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# ON THE APPROXIMABILITY OF THE LOGARITHMS OF ALGEBRAIC NUMBERS

by Pieter L. CIJSOUW

## 1. Introduction.

Let $\alpha$ be a non-zero algebraic number, and let $\log \alpha$ be any (fixed) branch of the logarithm of $\alpha$, with $\log \alpha \neq 0$. It is well known that then $\log \alpha$ is transcendental. Let $\xi$ denote any algebraic number, and denote the degree and the height of $\xi$ by N and H, respectively (we use the height of an algebraic number in the sense of the maximal absolute value of the coefficients of its minimal defining polynomial). Then clearly $|\log \alpha - \xi| \neq 0$. We will look for positive lower bounds for $|\log \alpha - \xi|$, expressed in N and H.

In the literature, such lower bounds have been given from two different points of view. At first, there exist bounds which are sharp with respect to H and not necessarily to N ; important estimates of this kind have been proved by K. MAHLER in several papers. Secondly, there are bounds which are good when N and $\log H$ have about the same order of magnitude. For this case, N. I. FEL'DMAN has given the following inequalities (see [4] and [5]) :

$$(1) \qquad |\log \alpha - \xi| > \exp\{-C_1 N^2(1+\log N)(1+N \log N+\log H) \log(2+N \log N+\log H)\}$$

for all algebraic $\xi$, and

$$(2) \qquad |\log \alpha - \xi| > \exp \{- C_2 N^2 \log H(1 + \log N)^2\}$$

for all algebraic numbers $\xi$ with $N < (\log H)^{1/4}$.

Here, $C_1$ and $C_2$ are positive numbers, depending only on $\alpha$ and on the used branch of the logarithm. The author of this note proved (partly using the second estimate of FEL'DMAN) that

$$(3) \qquad |\log \alpha - \xi| > \exp \{- C_3 N^2(N + \log H)(1 + \log N)^2\}$$

for all algebraic numbers $\xi$ ; see [1] or [2] ; here again $C_3 > 0$ depends only on $\alpha$ and the logarithm.

In this note, the following theorem will be proved :

THEOREM. - Let $\alpha$ be a non-zero algebraic number, and let $\log \alpha$ be an arbitrary value of the logarithm of $\alpha$ with $\log \alpha \neq 0$. Then there exists a positive number $C_4$, depending only on $\alpha$ and on the branch of the logarithm, such that

$$(4) \qquad |\log \alpha - \xi| > \exp \{- C_4 \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

for all algebraic number $\xi$, where N and H denote the degree and the height of $\xi$.

We remark that the inequality (4) belongs to the mentioned second kind of estimates. Further, it is clear that (4) improves (1), (2) and (3). Finally, it is noted here that from (4) follows that $\mathbb{Q}(\log \alpha)$ has a transcendence type at most exactly 3 .

The proof of the theorem will be given by A. O. GEL'FOND's method, as used also by N. I. FEL'DMAN in the quoted papers. The improvements are due to certain refinements in additional ideas, developed in their original form by N. I. FEL'DMAN in [5] and some other papers.

## 2. Lemmas.

The first three lemmas give variants of well known properties. Proofs or references to proofs can be found in [2].

LEMMA 1. - Let $\xi$ be algebraic of degree $N$ and height $H$. Let $n \geq 0$ be an integer, and let $\alpha_i$ be algebraic of degree $d_i$ and height $h_i$ $(i = 1, \ldots, n)$. Put $d = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]$ if $n \geq 1$ and $d = 1$ if $n = 0$. Let

$$P(z_0, z_1, \ldots, z_n) = \sum_{i_0=0}^{N_0} \sum_{i_1=0}^{N_1} \ldots \sum_{i_n=0}^{N_n} p_{i_0 i_1 \ldots i_n} z_0^{i_0} z_1^{i_1} \ldots z_n^{i_n}$$

be a polynomial with integral coefficients whose sum of absolute values is at most $B$. Then $P(\xi, \alpha_1, \ldots, \alpha_n) = 0$ or

$$(5) \qquad |P(\xi, \alpha_1, \ldots, \alpha_n)| > B^{-dN} e^{-dN_0(N+\log H)} \exp\{-dN \sum_{i=1}^{n} \frac{N_i(d_i+\log h_i)}{d_i}\} .$$

LEMMA 2. - Let $P_{\rho\sigma}(z_0, z_1, \ldots, z_n)$ for $\rho = 1, \ldots, r$ and $\sigma = 1, \ldots, s$ be polynomials with integral coefficients, such that the sum of the absolute values of the coefficients of each polynomial is at most $B$, and such that the degree in $z_i$ of each polynomial is at most $N_i$ $(i = 0, 1, \ldots, n)$. Let $\xi$ be algebraic of degree $N$ and height $H$. Let $\alpha_i$ be algebraic of degree $d_i$ and height $h_i$ $(i = 1, \ldots, n)$. Put $d = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]$ if $n \geq 1$ and $d = 1$ if $n = 0$, and let $C$ be a positive even integer. If

$$(6) \qquad s \geq 4rd$$

and

$$(7) \qquad C^N \geq (Bs)^N e^{2(N_0+N)(N+\log H)} \exp\{2N \sum_{i=1}^{n} \frac{N_i(d_i + \log h_i)}{d_i}\} ,$$

then there exist integers $c_{\sigma\nu}$ $(\sigma = 1, \ldots, s ; \nu = 0, 1, \ldots, N-1)$, not all zero, such that $|c_{\sigma\nu}| \leq C$ for $\sigma = 1, \ldots, s$ and $\nu = 0, 1, \ldots, N-1$ and such that

$$(8) \qquad \sum_{\sigma=1}^{s} \sum_{\nu=0}^{N-1} c_{\sigma\nu} \xi^\nu P_{\rho\sigma}(\xi, \alpha_1, \ldots, \alpha_n) = 0$$

for $\rho = 1, \ldots, r$.

LEMMA 3. - Let $F$ be an entire function, and let $T$ and $S$ be positive inte-

<u>gers, and</u> R <u>and</u> A <u>be real numbers such that</u> $R \geq 2S$ <u>and</u> $A > 2$ . <u>Then</u>

$$\max_{|z|\leq R}|F(z)| \leq 2 \max_{|z|\leq AR}|F(z)| \ (\tfrac{2}{A})^{TS} \tag{9}$$
$$+ (\tfrac{9R}{S})^{TS} \max_{t=0,1,\dots,T-1,s=0,1,\dots,S-1} \frac{1}{t!} |F^{(t)}(s)| \ .$$

In the sequel of this note, $T$ and $M$ are fixed positive integers, and the functions $g_m$ $(m = 0 , 1 ,\dots, M - 1)$ will be defined by

$$g_0(z) = 1 \quad \text{and} \quad g_m(z) = ([\tfrac{m}{T}]!)^{-T} (z + 1) \dots (z + m) \ .$$

The lemmas 4, 5 and 6 represent the new elements in the proof of the main theorem. For proofs of these lemmas, the reader is referred to [3].

LEMMA 4. - <u>For</u> $t = 0 , 1 ,\dots, T - 1$ , <u>the values</u> $g_m^{(t)}(z)$ <u>satisfy</u>

$$\frac{1}{t!} |g_m^{(t)}(z)| \leq e^{|z|+2m+3m \log T} \ . \tag{10}$$

LEMMA 5. - <u>There exists a positive integer</u> $d$ , <u>such that all numbers</u>

$$d(1/t!)g_m^{(t)}(s) \quad (m = 0,1,\dots,M-1 \ ; \quad t = 0,1,\dots,T-1 \ ; \quad s = 0,1,2,\dots)$$

<u>are integers, while</u>

$$d \leq e^{4M \log T} \ . \tag{11}$$

LEMMA 6. - <u>Let</u> $a$ <u>be a non-zero complex number. Let</u> $F$ <u>be the exponential polynomial</u>

$$F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} C_{km}\, g_m(z)\, e^{akz} \ ,$$

<u>where the</u> $C_{km}$ <u>are complex numbers. Put</u>

$$C = \max_{k,m}|C_{km}| \ , \quad \Omega = \max(1 \ , \ (K - 1)\ |a|) \ , \quad \omega = \min(1 \ , \ |a|) \ ,$$

<u>let</u> $S'$ <u>be a positive integer, and define</u> $E$ <u>by</u>

$$E = \max_{t=0,1,\dots,T-1,s=0,1,\dots,S'-1} \frac{1}{t!} |F^{(t)}(s)| \ .$$

<u>If</u>

$$TS' \geq 2KM + 15\Omega S' \ , \tag{12}$$

<u>then</u>

$$C \leq \frac{3}{2}\, (2e)^M\, (\tfrac{6}{\omega K})^{kM}\, \{\max(6\Omega \ , \ \tfrac{3KM}{4S'})\}^{KM}\, 18^{TS'}\, E \ . \tag{13}$$

<u>Proof of the theorem.</u> - In what follows, $x$ denotes a positive number which is as large in comparison to the degree $d$ of $\alpha$ , the height $h$ of $\alpha$ and to $|\log \alpha|$, that all inequalities we need in the proof will hold. Further, $c_1$ , $c_2$ , ... will be positive numbers depending only on $d$ , $h$ and $|\log \alpha|$ . We prove that for all algebraic $\xi$ ,

$$|\log \alpha - \xi| > \exp\{ - x^{12} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}$$

Suppose that for some algebraic number $\xi$ , we have

$$(14) \qquad |\log \alpha - \xi| \leqslant \exp\{ - x^{12} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\} .$$

Choose the following integers :

$$K = [x^3 N] \qquad M = [x^6 \frac{N(1 + N \log N + \log H)}{(1 + \log N)^2}]$$

$$T = [x^6 \frac{N^2}{1 + \log N}] \qquad S = [x^2 \frac{1 + N \log N + \log H}{1 + \log N}]$$

$$S' = [x^4 \frac{1 + N \log N + \log H}{1 + \log N}] \qquad C = 2[\frac{1}{2} \exp\{x^7 \frac{N(1 + N \log N + \log H)}{1 + \log N}\}] .$$

Let $F$ be the exponential polynomial

$$(15) \qquad F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{\nu=0}^{N-1} C_{km\nu}\, \xi^\nu\, g_m(z)\, e^{k(\log \alpha)z} ,$$

where the numbers $C_{km\nu}$ are integers with absolute values at most $C$ ; they will be specified later. For $t = 0 , 1 , 2 , \ldots$ and $s = 0 , 1 , 2 , \ldots ,$ we have

$$F^{(t)}(s) = \sum_k \sum_m \sum_\nu C_{km\nu}\, \xi^\nu \sum_{\tau=0}^{m} \binom{t}{\tau} g_m^{(\tau)}(s)\, k^{t-\tau}(\log \alpha)^{t-\tau}\, \alpha^{ks} .$$

Define $\Phi_{ts}$ by

$$\Phi_{ts} = \sum_k \sum_m \sum_\nu C_{km\nu}\, \xi^\nu \sum_\tau \binom{t}{\tau} g_m^{(\tau)}(s)\, k^{t-\tau}\, \xi^{t-\tau}\, \alpha^{ks} .$$

For $0 \leqslant t - \tau \leqslant T - 1$ , we have from (14)

$$|(\log \alpha)^{t-\tau} - \xi^{t-\tau}| \leqslant T(|\log \alpha| + 1)^T\, |\log \alpha - \xi|$$
$$\leqslant \exp\{ - \frac{1}{2} x^{12} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\} .$$

It follows with lemma 4 that, for $t = 0 , 1 , \ldots , T - 1$ and $s = 0,1,\ldots,S'-1$

$$|F^{(t)}(s) - \Phi_{ts}| \leqslant \exp\{- \frac{1}{2} x^{12} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

$$(16) \qquad \times\, KMNC(|\log \alpha|+1)^N\, 2^T\, T^T\, e^{S'+2M+3M \log T}\, K^T(|\alpha|+1)^{KS'}$$

$$\leqslant \exp\{c_1 x^7 \frac{N(1+N \log N+\log H)}{1 + \log N} - \frac{1}{2} x^{12} \frac{N^2(1+N \log N+\log H)}{1 + \log N}\}$$

$$\leqslant \exp\{- x^{11} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\} .$$

Let $d$ be the number introduced in lemma 5. Then $d\Phi_{ts}$ $(t = 0 , 1 ,\ldots, T - 1 ;$ $s = 0 , 1 ,\ldots, S - 1)$ obviously has the form

$$d\Phi_{ts} = \sum_k \sum_m \sum_\nu C_{km\nu}\, \xi^\nu\, P_{tskm}(\xi , \alpha)$$

where $P_{tskm}$ has integral coefficients. According to lemma 2, applied with $n = 1$ , $\alpha_1 = \alpha$ , $N_0 = T$ , $N_1 = KS$ , $B \leqslant \exp\{c_2 x^6 \log x\, (N(1+N \log N+\log H))/(1+\log N)\}$ , $r = TS$ , $s = KM$ , we can choose the numbers $C_{km\nu}$ as integers, not all zero, such that $|C_{km\nu}| \leqslant C$ for all used $k$ , $m$ and $\nu$ and such that

$$(17) \qquad d\Phi_{ts} = 0 \quad (t = 0 , 1 ,\ldots, T - 1 ; \ s = 0 , 1 ,\ldots, S - 1) .$$

It follows that $\Phi_{ts} = 0$ and that, by (16),

$$(18) \qquad |F^{(t)}(s)| \leqslant \exp\{ - x^{11} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\} .$$

Subsequently, we apply lemma 3 to $F$ with $R = S'$ and $A = 6N$. Since, from (15) and (10),

$$\max_{|z|\leqslant 6NS'} |F(z)| \leqslant \exp\{c_3 x^7 \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

we obtain from (9) and (18)

$$(19) \qquad \max_{|z|\leqslant S'} |F(z)| \leqslant \exp\{ - \frac{1}{2} x^8 \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\} .$$

By Cauchy's theorem, this implies for $t = 0 , 1 ,\ldots, T - 1$ and $s=0,1,\ldots,S'-1$ that

$$(20) \quad |F^{(t)}(s)| \leqslant T^T S' \max_{|z|\leqslant S'} |F(z)| \leqslant \exp\{- \frac{1}{3} x^8 \frac{N^2(1+N \log N+\log H)}{1 + \log N}\} .$$

Together with (16) and (11), we obtain

$$(21) \qquad |d\Phi_{ts}| \leqslant \exp\{ - \frac{1}{4} x^8 \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

for $t = 0 , 1 ,\ldots, T - 1$ ; $s = 0 , 1 ,\ldots, S' - 1$ .

But $d\Phi_{ts}$, for these values for $t$ and $s$, is a polynomial with integral coefficients in the algebraic numbers $\xi$ and $\alpha$. We apply lemma 1 to $d\Phi_{ts}$ with $n = 1$, $\alpha_1 = \alpha$, $N_0 = N + T$, $N_1 = KS'$ and $B \leqslant \exp\{c_4 x^7 \frac{N(1+N \log N+\log H)}{1 + \log N}\}$. We thus obtain that either $d\Phi_{ts} = 0$ or

$$(22) \qquad |d\Phi_{ts}| > \exp\{ - c_5 x^7 \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

for $t = 0 , 1 ,\ldots, T - 1$ ; $s = 0 , 1 ,\ldots, S' - 1$ .

Since (21) and (22) are incompatible if $x$ is large enough, it follows that $d\Phi_{ts} = 0$, $\Phi_{ts} = 0$ and, by (16),

$$(23) \qquad |F^{(t)}(s)| \leqslant \exp\{ - x^{11} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

for $t = 0 , 1 ,\ldots, T - 1$ ; $s = 0 , 1 ,\ldots, S' - 1$ .

We are now ready to apply lemma 6 with $a = \log \alpha$ and $C_{km} = \sum_\nu C_{km\nu} \xi^\nu$. By the inequality

$$(24) \qquad \Omega \leqslant c_6 x^3 N$$

condition (12) is easily checked. By the same inequality, (13) reduces to

$$(25) \qquad C \leqslant (c_7 \frac{M}{S'})^{KM} 18^{TS'} E .$$

From (23), we thus obtain

$$(26) \qquad |\sum_{\nu=0}^{N-1} C_{km\nu} \xi^\nu| \leqslant \exp\{ - x^{10} \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\}$$

for $k = 0 , 1 ,\ldots, K - 1$ ; $m = 0 , 1 ,\ldots, M - 1$ .

Finally, we apply lemma 1 for a second time, with $n = 0$ and $B \leqslant NC$. We obtain that either $\sum_\nu C_{km\nu} \xi^\nu = 0$ or

$$(27) \qquad |\sum_\nu C_{km\nu} \xi^\nu| > \exp\{ - c_8 x^7 \frac{N^2(1 + N \log N + \log H)}{1 + \log N}\} .$$

Thus, $\sum_\nu C_{km\nu} \xi^\nu = 0$ for all $k$ and $m$. Since $1 , \xi ,\ldots, \xi^{N-1}$ are linearly independent, it follows that all $C_{km\nu}$ are zero. The contradiction to the choice of these integers proves the theorem.

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