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A COERCIVENESS INEQUALITY FOR A CLASS OF NONELLIPTIC OPERATORS AND ITS APPLICATIONS

by C. H. WILCOX

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§ 0 INTRODUCTION.

This exposition describes a coerciveness inequality for a class of nonelliptic operators due to J. R. Schulenberger and C. H. Wilcox [3], [5], its proof and some of its applications.

The nonelliptic operators considered here occur in the study of symmetric hyperbolic systems of the form

$$E(x)\, D_t u + \sum_{j=1}^{n} A_j D_j u = 0 \tag{0.1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}^1$, $D_j = \partial/\partial x_j$, $D_t = \partial/\partial t$, $u = u(x,t)$ is an $m \times 1$ (column) matrix over $\mathbb{C}$, the coefficients $E(x)$, A_1, $A_2, \ldots, A_n$ are $m \times m$ Hermitian matrices over $\mathbb{C}$, $E(x)$ is positive definite and $A_1, \ldots, A_n$ are constant. This class of equation provides a unified description of the wave equations of classical physics. Examples include the elastic waves in a variety of inhomogeneous anisotropic media [7].

The systems (0.1) can be written in the Schrödinger form

$$i D_t u = \Lambda u \tag{0.2}$$

where

$$\Lambda = -i\, E(x)^{-1} \sum_{j=1}^{n} A_j D_j \tag{0.3}$$

is formally selfadjoint with respect to the inner product

$$(u,v) = \int_{\mathbb{R}^n} u(x)^* E(x)\, v(x)\, dx . \tag{0.4}$$

(Here u^* denotes the Hermitian adjoint of u). It will be assumed that $E(x)$ is Lebesgue measurable, bounded and unifomly positive definite on $\mathbb{R}^n$ It follows that

$$\mathcal{H} = \{u : u(x) \text{ is L measurable, } (u,u) < \infty\} \tag{0.5}$$

is a Hilbert space and if

(0.6) $$D(\Lambda) = \mathcal{H} \cap \{u : \Lambda u \in \mathcal{H}\}$$

then $\Lambda : \mathcal{H} \to \mathcal{H}$ is selfadjoint [7]. This implies that

(0.7) $$U(t) = \exp(-it\Lambda) = \int_{\mathbb{R}^1} \exp(-it\lambda)\, d\pi(\lambda)$$

is a solution operator for the Cauchy problem for (0.1).

For the study of the properties of $U(t)$ it is important to know whether Λ is coercive on $D(\Lambda)$. It is well known that if Λ is coercive on $D(\Lambda)$ then Λ must be elliptic ; cf. [3]. This means that if

(0.8) $$\Lambda(p,x) = E(x)^{-1} \sum_{j=1}^{n} A_j p_j$$

denotes the symbol of Λ then

(0.9) $$\operatorname{rank} \Lambda(p,x) = m \text{ for all } p \in \mathbb{R}^n - \{0\} \text{ and } x \in \mathbb{R}^n \quad .$$

Unfortunately, the operators Λ that arise in classical physics are not elliptic [3]. However, most of them have the weaker property

(0.10) $$\operatorname{rank} \Lambda(p,x) = m - k \quad \text{for all } p \in \mathbb{R}^n - \{0\} \text{ and } x \in \mathbb{R}^n ,$$

where k is an integer [3]. Such operators will be said to have <u>constant deficit</u> k. This property repleaces ellipticity in the coerciveness theorem described below.

§ 1. THE COERCIVENESS THEOREM.

Theorem 1.1 : Assume that

(1.1) $E(x)$ and $D_j E(x)$ are continuous and bounded on $\mathbb{R}^n$.

(1.2) $\lim_{|x|\to\infty} E(x) = E_0$ exists, uniformly in $x/|x|$.

(1.3) $\operatorname{rank} \sum_{j=1}^{n} A_j p_j = m-k$ for all $p \in \mathbb{R}^n - \{0\}$.

Then Λ is coercive on $N(\Lambda)^{\perp}$, the orthogonal complement in $\mathcal{H}$ of the nullspace $N(\Lambda)$. This means that

(1.4) $$D(\Lambda) \cap N(\Lambda)^{\perp} \subset \mathcal{L}^1_{\mathcal{H}} \equiv \mathcal{H} \cap \{u : D_j u \in \mathcal{H},\ j=1,\dots,n\}$$

and there exists a constant $c>0$ such that

(1.5) $$\sum_{j=1}^{n} \|D_j u\|^2 \le c^2 \left(\|\Lambda u\|^2 + \|u\|^2\right) \text{ for all } u \in D(\Lambda) \cap N(\Lambda)^{\perp} .$$

Theorem 1.2 : If the hypotheses of theorem 1.1 hold and

(1.6) $D^{\alpha} E(x)$ is continuous and bounded on $\mathbb{R}^n$ for $0 \le |\alpha| \le q$

then

(1.7) $$D(\Lambda^q) \cap N(\Lambda)^{\perp} \subset \mathcal{L}^q_{\mathcal{H}} \equiv \mathcal{H} \cap \{u : D^{\alpha} u \in \mathcal{H} \text{ for } |\alpha| \le q\}$$

and there exists a constant $c_q > 0$ such that

(1.8) $$\sum_{|\alpha| \le q} \|D^{\alpha} u\|^2 \le c_q^2 \left(\|\Lambda^q\|^2 + \|u\|^2\right) \text{ for all } u \in D(\Lambda^q) \cap N(\Lambda)^{\perp} .$$

The proofs of these results are given in [3] and [5]. The idea of these proofs are sketched below.

§ 2. A PROOF OF THE COERCIVENESS THEOREM.

The idea of the proof in [3] is to construct an augmental operator

(2.1) $$\Lambda'' = \begin{pmatrix} \Lambda' \\ \Lambda \end{pmatrix}$$

where

(2.2) $$\Lambda' : \mathcal{K} \to \mathcal{K}$$

is chosen so that

(2.3) $$\Lambda'' \text{ is elliptic },$$

and

(2.4) $$\Lambda' \Lambda = 0 .$$

Condition (2.3) implies that Λ'' is coercive on its domain ; that is

(2.5) $$D(\Lambda'') = D(\Lambda') \cap D(\Lambda) \subset \mathcal{L}^1_{\mathcal{K}}$$

and

(2.6) $$\sum_{j=1}^{n} \|D_j u\|^2 \leq c^2(\|\Lambda'' u\|^2 + \|u\|^2)$$
$$= c^2(\|\Lambda' u\|^2 + \|\Lambda u\|^2 + \|u\|^2) \text{ for all } u \in D(\Lambda'') .$$

Moreover, (2.4) implies

(2.7) $$\overline{R(\Lambda)} \subset N(\Lambda') .$$

Also $N(\Lambda) = R(\Lambda^*)^{\perp} = R(\Lambda)^{\perp}$, whence $R(\Lambda) = N(\Lambda)^{\perp}$. Thus

(2.8) $$N(\Lambda)^{\perp} \subset N(\Lambda') .$$

Combining (2.5), (2.6) and (2.8) gives the coerciveness of Λ on $N(\Lambda)^{\perp}$; i.e. (1.4), (1.5)

The difficulty with the program outlined above is that, in general, there is no first order differential operator Λ' which satisfies (2.3) and (2.4). In [5] Λ' is constructed as follows.

Define

$$\hat{P}_o(p) = -\frac{1}{2\pi i} \int_\Gamma (\Lambda_o(p) - \zeta)^{-1} d\zeta , \quad \Lambda_o(p) = E_o^{-1} \sum_{j=1}^{n} A_j p_j \quad , \tag{2.9}$$

where Γ is a circle about $\zeta = 0$ which contains no nonzero eigenvalue of $\Lambda_o(p)$. The possibility of doing this follows from the constant deficit condition (1.3) ; see [5]. Then

$$\Lambda_o(p)\hat{P}_o(p) = \hat{P}_o(p)\,\Lambda_o(p) = 0 \ . \tag{2.10}$$

Define

$$P_o = \Phi^* \hat{P}_o \Phi \ , \quad \Phi = \text{ Fourier transform} \quad , \tag{2.11}$$

so that

$$P_o \Lambda_o = 0 \ . \tag{2.12}$$

Define

$$\Lambda' = (-\Delta)^{1/2} P_o E_o^{-1} E \tag{2.13}$$

with symbol

$$\Lambda'(p,x) = |p|\,\hat{P}_o(p)\,E_o^{-1}\,E(x) \ . \tag{2.14}$$

In general, Λ' is not a differential operator. However, rank $\Lambda'(p,x) =$ rank $\hat{P}_o(p) = k$ for $p \neq 0$ and rank $\Lambda''(p,x) = m$ for $p \neq 0$, $x \in \mathbb{R}^n$. Moreover. $\Lambda'\Lambda = (-\Delta)^{1/2} P_o \Lambda_o = 0$. The proof of theorem 1.1 in [3] is based on these two properties and follows the general plan of the usual proof of Gårding's inequality.

The proof of theorem 1.2 is based on theorem 1.1 and an induction of q. The details are given in [3, § 6].

L. Sarason [2] has recently given another proof of theorem 1.1 which is shorter and technically easier than the one outlined above. An outline of his proof is given below.

§ 3. A SECOND PROOF DUE TO L. SARASON.

Notation : $u : \mathbb{R}^n \to \mathbb{C}^m$ represents an $m \times 1$ matrix-valued function.

(3.1) $$(u,v)_o = \int_{\mathbb{R}^n} u(x)^* v(x)\, dx \ .$$

(3.2) $$\mathcal{L}_{2,m}(\mathbb{R}^n) = \{u : u(x) \text{ is L measurable}, \ (u,u)_o < \infty\} \ .$$

(3.3) $$\mathcal{L}^q_{2,m}(\mathbb{R}^n) = \mathcal{L}_{2,m}(\mathbb{R}^n) \cap \{u : D^\alpha u \in \mathcal{L}_{2,m}(\mathbb{R}^n), \ 0 \le |\alpha| \le q\} \ .$$

(3.4) $$(u,v)_q = \int_{\mathbb{R}^n} \sum_{|\alpha| \le q} D^\alpha u(x)^* D^\alpha v(x)\, dx \ .$$

(3.5) $$A = -i \sum_{j=1}^{n} A_j D_j$$

(3.6) $$D(A) = \mathcal{L}_{2,m}(\mathbb{R}^n) \cap \{u : A u \in \mathcal{L}_{2,m}(\mathbb{R}^n)\} \ .$$

It follows that $A : \mathcal{L}_{2,m}(\mathbb{R}^n) \to \mathcal{L}_{2,m}(\mathbb{R}^n)$ is selfadjoint.

(3.7) $$N(A) = \mathcal{L}_{2,m}(\mathbb{R}^n) \cap \{u : A u = 0\}$$

(3.8) $$N(A)^\perp = \mathcal{L}_{2,m}(\mathbb{R}^n) \cap \{u : (u,v)_o = 0 \ \forall \ v \in N(A)\}$$

(3.9) $$(E\,N(A))^\perp = \mathcal{L}_{2,m}(\mathbb{R}^n) \cap \{u : (u, E\,v)_o = 0 \ \forall \ v \in N(A)\} \ .$$

It is evident that

(3.10) $$D(\Lambda) = D(A) \quad , \quad N(\Lambda) = N(A), \quad \text{and}$$

(3.11) $$N(\Lambda)^\perp = (E\,N(A)) \quad .$$

It follows from these relations and the equivalence of the norms $\|u\|$ and $\|u\|_o$ that theorem 1.1 is equivalent to

Theorem 3.1 : Under the hypotheses of theorem 1.1

(3.12) $$D(A) \cap (E\,N(A))^\perp \subset \mathcal{L}^1_{2,m}(\mathbb{R}^n)$$

and there exists a constant $c > 0$ suche that

(3.13) $$\|u\|_1 \le c(\|A\,u\|_o + \|u\|_o) \ \text{ for all } u \in D(A) \cap (E\,N(A))^\perp \ .$$

Sarason's proof of theorem 3.1 is based on the following three lemmas.

Lemma 3.2 : rank $A(p) = m - k$ for all $p \in \mathbb{R}^n - \{0\} \Rightarrow A$ is coercive on $N(A)^\perp$; that is

(3.14) $$D(A) \cap N(A)^\perp \subset \mathcal{L}^1_{2,m}(\mathbb{R}^n)$$

and there exists a constant $c > 0$ such that

(3.15) $$\|u\|_1 \le c(\|A u\|_o + \|u\|_o) \quad \text{for all } u \in D(A) \cap N(A)^\perp \quad .$$

This result can be proved easily by Fourier analysis ; see for example [3].

Lemma 3.3 : Let $u \in \mathcal{L}^1_{2,m}(\mathbb{R}^n) \cap (E\,N(A))^\perp$ and let

(3.16) $$u = u_1 + u_2 \quad , \quad u_1 \in N(A)^\perp \quad , \quad u_2 \in N(A) \quad .$$

Then there exists a constant C, depending on $\max_{x \in \mathbb{R}^n} \left(|E(x)| + \sum_{j=1}^{n} |D_j E(x)| \right)$, such that

(3.17) $$\|u_2\|_1 \le C \; \|u_1\|_1 \quad .$$

This result expresses the main idea of Sarason's proof. Let

(3.18) $$j_\varepsilon(x) = \varepsilon^{-n} j(\varepsilon x)$$

be a mollifier in the sense of Friedrichs and

(3.19) $$J_\varepsilon u(x) = \int_{\mathbb{R}^n} j_\varepsilon(x - y)\, u(y)\, dy \quad .$$

Lemma 3.4 : If $u \in D(A) \cap (E\,N(A))^\perp$ and

(3.20) $$u_\varepsilon = E^{-1} J_\varepsilon E u$$

then $u_\varepsilon \in \mathcal{L}^1_{2,m}(\mathbb{R}^n) \cap (E\,N(A))^\perp$ and

(3.21) $$\lim_{\varepsilon \to 0} u_\varepsilon = u \quad \text{and} \quad \lim_{\varepsilon \to 0} A u_\varepsilon = A u \quad \text{in} \quad \mathcal{L}_{2,m}(\mathbb{R}^n) \quad .$$

Proof of theorem 3.1 : Under the hypotheses of lemma 3.3, $A u_1 = A u$ and $\|u_1\|_o \leq \|u\|_o$. Thus lemma 3.2 implies

$$\|u_1\|_1 \leq c(\|A u_1\|_o + \|u_1\|_o) \leq c(\|A u\|_o + \|u\|_o) \quad . \tag{3.22}$$

Combining this with (3.17) gives

$$\begin{aligned} \|u\|_1 &\leq \|u_1\|_1 + \|u_2\|_1 \leq (1 + C)\, \|u_1\|_1 \\ &\leq (1 + C)\, c(\|A u\|_o + \|u\|_o) \quad ; \end{aligned} \tag{3.23}$$

that is, (3.13) for $u \in \mathcal{L}^1_{2,m} \cap (E\, N(A))^{\perp}$. It follows from lemma 3.4 that if $u \in D(A) \cap (E\, N(A))^{\perp}$ then

$$\|u_\varepsilon\|_1 \leq c(\|A u_\varepsilon\|_o + \|u_\varepsilon\|_o) \quad . \tag{3.24}$$

The proof of theorem 3.1 is completed by making $\varepsilon \to 0$ and using (3.21).

§ 4. APPLICATIONS OF THE COERCIVENESS THEOREM.

1°) Regularity theory for the Cauchy problem

$$u(x,t) = \exp(-it\Lambda)\, f(x) = U(t)\, f(x) \tag{4.1}$$

is the solution of the Cauchy problem

$$D_t u = -i \Lambda u \;, \quad u(x,0) = f(x), \quad x \in \mathbb{R}^n \; . \tag{4.2}$$

Problem : Find conditions on f and Λ which guarantee that

$$D_t^{\alpha_o} D_1^{\alpha_1} \ldots D_n^{\alpha_n}\, u(x,t) \tag{4.3}$$

exist (in some sense).

Now

$$\mathcal{H} = N(\Lambda) \oplus N(\Lambda)^{\perp} \quad \text{reduces } \Lambda \; . \tag{4.4}$$

Moreover

$$f \in D(\Lambda^q) \Rightarrow f = f' + f'' \quad \text{where}$$
(4.5)
$$f' \in N(\Lambda) \quad \text{and} \quad f'' \in D(\Lambda^q) \cap N(\Lambda)^{\perp} .$$

Thus

$$u(x,t) = U(t)\, f'(x) + U(t)\, f''(x)$$
(4.6)
$$= u'(x,t) + u''(x,t)$$

where

$$u'(x,t) = f'(x) \quad \text{for all} \quad t \in \mathbb{R}^1, \text{ and}$$
(4.7)
$$u''(x,t) \in D(\Lambda^q) \cap N(\Lambda)^{\perp} \quad \text{for all} \quad t \in \mathbb{R}^1 \quad .$$

Application of theorem 1.2 implies that the derivatives (4.3) exist for $t \in \mathbb{R}^1$ and $\alpha_o + \alpha_1 + \ldots + \alpha_n \le q$. In fact, we can prove

<u>Theorem 4.1</u> : Under the hypotheses of theorem 1.2, if $f \in \mathcal{L}^q_{\mathcal{H}}$ then

(4.8) $$D_t^{\alpha_o} D_1^{\alpha_1} \ldots D_n^{\alpha_n} u(x,t) \in C(\mathbb{R}^1;\mathcal{H}) \quad \text{for } \alpha_o + \alpha_1 + \ldots + \alpha_n \le q \quad .$$

<u>Corollary 4.2</u> : If $q \ge [n/2] + 1 + 1$ then

(4.9) $$u(x,t) \in C^1(\mathbb{R}^{n+1}) \quad .$$

This follows from (4.8) and the Sobolev imbedding theorems.

2°) <u>Other applications</u>

The coerciveness theorem has been used to study the point spectrum and continuous spectrum of Λ [4] and to prove the existence and completeness of the wave operators in scattering theory for Λ [1,6].

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