Séminaire Choquet.
Initiation à l'analyse

Ivan Singer

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Séminaire Choquet. Initiation à l'analyse, tome 6, n° 2 (1966-1967), exp. n° 12, p. 1-8
<http://www.numdam.org/item?id=SC_1966-1967__6_2_A2_0>

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Séminaire CHOQUET
(Initiation à l'Analyse)
6e année, 1966/67, n° 12

16 février 1967

SOME OPEN PROBLEMS ON BEST APPROXIMATION IN NORMED LINEAR SPACES

by Ivan SINGER

[according to a draft by Ivan SINGER]

Notations.

E : normed linear space (real or complex).

G : linear subset of E .

$x \in E$, $g_0 \in G$: best approximation of x if

$$\|x - g_0\| = \inf_{g \in G} \|x - g\| \quad ,$$

or nearest point.

G is a Čebyšev subset, if each $x \in E$ has exactly one (i. e. one and only one) best approximation $g_0 \in G$.

In particular, if G is a linear subspace, G is a Čebyšev subspace.

1. - In 1957, V. KLEE [13] has raised the following problem (in somewhat different terms) : Does every m-dimensional Banach space E , where $3 \leqslant m < \infty$, possess a one-dimensional Čebyšev subspace G ($\dim E = n$, $\dim G = 1$) ?

Geometrically : Does E possess a line G through the origin such that there exists no segment on the unit sphere $\{x \in E \mid \|x\| = 1\}$ parallell to G ? If the unit cell of E is a polyhedron (e. g. take a cube), the answer is obviously affirmative. For the usual euclidean norm, the answer is again obviously affirmative.

History of the problem. - Student → KLEE himself → KAKUTANI, CHOQUET and others.

For $m = 3$, T. J. McMINN [15] has proved that the answer is affirmative. Another proof of this result has been given by A. S. BESICOVITCH [1].

For $m \geqslant 3$, incorrect proofs have been given by G. EWALD [7], G. ROGERS, EWALD-ROGERS and others.

2. - In 1964, A. L. GARKAVI [10] has posed the following problem : Does there exist a separable Banach space which has no Čebyšev subspace ?

Since, in every separable conjugate Banach space $E = B^*$, the unit cell S_E has at least one exposed point ($x \in E$, $\|x\| = 1$, such that there exists a support hyperplane H of $S_E = \{x \in E \mid \|x\| \leqslant 1\}$ satisfying $H \cap S_E = \{x\}$), one must seek for the space E among those separable Banach spaces which are not isometric to any conjugate space.

GARKAVI has constructed the following example of a non-separable Banach space E which has no Čebyšev subspace : If I is a set of cardinality $\overline{\overline{I}} > c$, then the space $E = E(I)$ of all bounded families $x = \{\xi_i\}_{i \in I}$ of scalars which have at most a countable number of non-zero "coordinates" ξ_i, endowed with the usual vector operations and with the norm $\|x\| = \sup_{i \in I} |\xi_i|$, has no Čebyšev subspace.

V. KLEE and myself have given (unpublished) the following example of a separable non-complete normed linear space E which has no Čebyšev subspace : the dense subspace of c_0 consisting of all almost-zero sequences (i. e. whose coordinates are $= 0$, except a finite number of them).

It is well known that the space $E = c_0$ has no Čebyšev subspaces of infinite dimension, but it has such subspaces of any finite dimension. It is also known that $E = L^1([0 , 1])$ has no Čebyšev subspace of finite dimension or of finite codimension, but yet it has Čebyšev subspaces. Thus it is natural to ask the following problem : How to combine the spaces c_0 and $L^1([0 , 1])$ in order to obtain a separable Banach space which has no Čebyšev subspace ? [The spaces c_0 and $L^1([0 , 1])$ are not isometric, not isomorphic even, to any conjugate Banach space.]

3. - A related question of GARKAVI : Does the space $C([0 , 1])$ possess a Čebyšev subspace of infinite dimension and infinite codimension

$$(\operatorname{codim} G = \dim E/G) \quad ?$$

It is well known that it has Čebyšev subspaces of any finite dimension, and Čebyšev subspaces of codimension 1, but has no Čebyšev subspaces of finite codimension $n \geqslant 2$.

4. - Characterize (topologically) those compact spaces Q for which the complex space $C(Q)$ (of all continuous functions on Q with $\|x\| = \max_{q \in Q} |x(q)|$) contains a Čebyšev subspace of finite dimension $n > 1$.

This problem presents some interest from the following point of view : Although the classical theorem of Banach-Stone,

$$Q_1 \sim Q_2 \iff C(Q_1) \equiv C(Q_2) \quad ,$$

shows, theoretically, that the vectorial-metric properties of the spaces $C(Q)$ are completely determined by the topological properties of the compact spaces Q and conversely yet the effective, explicit study of this interdependence still presents many open problems, and a solution of the above problem might be considered as a contribution to this study.

According to the classical Haar (real) - Kolmogorov (complex) theorem, an n-dimensional linear subspace

$$G = [x_1 , \dots , x_n] \subset C(Q)$$

is a Čebyšev subspace if, and only if, $x_1 , \dots , x_n$ is a Čebyšev system on Q (i. e. each linear combination $\sum_{i=1}^{n} \alpha_i x_i$ has at most $n - 1$ zeros on Q). Thus the above problem amounts to the following : Characterize (topologically) those compact spaces Q which admit a Čebyšev system $x_1 , \dots , x_n$ with $n > 1$.

In the case of real scalars (i. e. $C_R(Q)$), S. MAZUR has conjectured that if a compact space Q admits a real Čebyšev system $x_1 , \dots , x_n$ with $n > 1$, then Q is homeomorphic to a subset of a circle. This conjecture has been proved for the first time by J. C. MAIRHUBER [16] under the additional hypothesis that Q is a subset of a k-dimensional euclidean space. For general compact spaces, it has been proved, independently, by K. SIEKLUCKI [20] and P. C. CURTIS [4]. A more simple proof has been given by I. J. SCHOENBERG and C. T. YANG [19].

For the case of complex scalars, only the following partial result of I. J. SCHOENBERG and C. T. YANG (loc. cit.) is known : If a compact space Q admits a complex Čebyšev system $x_1 , \dots , x_n$ with $n > 1$, and if Q is homeomorphic to a subset of a finite polyhedron, then Q is homeomorphic to a subset of the euclidean plane.

Other problems.

1° Similar problem for complex $L^1(T , \nu)$ spaces. For real $L^1_R(T , \nu)$, A. L. GARKAVI [10] has proved ...

2° Extension to $\dim P_G(x) \leqslant k$ $(x \in E \setminus G)$.

5. - Characterize (topologically) those compact spaces Q for which the real space $C_R(Q)$ [or the complex space $C(Q)$] contains a Čebyšev subspace of finite codimension $n \geqslant 1$.

Concerning this problem, there exist only partial results of R. R. PHELPS [18]

and A. L. GARKAVI ([8], [9] and [10]), giving sufficient conditions or necessary conditions on Q, in order that $C_R(Q)$ contain a Čebyšev subspace of codimension $\geqslant 1$. Let us recall them :

1° Sufficient conditions.

(a) Q compact separable (in particular, compact metric) implies

$$C_R(Q) \supset G \quad (\text{Čebyšev}) , \qquad \operatorname{codim} G = 1 .$$

(b) Q compact separable, satisfying $Q = \overline{Q \setminus Q'}$ (i. e. Q = the closure of the set of its isolated points), implies

$$C_R(Q) \supset G \quad (\text{Čebyšev}) , \qquad \operatorname{codim} G = n \quad (n = 1 , 2 , \ldots) .$$

The compact spaces Q satisfying this condition (in particular, for Q countable) may also contain connected closed infinite subsets (continua).

There exist examples showing that the above conditions are not necessary. However, at the International Congress of Mathematicians [1966. Moscow], A. L. GARKAVI has announced that if Q is also metric, then the condition of (b) is both necessary and sufficient in order that $C_R(Q)$ contain Čebyšev subspaces of any finite codimension $n \geqslant 1$. Thus, for metric compact spaces, the problem is solved.

2° Necessary conditions.

(c) $C_R(Q) \supset G$ (Čebyšev) and $\operatorname{codim} G < \infty$ imply that Q has at most a countable number of disjoint open subsets (imply, in particular, at most a countable number of isolated points).

(d) $C_R(Q) \supset G$ (Čebyšev) and $\operatorname{codim} G = n > 1$ imply that Q contains no open connected infinite subset.

(e) Corollary of (d) : when either Q is finite, or Q has an infinity of connected components.

6. - Characterize those Banach spaces E which contain no closed linear subspace G with $\operatorname{codim} G = n < \infty$, with the following property :

The set of all best approximation of x ,

$$P_G(x) = \emptyset \qquad (x \in E \setminus G) ;$$

or, equivalently : such that there does not exist

$$z \in E \setminus \{0\} \quad \text{with} \quad z \perp G , \qquad \|z - g\| \geqslant \|g\| \quad (g \in G) .$$

For $n = 1$, R. C. JAMES [12] has shown (the proof is very difficult) that this class coincides with the class of reflexive Banach spaces [his first proof for separable spaces [11] was not correct, a student of V. KLEE has discovered an irremediable error in it]. I have posed this problem to Professor JAMES at the International Congress of Mathematicians in Moscow, 1966.

7. - Čebyšev set G : each $x \in E$ has exactly one best approximation $g_0 \in G$. It is obvious that in a reflexive strictly convex Banach space, every convex set is a Čebyšev set. I. S. MOTZKIN [17] and others have proved that, inversely, in a finite-dimensional smooth Banach space (i. e. for every $x \in E$, there exists exactly one $f \in E^*$ with $\|f\| = 1$, $f(x) = \|x\|$), every Čebyšev set is convex. In connection with this result, the following problem arises naturally : Characterize those Banach spaces in which every Čebyšev set is convex.

1° The case of n-dimensional Banach spaces, where $n < \infty$. N. V. EFIMOV and S. B. STEČKIN [5] and V. KLEE ([14] or a Conference at the Colloquium on Convexity [1965. Copenhagen] to appear) have claimed that the converse of the above is also true, i. e. that a finite-dimensional Banach space in which every Čebyšev set is convex must be smooth. For $n = 2$, this is obviously true. However, A. BRØNDSTED ([2], [3]) has shown that for $n \geqslant 3$ this is false ; namely, for every integer $n \geqslant 3$, there exists a non-smooth n-dimensional Banach space in which every Čebyšev set is convex. BRØNDSTED [3] has also solved the above characterization problem for $n = 3$, by proving that every Čebyšev set in E (with $\dim E = 3$) is convex if, and only if, every exposed point of $S_E = \{x \in E \mid \|x\| \leqslant 1\}$ is a smooth point of S_E .

For $n > 3$, the problem is still open.

2° The case of infinite-dimensional Banach spaces. In this case, considerably less is known. Not only the possibility of extending MOTZKIN's result to infinite-dimensional smooth Banach spaces is unknown, but even the answer to the following question is unknown : In a Hilbert space $\mathcal{H}$, is every Čebyšev set necessarily convex ? The best result known until the present is the following theorem of L. P. VLASOV [21], which contains as particular cases results of V. KLEE, N. V. EFIMOV and S. B. STEČKIN : In a smooth Banach space, every boundedly compact (i. e. every bounded closed subset of it is compact) Čebyšev set is convex. The proof is very short and elegant, it makes use of the Schauder fixed point theorem.

Since the condition of being boundedly compact, is too restrictive, N. V. EFIMOV and S. B. STECKIN [6] have introduced the notion of approximative compactness : a

set $G \subset E$ is called approximatively compact if, for every $x \in E$ and every sequence

$$\{g_n\} \subset G \qquad \text{with} \quad \lim_{n\to\infty} \|x - g_n\| = \rho(x , G) \quad ,$$

there exists a subsequence $\{g_{n_k}\}$ converging to an element of G (e. g. in a uniformly convex Banach space E, every weakly closed subset is approximatively compact). N. V. EFIMOV and S. B. STEČKIN [6] have proved that in a smooth uniformly convex Banach space E, a Čebyšev set G is convex if, and only if, it is approximatively compact. Hence (by the above), for a set G in a smooth uniformly convex Banach space, the following statements are equivalent :

(1) G is convex and closed ;
(2) G is Čebyševian and weakly closed ;
(3) G is Čebyševian and approximatively compact.

We have thus useful infinite-dimensional characterizations of closed convex sets in terms of Čebyšev sets, which show, among others, that the problem of convexity of Čebyšev sets in smooth uniformely convex Banach spaces is equivalent to each of the following problems :

(a) Is a Čebyšev set weakly closed ?
(b) Is a Čebyšev set approximatively compact ?

Instead of imposing supplementary conditions directly on the Čebyšev set G in order to be able to conclude that it is convex, one can impose conditions on the mapping π_G (assigning to each x the unique best approximation of x in G ; this is well defined, once G is a Čebyšev set). Thus, V. KLEE has proved [14] that, if G is a Čebyšev set in a smooth and reflexive Banach space E, and if each $x \in E \setminus G$ admits a neighbourhood $V(x)$ such that $(\pi_G)_{V(x)}$ is both continuous and weakly continuous, then G is convex. The condition of continuity of the mapping π_G, where G is a Čebyšev set, does not imply the weak continuity of π_G, as shown by the example

$$E = (\text{some Hilbert space}) , \qquad G = S_E \quad ;$$

on the other hand, it is not known whether the weak continuity of π_G, where G is a Čebyšev set, implies its strong continuity.

The problem of convexity of Čebyšev sets in Hilbert spaces $\mathcal{H}$ is also related to the following problem on farthest points : If a closed convex set $A \subset \mathcal{H}$ has the property that every $y \in \mathcal{H}$ admits a unique farthest point in A (i. e. which has maximal distance from x), does A necessarily consist of one simple point ?

A. F. FICKEN and V. KLEE [14] have shown that if the answer to this question would be affirmative, then every Čebyšev set in $\mathcal{H}$ would be convex.

Since the problem of convexity of Čebyšev sets $G \subset \mathcal{H}$ is open, it is natural to pose the problem of convexity of sets A belonging to some larger classes of sets. V. KLEE (Conference at the Colloquium on Convexity [1965. Copenhagen]) has considered two such classes, C_1 and C_2. Let us mention C_1 : A set G, in a normed linear space E, is called a C_1-set if G is semi-Čebyševian (i. e. each $x \in E$ has at most one best approximation in G) and closed. Every boundedly compact C_1-set in a smooth Banach space E is easily shown to be convex. However, V. KLEE has shown, that, in every infinite dimensional Hilbert space $\mathcal{H}$, there exists a C_1-set G whose complementary set $\mathcal{H} \setminus G$ is non-void, bounded and convex (hence G is non-convex).

Some other interesting related problems have been raised in the same paper of V. KLEE.

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