$u = -\sum_j \varphi_{jz_j} \tag{10}$$

is the unique solution of (2) which is orthogonal to $\mathcal{H}(\Omega)$.

Proposition : If the $\overline{\partial}$-Neumann is subelliptic at $x_o \in$ b Ω, if $(\alpha_1,\ldots,\alpha_n)$ satisfies (3) and if $\alpha_j \in H^s_{loc}(x_o)$ then $u \in H^{s+1}_{loc}(x_o)$, where $u \perp \mathcal{H}(\Omega)$ and u satisfies (2). In particular we have

$$U \cap s.s.(u) \subset U \cap \bigcup_j s.s.(\alpha_j). \tag{11}$$

Hence if the $\bar{\partial}$-Neumann problem is subelliptic at all points in b Ω then II is settled affirmatively. From Catlin's result it then follows that subellipticity cannot hold at $x_0 \in$ b Ω if there is a complex-analytic curve through x_0 contained in b Ω. Catlin also obtained a quantitative measure of the dependence of subelliplicity on how close curves through x_0 can be to b Ω.

Theorem (Catlin) : Suppose Ω is pseudo-convex, $x_0 \in$ b Ω and (8) holds for some $\varepsilon > 0$. Suppose further that there is a complex-analytic curve V with $x_0 \in V$, which has the property that there exists a neighborhood U of x_0 and constants $C > 0$, $\eta > 0$ such that

$$|r(z)| \leq C|z - x_0|^{\eta} \tag{12}$$

for all $z \in U \cap V$. Then $\varepsilon \leq \frac{1}{\eta}$.

Definition : Suppose $x_0 \in$ b Ω, let $\mathcal{V}(x_0) = \mathcal{V}$ denote the set of germs of complex analytic curves through x_0. The order of x_0 denoted by $\mathcal{O}(x_0)$ is defined by

$$\mathcal{O}(x_0) = \sup_{V \in \mathcal{V}} \mathcal{O}(x_0, V) , \tag{13}$$

where $\mathcal{O}(x_0, V)$, the order of contact of V to b Ω at x_0, is defined by

$$\mathcal{O}(x_0, V) = \sup\{\eta \mid \exists \text{ a neighborhood U of } x_0 \text{ and } C > 0 \text{ so that (12) holds for all } z \in U \cap V\}. \tag{14}$$

Denoting by $\mathcal{W}$ the set of germs of non-singular complex-analytic curves at x_0 we define reg $\mathcal{O}(x_0)$ the regular order of x_0 by

$$\text{reg } \mathcal{O}(x_0, V) = \sup_{V \in \mathcal{W}} \mathcal{O}(x_0, V). \tag{15}$$

It is clear that for n = 1 we have $\mathcal{O}(x_0) = \text{reg } \mathcal{O}(x_0) = 1$ and that if n > 1 then $\mathcal{O}(x_0) \geq \text{reg } \mathcal{O}(x_0) \geq 2$. Furthermore, it is easy to show that for n = 2 and Ω pseudo-convex we have $\mathcal{O}(x_0) = \text{reg } \mathcal{O}(x_0)$. However for $\Omega \subset \mathbb{C}^3$, we have the following example (studied by Bloom and Graham), let Ω be given by

$$r(z) = \text{Re}(z_3) + |z_1^2 - z_2^3|^2 + \exp(-1/|z|^2) \tag{16}$$

then for $x_o \neq 0$ we have $\mathcal{O}(x_o) = \text{reg } \mathcal{O}(x_o) = 2$ and when $x_o = 0$, we have reg $\mathcal{O}(o) = 6$ and $\mathcal{O}(o) = \infty$.

D'Angelo has shown that for pseudo-convex domains if reg $\mathcal{O}(x_o) \leq 4$ than $\mathcal{O}(x_o) = \text{reg } \mathcal{O}(x_o)$.

Returning to subellipticity for pseudo-convex domains the convex of Catlin's theorem holds if $n \leq 2$ and if $\mathcal{O}(x_o)=2$, that is in those cases when $\mathcal{O}(x_o) < \infty$, then subellipticity holds at x_o for all $\varepsilon \leq \frac{1}{\mathcal{O}(x_o)}$. A recent example of d'Angelo shows that this is not true in general. Let $\Omega \subset \mathbb{C}^n$ for $n \geq 3$ defined by :

$$r(z) = \text{Re}(z_n) + \sum_{j=1}^{n-2} |z_j^m - z_n z_{j+1}|^2 + |z_{n-1}^m|^2 , \tag{17}$$

for this domain $\mathcal{O}(x_o) = 2m$ when $x_o = (0,\ldots,0)$, $\mathcal{O}(x_o) = 2$ when $z_{n-1}(x_o) \neq 0$ and $\mathcal{O}(x_o) = (2m)^n$ when $x_o = (0,\ldots,0,i\delta)$ whenever $\delta \in \mathbb{R} - \{0\}$. Catlin shows that subellipticity at $x_o = (0,\ldots,0)$ holds for $\varepsilon \leq (2m)^{-n}$ but does not hold for $\varepsilon > (2m)^{-n}$. Using the phenomena involved in d'Angelo's example Catlin has constructed for each ε_o, with $0 < \varepsilon_o \leq \frac{1}{4}$, a domain such that subellipticity holds with $\varepsilon = \varepsilon_o$ but does not hold for $\varepsilon > \varepsilon_o$. Furthermore he constructs a domain such that subellipticity holds for all $\varepsilon < \varepsilon_o$ but does not hold for $\varepsilon = \varepsilon_o$.

The major problem now is whether $\mathcal{O}(x_o) < \infty$ implies subellipticity on pseudo-convex domain. Here we will briefly describe the sufficient conditions that are known.

Definition : If $x_o \in b\,\Omega$ we define a succession of ideals $I_1(x_o) \subset I_2(x_o) \subset \ldots \subset I_k(x_o) \subset C^\infty(x_o)$, where $C^\infty(x_o)$ denotes the ring of germs of C^∞ functions at x_o. We define by

$$\lambda = \det \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} \\ r_{\bar{z}_1} & r_{z_1\bar{z}_1} & \cdots & r_{z_n\bar{z}_1} \\ \vdots & \vdots & & \vdots \\ r_{\bar{z}_n} & r_{z_1\bar{z}_n} & \cdots & r_{z_n\bar{z}_n} \end{pmatrix} \tag{18}$$

Let $I_1(x_o) = \sqrt[\mathbb{R}]{(\lambda)}$, where (λ) denotes the ideal generated by λ and $\sqrt[\mathbb{R}]{\ }$ is the real radical defined as follows. If $J \subset C^\infty(x_o)$ is an ideal then $\sqrt[\mathbb{R}]{J} = \{f \in C^\infty(x_o) \mid \exists\ m \text{ and } g \in J \text{ with } |f|^m \leq |g|\}$. To define $I_k(x_o)$ consider for any $f^{(1)},\ldots,f^{(n+1)} \in I_{k-1}(x_o)$ the $(n+1) \times 2(n+1)$ matrix

$$
(19) \qquad \begin{pmatrix}
0 & r_{z_1} & \cdots & r_{z_n} \\
r_{\bar{z}_1} & r_{z_1\bar{z}_1} & \cdots & r_{z_n\bar{z}_1} \\
\vdots & \vdots & & \\
r_{\bar{z}_n} & r_{z_1\bar{z}_n} & \cdots & r_{z_n\bar{z}_n} \\
0 & f^{(1)}_{z_1} & \cdots & f^{(1)}_{z_n} \\
 & & & \\
0 & f^{(n+1)}_{z_1} & \cdots & f^{(n+1)}_{z_n}
\end{pmatrix}
$$

Let $D^k(f^{(1)},\ldots,f^{(n+1)})$ denote the set of all determinants of $(n+1)\times(n+1)$ minors of the above matrix. Then we define $I_k(x_o)$ by

$$
(20) \qquad I_k(x_o) = \sqrt[\mathbb{R}]{(I_{k-1}(x_o), \cup D^k(f^{(1)},\ldots,f^{(n+1)}))} ,
$$

where the union is taken over all $(n+1)$-tuples in $I_{k-1}(x_o)$.

[Theorem : If Ω is pseudo-convex and if $1 \in I_k(x_o)$ the subellipticity holds at x_o.

In case r is real-analytic in a neighborhood of x_o it can be shown (using a result of Diederich and Fornaess) that for pseudo-convex domain the condition $1 \in I_k(x_o)$ for some k is equivalent to the non-existence of complex-analytic curves through x_o which lie in b Ω.

Recent result of d'Angelo give a new analysis of the notion of order of x_o and in many cases give sharp bounds for the $\mathcal{O}(x)$ in terms of $\mathcal{O}(x_o)$ when x is near x_o.

References : The recent results of Catlin and d'Angelo are not yet published. For the other references and a more detailed survey see : "Several complex variables from the point of view of linear partial differential equations" by J.J. Kohn in Proc. Conf. on PDE and Diff. Geom. 1980, Beijing. Acad. Sinica.