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THE STRUCTURE OF ω -REGULAR SEMIGROUPS

by Janet AULT and Mario PETRICH

1. - Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and (or) ideals have been imposed. On the one hand, there is the theorem of REES [7], giving the structure of completely 0-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and $\mathfrak{J}$ - and $\mathfrak{J}_1$ -regular semigroups [4]. On the other hand, with very different restrictions, REILLY [8] has determined the structure of bisimple ω -semigroups and, independently of each other, KOCHIN [1] of inverse simple ω -semigroups, and MUNN [5] of inverse ω -semigroups.

An ω -chain with zero is a poset $\{e_i \mid i \geq 0\} \cup 0$, with $e_i > e_j$ if $i < j$, and $0 < e_i$ for all i , j . We call a regular semigroup S ω -regular, if S has a zero, and the poset of its idempotents is an orthogonal sum [2] of ω -chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. - An ω -regular semigroup can be uniquely written as an orthogonal sum of ω -regular prime (i. e., with 0 a prime ideal) semigroups. This reduces the problems of structure and isomorphism to ω -regular prime semigroups. We distinguish three cases :

- (i) 0-simple,
- (ii) Prime with a proper 0-minimal ideal,
- (iii) Prime without a 0-minimal ideal.

Case (i) is the most difficult (and interesting), and includes a variety of special cases some of which reduce to those constructed by REILLY [8], KOCHIN [1], and MUNN [5], [6].

3. - Let A be a nonempty set, d be a positive integer, V be a semigroup which is a chain of d groups $G_0 > G_1 > \dots > G_{d-1}$, and σ be a homomorphism of V into G_0 . Let $w : A \rightarrow \{0, 1, \dots, d-1\}$ be any function, denoted by $w : \alpha \rightarrow w_\alpha$. For $\alpha \in A$, $0 \leq i, j < d$, define $\langle \alpha, i \rangle$ by

$$\langle \alpha, i \rangle \equiv w_\alpha + i \pmod{d}, \quad 0 \leq \langle \alpha, i \rangle < d,$$

and define $[i, \alpha, j]$ to satisfy

$$[i, \alpha, j]d = (i - j) - (\langle \alpha, i \rangle - \langle \alpha, j \rangle) .$$

Construction 1. - On the set

$$S = \{(\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, m, n \geq 0, g \in V\} \cup 0 ,$$

define a multiplication by, for $g_i \in G_i, g_j \in G_j, v = n - s - [i, \beta, j]$,

$$(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma)$$

$$= \begin{cases} (\alpha, m - [i, \alpha, j] - v, (g_i \sigma^{-v})g_j, t, \gamma), & \text{if } v < 0, \\ & \text{or } v = 0, i \leq j, \\ (\alpha, m, g_i(g_j \sigma^v), t + [i, \gamma, j] + v, \gamma), & \text{if } v > 0, \\ & \text{or } v = 0, i > j, \end{cases}$$

and all other products are equal to 0. The set S , with this multiplication, will be denoted by $\mathcal{O}(A, w; V, \sigma)$.

Construction 2. - On the set

$$S' = \{(\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, m - w_\alpha \equiv n - w_\beta \equiv i \pmod{d}, g \in G_i\} \cup 0 ,$$

define a multiplication by, for $g_i \in G_i, g_j \in G_j, v = n' - s' - [i, \beta, j]$, where $n = n'd + n'', s = s'd + s'', 0 \leq n'', s'' < d$,

$$(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma)$$

$$= \begin{cases} (\alpha, m + s - n, (g_i \sigma^{-v})g_j, t, \gamma), & \text{if } n \leq s, \\ (\alpha, m, g_i(g_j \sigma^v), t + n - s, \gamma), & \text{if } n > s, \end{cases}$$

and all other products are equal to 0. The set S' , with this multiplication, will be denoted by $\mathcal{O}[A, w; V, \sigma]$.

The following is our fundamental result.

THEOREM 1. - For a groupoid S , the following statements are equivalent:

- (i) S is a 0-simple w -regular semigroup;
- (ii) S is isomorphic to $\mathcal{O}(A, w; V, \sigma)$;
- (iii) S is isomorphic to $\mathcal{O}[A, w; V, \sigma]$.

The proof of "(i) $\implies$ (ii)" consists of "introducing coordinates" into various $\mathcal{P}$ - and $\mathcal{R}$ -classes, and of constructing the homomorphism σ ; it is quite long, and

is broken into a sequence of lemmas. For "(ii) $\Rightarrow$ (iii)", one finds a suitable isomorphism, while "(iii) $\Rightarrow$ (i)" consists of a verification of the defining properties of a 0-simple ω -regular semigroup.

Define the top of S in the theorem by $\mathfrak{I}(S) = \{a \in S \mid e \mathcal{L} a, a \mathcal{R} f \text{ for some maximal idempotents } e, f\} \cup 0$. Then $\mathfrak{I}(S)$ is a primitive inverse semigroup. It follows from the proof that we can always suppose that $w_\alpha = 0$ for some $\alpha \in A$. Call S balanced, if any two maximal idempotents of S are 0-equivalent.

THEOREM 2. - The following conditions on a 0-simple ω -regular semigroup S are equivalent :

- (i) S is balanced ;
- (ii) S admits a representation as in theorem 1, with $w_\alpha = 0$ for all $\alpha \in A$;
- (iii) $\mathfrak{I}(S)$ is a Brandt semigroup ;
- (iv) S is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(K ; A, A ; \Delta)$ over a simple inverse ω -semigroup K , Δ is the identity matrix.

The structure of the semigroup K in theorem 2 was determined by KOCHIN [1] and MUNN [5], the Rees matrix semigroups over bisimple inverse semigroups were studied in [3] (for the 0-simple case in the theorem, cf. [3], cor. 5.7, and [6], th. 4.2). Various other special cases include : 0-bisimple, combinatorial, balanced, and combinations thereof.

4. - For the remaining cases, we will need the following.

Construction 3. - Let Y be a tree semilattice satisfying one of the two conditions :

- (1) Y has a zero ζ , and all elements of Y are of finite height ;
- (2) Y has no zero, and is of locally finite length.

To every non-zero element α of Y , associate a Brandt semigroup S_α , suppose that the family $\{S_\alpha\}$ is pairwise disjoint, and that a homomorphism $\varphi_\alpha : S_\alpha \rightarrow S_{\bar{\alpha}}$ is given, where $\bar{\alpha}$ is the unique element of Y covered by α , with the properties :

- (i) $S_\alpha \varphi_\alpha \cap S_\beta \varphi_\beta = 0$, if $\bar{\alpha} \neq \bar{\beta}$;
 - (ii) For every infinite ascending chain $\alpha_1 < \alpha_2 < \dots$ in Y , and every $a \in S_{\alpha_1}$, there exists α_k such that $a \notin S_{\alpha_k} \varphi_{\alpha_k} \varphi_{\alpha_{k-1}} \dots \varphi_{\alpha_2}$.
- Let $\psi_{\alpha, \alpha}$ be the identity mapping on S_α , and for $\alpha > \beta$, let
- $$\psi_{\alpha, \beta} = \varphi_\alpha \varphi_{\alpha_1} \dots \varphi_{\alpha_n}, \quad \text{where } \alpha > \alpha_1 > \dots > \alpha_n > \beta.$$

Let

$$S = \left(\bigcup_{\alpha \in Y \setminus \zeta} (S_\alpha \setminus 0_\alpha) \right) \cup 0, \quad ,$$

where ζ is the zero of Y (if Y has one), and 0 is an element not contained in any S_α ; and on S define the multiplication $\star$ by

$$a \star b = (a\psi_{\alpha, \alpha\beta})(b\psi_{\beta, \alpha\beta}), \quad \text{if } \alpha\beta \neq \zeta \text{ and } (a\psi_{\alpha, \alpha\beta})(b\psi_{\beta, \alpha\beta}) \neq 0_{\alpha\beta} \text{ in } S_{\alpha\beta},$$

and all other products are equal to 0 . The set S , with this multiplication, will be called a Brandt tree, if Y has a zero and a rooted Brandt tree otherwise.

THEOREM 3. - A semigroup S is prime ω -regular and has a proper 0 -minimal ideal if, and only if, S is an ideal extension of a 0 -simple ω -regular semigroup I by a Brandt tree T determined by a 0 -restricted homomorphism of T into the top of I .

Such a homomorphism is completely determined by its restriction to the socle $\mathcal{S}(T)$ of T , so all such homomorphisms are given by 0 -restricted homomorphisms of $\mathcal{S}(T)$ into $\mathcal{I}(I)$, both of which are primitive inverse semigroups, and are easy to find explicitly.

THEOREM 4. - A groupoid S is a prime ω -regular semigroup without 0 -minimal ideals if, and only if, S is a rooted Brandt tree.

5. - The semigroups $\mathcal{O}(A, w; V, \sigma)$ and $\mathcal{O}[A, w; V, \sigma]$ do not seem to admit a neat isomorphism theorem, except in special cases. In the balanced case, using theorem 2, ([3], 4.1) and ([1], theor.4), we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in construction 3 are similar to those in [4], théorème 3.1, while isomorphisms of the semigroups in theorem 3 can be expressed by isomorphisms of I and T satisfying a commutative diagram.

REFERENCES

- [1] KOCHIN (B. P.). - The structure of inverse ideally-simple ω -semigroups [in Russian], Vestnik Leningr. Univ., t. 7, 1968, p. 41-50.
- [2] LALLEMENT (G.) and PETRICH (M.). - Décompositions I -matricielles d'un demi-groupe, J. Math. pure et appl., 9e série, t. 45, 1966, p. 67-117.
- [3] LALLEMENT (G.). - A generalization of the Rees theorem in semigroups, Acta Sc. Math. Szeged, t. 30, 1969, p. 113-132.
- [4] LALLEMENT (G.). - Structure d'une classe de demi-groupes réguliers, J. Math. pure et appl., 9e série, t. 48, 1969, p. 345-397.

- [5] MUNN (W. D.). - Regular ω -semigroups, Glasgow math. J., t. 9, 1968, p. 46-66.
- [6] MUNN (W. D.). - 0-bisimple inverse semigroups, J. of algebra (to appear).
- [7] REES (D.). - On semigroups, Proc. Cambridge phil. Soc., t. 36, 1940, p. 387-400.
- [8] REILLY (N. R.). - Bisimple ω -semigroups, Proc. Glasgow math. Assoc., t. 7, 1966, p. 160-167.

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