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CURVES AND THEIR FUNDAMENTAL GROUPS
[following Grothendieck, Tamagawa and Mochizuki]

by **Gerd FALTINGS**

1. INTRODUCTION

The fundamental group of a topological space is one of the most elementary invariants in algebraic topology. In fact many interesting topological spaces (for example negatively curved) are $K(\pi,1)$'s, and thus their fundamental groups classify them completely up to homotopy. Similarly hyperbolic spaces (constant curvature -1) of dimension at least three are up to isometry determined by their group (Mostow).

In SGA1, A. Grothendieck has defined the algebraic fundamental group of a scheme as the profinite group which classifies finite etale coverings. For a complex algebraic variety it is equal to the profinite completion of the topological fundamental group. However if the variety X is defined over a subfield K of $\mathbb{C}$ there is an additional structure. Namely the geometric fundamental group fits into an exact sequence

$$0 \longrightarrow \pi_1(X \otimes_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 0,$$

thus defining a homomorphism

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_K \overline{K})),$$

which determines the class of the extension if $\pi_1(X \otimes_K \overline{K})$ has trivial center.

We owe to A. Grothendieck the idea that this extension may encode many algebraic properties of X. Obviously one should restrict to varieties which classically would be $K(\pi,1)$'s, so that for example projective spaces should be excluded. Grothendieck conjectures the existence of a "natural" class of anabelian schemes" (over fields K), which should have the following properties:

a) X is anabelian $\Longleftrightarrow$ $X \otimes_K \overline{K}$ is anabelian.

b) If $X \to Y$ is proper and smooth (a fibration) and Y as well as the fibers are anabelian, so is X.

c) Hyperbolic curves are anabelian.

d) The moduli stacks $\mathcal{M}_{g,n}$ (classifying curves of genus g with n punctures) are anabelian.

He then conjectures that for anabelian schemes X over a field K which is finitely generated over $\mathbb{Q}$, the functor which sends X to its fundamental group $\pi_1(X)$ is fully faithful. Here maps $\pi_1(X) \to \pi_1(Y)$ should be (and will be for the sequel of this talk) $\pi_1(Y \otimes_K \overline{K})$-conjugacy-classes of maps of extensions

$$0 \longrightarrow \pi_1(X \otimes_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 0.$$

Especially rational points $X(K)$ should correspond bijectively to conjugacy-classes of sections

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X).$$

In many cases the Mordell-Weil theorem already implies that $X(K)$ injects into the space of conjugacy-classes of sections. In this talk I want to report on recent progress about these conjectures. Firstly A.Tamagawa has shown that for (projective and irreducible) curves X over a finite field K, one can characterise the sections s coming from K-rational points entirely in terms of the groups. We present this here (in somehow modified form) to introduce a very important technique, namely to consider not only X but also an infinite number of coverings of it. This dates back to earlier work of H. Nakamura, and has been used by S. Mochizuki to show our main result:

Namely if K is a finitely generated extension of $\mathbb{Q}_p$, then for hyperbolic curves X_1 and X_2 over K, any open map of extensions

$$\pi_1(X_1) \longrightarrow \pi_1(X_2)$$

is induced from a (unique and dominant) map

$$X_1 \longrightarrow X_2.$$

This implies the corresponding result for number fields.

The proof uses p-adic Hodge theory, and we sketch the relevant facts. Also for simplicity we restrict ourselves to finite extensions of $\mathbb{Q}_p$ (local fields) and projective curves. Finally we do not discuss the vast amount of literature dealing with the problem how to recover a field from its absolute Galois group (for example [P]). I thank S. Mochizuki for his help. The talk is based on his ideas. However in some arguments I have used different approaches, to further scientific diversity.

2. PRELIMINARIES

For a connected scheme X and a geometric point x of X, P. Grothendieck defined in SGA1 the algebraic fundamental group $\pi_1(X, x)$. It is a profinite compact topological group, such that the category of finite etale coverings is equivalent to that of finite sets with continuous $\pi_1(X, x)$-action. To a finite etale covering

$$f : Y \longrightarrow X$$

one associates the set $f^{-1}(x)$. Here connected coverings correspond to homogeneous $\pi_1(X, x)$-sets, and connected pointed coverings to open subgroups of $\pi_1(X, x)$. The fundamental group is a covariant functor for pointed maps. In general for different base-points the corresponding fundamental groups are isomorphic, the isomorphism being unique up to conjugation. Hence for arbitrary maps we obtain a functor into the category of groups with conjugacy classes of homomorphisms. If we neglect base-points we use the notation $\pi_1(X)$. For a scheme X of finite type over the complex numbers $\mathbb{C}$ the algebraic fundamental group is the profinite completion of the topological fundamental group of $X(\mathbb{C})$. So for example for a smooth connected projective curve X of genus g the fundamental group $\pi_1(X, x)$ is (profinitely) generated by $2g$ elements

$$a_1, \cdots, a_g, b_1, \ldots, b_g,$$

subject to the single relation

$$a_1 . b_1 . a_1^{-1} . b_1^{-1} . \ldots . a_g . b_g . a_g^{-1} . b_g^{-1} = 1.$$

It is known that one obtains the same group for any such curve over an algebraically closed field of characteristic zero. In positive characteristic the fundamental group is a quotient of the above. As a warmup we prove

1. *Lemma.— Suppose X is a smooth projective irreducible curve over an algebraically closed field K, of genus $g \geq 2$. Then $\pi_1(X, x)$ has trivial center.*

Proof.— Suppose σ lies in the center. Then for any connected finite etale covering $Y \to X$, σ defines an automorphism of Y which acts on the fundamental group $\pi_1(Y, y)$ by inner automorphism. Especially σ acts trivially on the etale cohomology $H^1(Y, \mathbb{Z}_\ell)$, ℓ any prime invertible in K. As Y has also genus bigger than one σ must be the identity on Y, and as this holds for all Y we have $\sigma = 1$.

2. **COROLLARY**.— *Suppose X is a smooth projective geometrically irreducible curve over an arbitrary field K, of genus $g \geq 2$. Consider the extension* [SGA1]

$$0 \longrightarrow \pi_1(X \otimes_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 0.$$

The isomorphism class of this extension is then uniquely determined by the induced map

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_K \overline{K})) \quad (= \textit{automorphisms/inner automorphisms}).$$

More precisely for two such curves X and Y, any conjugacy-class of surjective homomorphisms $\pi_1(X \otimes_K \overline{K}) \to \pi_1(Y \otimes_K \overline{K})$ respecting these Galois actions extends to a homomorphism of extensions $\pi_1(X) \to \pi_1(Y)$, unique up to conjugation by $\pi_1(Y \otimes_K \overline{K})$.

Proof.— $\pi_1(X)$ is the fibered product of $\mathrm{Aut}(\pi_1(X \otimes_K \overline{K}))$ and $\mathrm{Gal}(\overline{K}/K)$, over $\mathrm{Out}(\pi_1(X \otimes_K \overline{K}))$. Here as topology on $\mathrm{Aut}(\pi_1(X \otimes_K \overline{K}))$ one uses uniform convergence.

3. CURVES OVER FINITE FIELDS

In this section we suppose that K is a finite field, thus $\mathrm{Gal}(\overline{K}/K)$ is free cyclic generated by Frobenius. Suppose X is a projective smooth geometrically irreducible curve over K. Then the number of K-rational points of X is equal to the alternating sum of the traces of (geometric) Frobenius on ℓ-adic etale cohomology. Especially this number can be read of from the action of Frobenius on $\pi_1(X \otimes_K \overline{K})$ (even the maximal abelian quotient suffices). Now suppose given a section

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

of the projection. For any characteristic open subgroup H of $G = \pi_1(X \otimes_K \overline{K})$ we obtain a subgroup $H \cdot s(\mathrm{Gal}(\overline{K}/K))$ of $\pi_1(X)$, which corresponds to an etale covering $Y = Y(H) \to X$. Here Y is also geometrically irreducible, and $H \cdot s(\mathrm{Gal}(\overline{K}/K))$ is equal to $\pi_1(Y)$. Y is a K-model of the Galois-cover of $X \otimes_K \overline{K}$ defined by H. If in fact the section s is defined by a rational point $x \in X(K)$, then chosing x as base point we see that Y is a pointed covering, that is x lifts to a point $y \in Y(K)$. There is a converse:

3. **THEOREM** (A. Tamagawa).— *Suppose each $Y(H)$ has a K-rational point. Then s is defined by a unique point $x \in X(K)$.*

Proof.— We may assume that X has genus ≥ 1. The non trivial part is to show that s comes from a section if each $Y(H)(K)$ is non empty which we now assume. For each integer n let $H_n \subset G$ the intersection of all kernels of homomorphisms from G to $\mathbb{Z}/n\,.\,\mathbb{Z}$, and

$$f_n : Y_n = Y(H_n) \longrightarrow X$$

the corresponding etale covering of X. Then the pullback in etale cohomology

$$f_n^* : H^1(X \otimes_K \overline{K}, \mathbb{Z}/n\,.\,\mathbb{Z}) \longrightarrow H^1(Y_n \otimes_K \overline{K}, \mathbb{Z}/n\,.\,\mathbb{Z})$$

vanishes, and by flat duality the same holds for the pushforward (now in flat cohomology)

$$f_{n,*} : H^1(Y_n \otimes_K \overline{K}, \mu_n) \longrightarrow H^1(X \otimes_K \overline{K}, \mu_n).$$

That means the trace-map on Jacobians $J_n \to J$ vanishes on K-valued n-torsion-points, and after composing with a high enough power of Frobenius the trace becomes divisible by n, as a homomorphism of abelian varieties defined over K. Now if y and z are two points in $Y_n(K)$ their difference $y - z$ is a divisor of degree zero and defines a point in $J_n(K)$. Its image under the trace-map is fixed by Frobenius, and thus lies in $n\,.\,J(K)$. It is the line bundle defined by the divisor $f_n(y) - f_n(z)$. As $J(K)$ is finite we can find an integer n which annihilates it. Then $f_n(y)$ and $f_n(z)$ define the same line bundle, thus must coincide. That is $f_n(Y_n(K)) \subset X(K)$ contains at most one element.

Also any $Y(H)$ is as good as $X = Y(G)$ for that argument. Thus we find a characteristic $H' \subset H$ with $Y(H')(K) \to Y(H)(K)$ constant. As each $Y(H)(K)$ was supposed to be non empty we derive that the projective limit of all $Y(H)(K)$ contains precisely one element, given by a coherent system of K-points $y(H)$ of $Y(H)$. In turn these define a system of sections

$$s(H) : \mathrm{Gal}(\overline{K}/K) \longrightarrow H \cdot s(\mathrm{Gal}(\overline{K}/K)),$$

well defined up to H-conjugation and (up to that) coherent. Especially for any H, $s(G)$ is modulo H G-conjugate to s . By compactness of G it follows that $s(G)$ is conjugate to s, so s comes from the (unique) point $y(G)$.

Without uniqueness the proof simplifies, as the filtering projective limit of non empty finite sets is non empty itself.

4. GALOIS COHOMOLOGY AND DIFFERENTIALS

Suppose V is a complete discrete valuation ring with fraction field K of characteristic 0, uniformiser π, and residue field $k = V/\pi . V$ perfect of characteristic $p > 0$. Suppose furthermore that R is a V-algebra of finite type etale over $V[u,v]/(u.v - \pi)$ (semistable singularity), or a strict henselisation of such, or an adic completion. We denote by $\omega_{R/V} = R.du/u$ its universal (finite) module of relative logarithmic differentials. For simplicity we also assume that $R \otimes_V \overline{K}$ is an integral domain. This always holds if k is algebraically closed, as the integral closure of V in R is unramified over V.

We denote the fundamental group of $\mathrm{Spec}(R \otimes_V \overline{K})$ by Δ. If $\overline{R}$ denotes the normalisation of R in the union of all integral finite etale coverings of $\mathrm{Spec}(R \otimes_V \overline{K})$, then Δ acts continuously on $\overline{R}$, and $\overline{R}$ contains the subextension R_∞ obtained by adjoining all p-power roots of u and v (the local parameters which were assumed to exist), and also of π. It is shown in [F] that $\overline{R}$ is almost etale over this subextension, which means the following:

R_∞ is the union of regular rings R_n. If $S_\infty \subset \overline{R}$ is the normalisation of R_∞ in a finite etale covering of $R_\infty \otimes_V K$, then for big n this covering is defined over $R_n \otimes_V K$, and we denote the corresponding normalisation by S_n. As R_n is regular and S_n is normal of dimension ≤ 2 we know that S_n is a projective R_n-module. Thus the determinant of the trace form on S_n is an invertible ideal in R_n which divides a power of π (or p). Then almost etale means that this power can be chosen arbitrarily small, provided n becomes big enough.

The notion of almost etale goes back to Tate, who used it to compute the continuous cohomology of $\mathrm{Gal}(\overline{K}/K)$ acting on Tate twists of $\overline{K}^\wedge$ (p-adic completion). Namely if V_∞ denotes the normalisation of V in a ramified $\mathbb{Z}_p$-extension of K, then $\overline{V}$ is almost etale over V_∞. As a consequence one gets that $H^i(\mathrm{Gal}(\overline{K}/K), \overline{K}^\wedge(n))$ vanishes if $i > 1$ or $n \neq 0$, and has dimension one over K if $n = 0$, and $i = 0, 1$. Similarly we can (almost) compute the Galois cohomologies $H^*(\Delta, \overline{R}/p^n . \overline{R})$ and $H^*(\Delta, \overline{R}^\wedge)$ (continuous cohomology of p-adic completion). If for simplicity we denote by "=" equality up to p-torsion annihilated by any p-power with exponent $> 1/(p-1)$, then

$$H^0(\Delta, \overline{R}^\wedge) = (R \otimes_V \overline{V})^\wedge,$$

$$H^1(\Delta, \overline{R}^\wedge) \text{ “=” } \omega_{R/V} \otimes_V \overline{V})^\wedge(-1) \quad \text{(Tate-twist)}$$

$$H^i(\Delta, \overline{R}^\wedge) \text{ “=” } (0) \quad \text{for } i > 1.$$

As a consequence one can give a proof for the Hodge-Tate decomposition. Suppose X is a projective smooth curve over K. Then

$$H^1(X \otimes_K \overline{K}, \mathbb{Q}_p) \otimes \overline{K}^\wedge \cong H^1(X, \mathcal{O}_X) \otimes \overline{K}^\wedge \oplus H^0(X, \omega_{R/K}) \otimes \overline{K}^\wedge(-1).$$

Namely for the proof we may replace K by a finite extension and assume that X has a semistable model over V. Then for each small enough open affine Spec(R) in this model the Galois-cohomology $H^*(\Delta, \overline{R}^\wedge \otimes_V K)$ is computed by a canonical bar-complex. Covering the model by such affines, one constructs out of these a double complex (with additional Cech-type differentials), whose cohomology $\mathcal{H}^*(X)$ is independent of choices. Also from the "local" calculations (plus technical remarks, like that the cohomology of a projective formal scheme over $\overline{V}^\wedge$ is equal to its algebraic cohomology, or that a spectral sequence degenerates because "different Tate-twists do not interact"), we have

4. THEOREM.—

$$\mathcal{H}^*(X) = H^*(X, \mathcal{O}_X) \otimes \overline{K}^\wedge \oplus H^{*-1}(X, \omega_{X/K}) \otimes \overline{K}^\wedge(-1).$$

Also as $\mathbb{Z}_p \subset \overline{R}$, there is a canonical map

$$H^*(X \otimes_K \overline{K}, \mathbb{Q}_p) \otimes \overline{K}^\wedge \longrightarrow \mathcal{H}^*(X),$$

which is then shown to be an isomorphism.

Finally for any smooth $\mathbb{Q}_p$-sheaf $\mathbb{L}$ on X we can define $\mathcal{H}^*(X, \mathbb{L})$ as above, replacing $\overline{R}^\wedge$ by $\overline{R}^\wedge \otimes \mathbb{L}$, and again there is a natural map

$$H^*(X \otimes_K \overline{K}, \mathbb{L}) \otimes \overline{K}^\wedge \longrightarrow \mathcal{H}^*(X, \mathbb{L}).$$

One can show that this is always an isomorphism. For unipotent $\mathbb{L}$'s (that is repeated extensions of sheaves induced from the base) this follows already from the result for $\mathbb{Q}_p$, and this will be all we need in the sequel.

Following Mochizuki we consider (for a projective geometrically irreducible curve X over K) the maximal pro-p quotient $\pi_1(X \otimes_K \overline{K})^{(p)}$ of the geometric fundamental group. It is (as all pro-p groups) topologically nilpotent, and has over $\mathbb{Q}_p$ an algebraic hull G. Here G is the projective limit of unipotent algebraic groups over $\mathbb{Q}_p$, and thus entirely determined by its Lie-algebra $\mathbf{g}$. Abstractly G can be generated p-adically by $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ subject to the usual relation, and $\mathbf{g}$ by their logarithms (also subject to one relation). We denote by $Z^n(G)$ (respectively $Z^n(\mathbf{g})$) the descending central series, so

$$Z^0(G) = G, Z^0(G)/Z^1(G) = G^{ab} = T_p(J) \otimes \mathbb{Q}_p, \text{ etc.}$$

The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on G and $\mathbf{g}$ via outer automorphisms. These become a real action if we choose a section

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

of the projection, and anyway the quotients $\mathrm{gr}^n_Z(G) = \mathrm{gr}^n_Z(\mathbf{g})$ are canonical $\mathrm{Gal}(\overline{K}/K)$-modules. After base extension to $\overline{K}^\wedge$ these modules have a Hodge-Tate decomposition, with occuring Tate-twists by integers between 0 and n. As Galois linear extensions between vector spaces with different Tate-twists must split, this lifts for each n to a direct sum decomposition of $(\mathbf{g}/Z^n(\mathbf{g})) \otimes \overline{K}^\wedge$, which depends on the choice of s. However it defines a G-invariant filtration

$$E^m(\mathbf{g}/Z^n(\mathbf{g})) = \text{sum of all Tate-twists by integers } \geq m,$$

which is independent of the choice of s. Thus $\mathbf{g} \otimes \overline{K}^\wedge$ and $G(\overline{K}^\wedge)$ have canonical filtrations $E^m(\mathbf{g})$, respectively $E^m(G(\overline{K}^\wedge))$. The quotient $\mathbf{h} = \mathrm{gr}^0_E(\mathbf{g})$ can be easily shown (for example by our next arguments) to be a free pro-unipotent Lie-algebra in g generators, and also $H = \mathrm{gr}^0_E(G)$ is a free pro-unipotent group in g generators over $\overline{K}^\wedge$, on which $\mathrm{Gal}(\overline{K}/K)$ acts (depending on s). Mochizuki defines the section s to be Hodge-Tate if this action is base-extended from the trivial action (over K), that is if $\mathbf{h}$ is generated by its Galois-invariants. More generally s is called Hodge-Tate up to level n if this holds module $Z^n(\mathbf{h})$. He proves:

5. THEOREM.— *If s comes from a K-rational point, then it is Hodge-Tate.*

Proof.— Mochizuki uses a deformation argument to reduce to the ordinary case. Here we present an alternative, using the faithful representation of H (or $\mathbf{h}$) on the completed enveloping algebra of $\mathbf{h}$. As $\mathbf{h}$ is free in g generators this enveloping algebra is a free tensor-algebra.

Let $x \in X(K)$ induce s. Firstly $H^1(X, \mathcal{O}_X)$ classifies vector bundles on X which are trivialised in x, and extensions of $\mathcal{O}_X$ by $\mathcal{O}_X$. Thus if $\Omega = H^0(X, \omega_{X/K})$ denotes its dual, there exists a universal such extension

$$0 \longrightarrow \Omega \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

One computes that the map

$$H^1(X, \mathcal{H}om(\mathcal{E}_1, \mathcal{O}_X)) \longrightarrow H^1(X, \mathcal{H}om(\Omega \otimes_K \mathcal{O}_X, \mathcal{O}_X)) = (\Omega^{\otimes 2})^{\mathrm{dual}}$$

is an isomorphism, and gets a universal extension

$$0 \longrightarrow \Omega^{\otimes 2} \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow 0,$$

trivialised at x.

Continuing gives a projective system of extensions

$$0 \longrightarrow \Omega^{\otimes n} \otimes_K \mathcal{O}_X \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow 0.$$

Next assume that X admits a semistable model. We construct inductively $\overline{K}^\wedge$-valued smooth sheaves $\mathbb{L}_n$ on X, trivialised along the section x, such that for a small enough affine $\mathrm{Spec}(R)$ of the semistable model we have functorially in R

$$\mathbb{L}_n \otimes \overline{R}^\wedge \cong \mathcal{E}_n \otimes_R \overline{R}^\wedge.$$

The $\mathbb{L}_n$ correspond to continuous representations of $\pi_1(X, x)$ on finite dimensional $\overline{K}^\wedge$-vector spaces, whose restriction to $s(\mathrm{Gal}(\overline{K}/K))$ is the base-extension (from K to $\overline{K}^\wedge$) of the direct sum of all $\Omega^{\otimes m}$, $0 \leq m \leq n$. Furthermore they are extensions

$$0 \longrightarrow \Omega^{\otimes n} \otimes_K \overline{K}^\wedge \longrightarrow \mathbb{L}_n \longrightarrow \mathbb{L}_{n-1} \longrightarrow 0.$$

Start with the constant $\mathbb{L}_0 = \overline{K}^\wedge$. Given $\mathbb{L}_{n-1}$, the cohomology

$$H^1(X \otimes_K \overline{K}, \mathcal{H}om(\mathbb{L}_{n-1}, \overline{K}^\wedge)) = \mathcal{H}^1(X, \mathcal{H}om(\mathbb{L}_{n-1}, \overline{K}^\wedge))$$

has a Hodge-Tate decomposition, which (because of the isomorphisms $\mathbb{L}_n \otimes \overline{R}^\wedge = \mathcal{E}_n \otimes_R \overline{R}^\wedge$) is equal to

$$H^1(X, \mathcal{H}om(\mathcal{E}_{n-1}, \mathcal{O}_X)) \otimes_K \overline{K}^\wedge \oplus H^0(X, \mathcal{H}om(\mathcal{E}_{n-1}, \omega_{X/K})) \otimes_K \overline{K}^\wedge(-1).$$

If we identify the etale H^1 with isomorphism classes of smooth sheaves over $X \otimes_K \overline{K}$, the second direct summand measures the "local" extension obtained after tensoring with $\overline{R}^\wedge$. Thus to the first direct summand there corresponds a universal "locally trivial" extension $\mathbb{L}_n$ of the type as above, trivialialised at x. It is unique up to unique isomorphism, thus $\mathrm{Gal}(\overline{K}/K)$-invariant, and we are done. Now the $\mathbb{L}_n$

correspond to representations of the algebraic group G which must factor over H, and the corresponding representation of the Lie-algebra $\mathbf{h}$ on the projective limit is isomorphic to that on the completed enveloping algebra, hence faithful. As we know how $s(\mathrm{Gal}(\overline{K}/K))$ acts on it we derive that $\mathbf{h}$ is generated by its Galois invariants, so s is Hodge-Tate.

6. *Remark.*— a) Suppose

$$s' : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

is another section, and consider the extension

$$0 \longrightarrow \mathrm{gr}^1_Z(\mathbf{h}) \longrightarrow \mathbf{h}/Z^2(\mathbf{h}) \longrightarrow \mathrm{gr}^0_Z(\mathbf{h}) \longrightarrow 0,$$

with $\mathrm{Gal}(\overline{K}/K)$ acting via s'. As canonically

$$\mathrm{gr}^0_Z(\mathbf{h}) = \mathrm{Hom}_K(\Omega, \overline{K}^\wedge),$$
$$\mathrm{gr}^1_Z(\mathbf{h}) = \Lambda^2 \mathrm{gr}^1_Z(\mathbf{h}) = \mathrm{Hom}_K(\Lambda^2\Omega, \overline{K}^\wedge),$$

this corresponds to a class

$$c(s, s') \in \mathrm{Hom}_K(\Lambda^2\,\Omega, \Omega) \otimes_K H^1(\mathrm{Gal}(\overline{K}/K, \overline{K}^\wedge)),$$

which must vanish if s' is also Hodge-Tate.

One can compute $c(s, s')$ as follows:

Computing modulo $Z^1(G)$ the difference of s and s' defines a cohomology-class in

$$H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{gr}^0_Z(G)) = H^1(\mathrm{Gal}(\overline{K}/K, T_p(J) \otimes \mathbb{Q}_p)).$$

By the Hodge-Tate decomposition of $T_p(J) \otimes \mathbb{Q}_p$ this maps to a class

$$c^*(s, s') \in \Omega^{\mathrm{dual}} \otimes_K H^1(\mathrm{Gal}(\overline{K}/K, \overline{K}^\wedge)).$$

Mochizuki shows (by an easy calculation) that up to a normalising factor two the second class maps to the first if we let Ω^{dual} act on $\Lambda^2\,\Omega$ by interior multiplication. Especially if $g \geq 2$ then s' is Hodge-Tate up to level 2 if and only if $c^*(s, s')$ vanishes.

7. *Remark.*— b) For two such curves X and Y, and an open homomorphism of extensions

$$\alpha : \pi_1(X) \longrightarrow \pi_1(Y)$$

the composition $\alpha \circ s$ of α with a Hodge-Tate section

$$s : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(X)$$

is again Hodge-Tate. This follows because the induced map on Lie-algebras $\mathbf{h}_X \to \mathbf{h}_Y$ is surjective.

8. *Remark.*— c) With similar methods one can show that for semistable X the Lie-algebra $\mathbf{g}$ is (with the action *via* a section s coming from a point) the projective limit of log-crystalline representations of $\mathrm{Gal}(\overline{K}/K)$, as defined by J.-M. Fontaine.

5. COHOMOLOGICAL COMPUTATIONS

Suppose X is a projective smooth geometrically irreducible curve over a field K of characteristic zero (or at least different from p). Assume also that the genus of X is at least one. The p-adic etale cohomologies $H^*(X, \mathbb{Z}_p)$ and $H^*(X \otimes_K \overline{K}, \mathbb{Z}_p)$ then coincide with the continuous cohomologies of the corresponding fundamental groups. They are related to the projective limits of cohomologies modulo p^n via short exact sequences

$$0 \to \mathrm{proj.lim}^{(1)}\, H^{m-1}(X, \mathbb{Z}_p/p^n.\mathbb{Z}) \to H^m(X, \mathbb{Z}_p) \to \mathrm{proj.lim}\, H^m(X, \mathbb{Z}/p^n.\mathbb{Z}) \to 0.$$

Here the derived projective limit $\mathrm{proj.lim}^{(1)}$ vanishes if the projective system satisfies a Mittag-Leffler condition, which will be the case in all our applications. As usual $\mathbb{Q}_p$-adic cohomology is defined by tensoring with that field.

Let $J^{(1)}$ denote the homogeneous space (under the Jacobian J of X) which classifies isomorphism classes of line-bundles of degree one on X. Note that a K-rational point of $J^{(1)}$ does not necessarily correspond to an actual line bundle of degree 1. (See below for some remarks about that topic.)

The fundamental group $\pi_1(J^{(1)})$ is the quotient of $\pi_1(X)$ under the commutator subgroup of $\pi_1(X \otimes_K \overline{K})$, with the surjection induced by the canonical map from X to $J^{(1)}$. As before denote by $\pi_1(J^{(1)})^{(p)}$ its quotient where the geometric fundamental group is replaced by its maximal pro-p-quotient. This is an extension of $\mathrm{Gal}(\overline{K}/K)$ by $T_p(J)$. Let us study sections s_{ab} of the projection $\pi_1(J^{(1)})^{(p)} \to \mathrm{Gal}(\overline{K}/K)$. Assume that X has genus $g \geq 2$, and note that the adjoint action of $\pi_1(X)$ on $H/Z^2(H)$ and $\mathbf{h}/Z^2(\mathbf{h})$ factors through $\pi_1(J^{(1)})^{(p)}$. Thus we define s_{ab} to be Hodge-Tate if $\mathbf{h}/Z^2(\mathbf{h})$ is generated by its invariants under $s_{ab}(\mathrm{Gal}(\overline{K}/K))$. This is the case if s is defined by a K-rational point of X. In fact it suffices if s_{ab} comes from a K-rational point of $J^{(1)}$:

We may extend K and assume that s_{ab} is defined by a divisor of degree one on X, and that this divisor is a linear combination of K-rational points of X. But each such point defines a section s_{ab} which is Hodge-Tate. Also for one such Hodge-Tate section s_{ab}, another section is Hodge-Tate if and only if the difference is a Hodge-Tate class in $H^1(\mathrm{Gal}(\overline{K}/K), T_p(J))$, that is lies in the kernel of the map to $H^1(\mathrm{Gal}(\overline{K}/K), \Omega \otimes_K \overline{K}^{\wedge})$. This implies our claim, and also the well known fact that line-bundles of degree zero define Hodge-Tate classes in $H^1(\mathrm{Gal}(\overline{K}/K), T_p(J))$.

One may ask for a converse. Some approximation of such a converse holds if K is a local field, that is a finite extension of $\mathbb{Q}_p$. Namely if $J^{(m)}$ denotes the homogeneous J-space classifying line-bundles of degree n, multiplication by n induces

$$n. : J^{(1)} \longrightarrow J^{(n)}.$$

The induced map on geometric fundamental-groups is also multiplication by n.

9. PROPOSITION.— *Assume that K is a finite extension of* $\mathbb{Q}_p$, *X a curve of genus* $g \geq 2$, *and*

$$s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m)})^{(p)}$$

a section which is Hodge-Tate. Then there exists a positive integer n prime to p, such that

$$n \,.\, s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m\cdot n)})^{(p)}$$

is induced by a point in $J^{(m\cdot n)}(K)$.

Proof.— The key fact is of course the theorem of Bloch and Kato ([BK], example 3.11) that up to torsion $J(K)$ is isomorphic to the Hodge-Tate classes in $H^1(\mathrm{Gal}(\overline{K}/K), T_p(J))$. If $J^{(m)}$ has a K-rational point this implies the assertion, except for the fact that n can be chosen prime to p. However replacing m by a multiple we can find such a rational point. If n is divisible by p, we note that

$$p\cdot : J^{(n/p)} \longrightarrow J^{(n)}$$

is a principal homogeneous space with group $J[p]$ (p-division-points). The obstruction to lift a K-rational point $x \in J^{(n)}(K)$ to $J^{(n/p)}$ is a class in

$$H^1(\mathrm{Gal}(\overline{K}/K), J[p]) = H^1(\mathrm{Gal}(\overline{K}/K), T_p(J)/p\,.\,T_p(J)).$$

It coincides with the obstruction to lift the corresponding section

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(n)})$$

to $\pi_1(J^{(n/p)})$ and lifts of sections correspond uniquely to lifts of points (the geometric fibre over x is $\pi_1(J^{(n)})/\pi_1(J^{(n/p)})$, with $\mathrm{Gal}(\overline{K}/K)$-action *via* the section corresponding to x). Thus if we can lift sections we can lift points, and this allows to remove factors p from n.

The conclusion of the proposition merits its own name. It works for an arbitrary ground field K of characteristic zero.

10. DEFINITION.— *A section*

$$s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m)})^{(p)}$$

is called geometric up to torsion if there exists a positive integer n *such that*

$$n\,.\,s_{ab} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \pi_1(J^{(m\cdot n)})^{(p)}$$

comes from a point in $J^{(m\cdot n)}(K)$.

¿From the previous it follows that we may always choose n prime to p, and that it suffices if the condition holds over a finite extension of K (use a trace argument).

In fact we would like this point in $J^{(m\cdot n)}(K)$ to be represented by an actual line-bundle $\mathcal{L}$ on X. There is an obstruction in the Brauer-group of K which has to vanish. Namely there exists over a Galois-extension of K a line-bundle $\mathcal{L}$ which represents the point in $J^{(m\cdot n)}$ and which is isomorphic to all its Galois-conjugates. However it may happen that these isomorphisms cannot be arranged to form a descent-datum, thus the obstruction. Nevertheless one checks that it is annihilated by the dimensions of cohomology-groups $H^i(X, \mathcal{L})$ and also by the Euler characteristic, that is by $m \cdot n + 1 - g$. Especially if p divides $g - 1$ and if m is prime to p, we may replace n by a further multiple (still prime to p) to get that $n \cdot s_{ab}$ is represented by an actual line-bundle $\mathcal{L}$, of degree $m \cdot n$.

The rest of the section becomes a little bit technical. We assume given two curves X_1 and X_2, of genus ≥ 2, and an open map of extensions

$$\alpha : \pi_1(X_1) \longrightarrow \pi_1(X_2),$$

inducing

$$\alpha_{ab} : \pi_1(J_1^{(m)})^{(p)} \longrightarrow \pi_1(J_2^{(m)})^{(p)}.$$

The base field will be an extension K^+ of a local p-adic field K, and the curves as well as the morphism α will be definable over a finite extension K_1 of K. In fact K^+ will also be a p-adic field with integers, V^+ a complete discrete valuation-ring dominating V but with residue field k^+ a function-field in one variable over k.

Recall that sections have the uniform p-adic topology.

11. PROPOSITION.— *Suppose*

$$s_{ab} : \mathrm{Gal}(\overline{K}^+/K^+) \longrightarrow \pi_1(J_1^{(1)})^{(p)}$$

is a section which is geometric up to torsion. Then

$$\alpha_{ab} \circ s_{ab} : \mathrm{Gal}(\overline{K}^+/K^+) \longrightarrow \pi_1(J_2^{(1)})^{(p)}$$

is the p-adic limit of such sections.

Proof.— We may pass to finite extensions of K and K^+ and assume that α, X_1, X_2 are defined over K, and the curves have semistable reduction. Furthermore the assertion holds if we replace K^+ by K, and as we may assume that X_1 has a K-rational point, we have to show the following claim:

α_{ab} preserves up to torsion the images of

$$J(L) \otimes \mathbb{Z}_p \longrightarrow H^1(\mathrm{Gal}(\overline{K}^+/K^+), T_p(J)), \quad (J = J_1, J_2).$$

For this we may replace J_i by the connected component of its Neron-model and then by the Raynaud-extensions G_i [see for example [C], Ch. II,1]. These are the extensions of abelian varieties by tori which define the same p-adic formal schemes as J_i. It is known that α_{ab} preserves $T_p(G_i) \subset T_p(J_i)$ (the quotient is maximal with the property that $\mathrm{Gal}(\overline{K}/K)$ acts *via* a finite unramified quotient), so it is enough to give a good characterisation of the images of $G_i(V^+) \otimes \mathbb{Z}_p$ in $H^1(\mathrm{Gal}(\overline{K}^+/K^+), T_p(G_i))$.

So suppose G is such a semiabelian variety, with associated p-divisible group G_∞. There is a map

$$\lambda : G(V^+) \longrightarrow \mathrm{Ext}_{V^+}(\mathbb{Q}_p/\mathbb{Z}_p, G_\infty) \quad (= \text{extensions of } p\text{-divisible groups over } V^+)$$

"which sends $g \in G(V^+)$ to its p^n-division-points". Also there is a variant of λ with V^+ replaced by k^+, and a reduction map (on both sides). It is classical that we get an isomorphism on the kernels of the reduction maps, that is extensions trivial over k^+ correspond to elements of $G(V^+)$ which are trivial modulo the maximal ideal (these are also R-points of the associated formal group to G_∞). Thus the image of $G(V^+)$ consists of extensions lifting elements of $\lambda(G(k^+))$. Similarly the $\mathbb{Z}_p$-submodule generated by $G(V^+)$ consists of lifts of $\lambda(G(k^+)\otimes\mathbb{Z}_p)$. Now apply this to our situation. By a theorem of Tate α extends to a morphism of p-divisible groups (denoted for simplicity by the same name)

$$\alpha : G_{1,\infty} \longrightarrow G_{2,\infty}.$$

Its reduction modulo the maximal ideal is a homomorphism of p-divisible groups over k. By a variant of a (different) theorem of Tate it lies in $\mathrm{Hom}_k(G_1, G_2)\otimes\mathbb{Z}_p$. Thus

$$\alpha(G_1(k^+)\otimes\mathbb{Z}_p) \subset G_2(k^+)\otimes\mathbb{Z}_p),$$

and α also respects the $\mathbb{Z}_p$-lattices generated by $\lambda(G_i(V^+))$. Finally apply the obvious maps from $\mathrm{Ext}_W(\mathbb{Q}_p/\mathbb{Z}_p, G)$ to $H^1(\mathrm{Gal}(\overline{K}^+/K^+), T_p(G))$ to obtain the assertion.

Let us sketch a proof for the mentioned variant of Tate's theorem:

12. THEOREM (Tate).— *Suppose k is a finite field of characteristic p, G_1 and G_2 semiabelian varieties over k. Then the map*

$$\mathrm{Hom}_k(G_1, G_2)\otimes\mathbb{Z}_p \longrightarrow \mathrm{Hom}_k(G_1(p^\infty), G_2(p^\infty))$$

is an isomorphism.

Proof.— One can check that any homomorphism of p-divisible groups respects the Tate-modules of the subtori (eigenvalues of Frobenius on crystalline cohomology), and that the result holds for tori. Also extensions of an abelian variety A by $\mathbb{G}_m$ are classified by $A^t(k)$ (dual abelian variety) which is a finite abelian group so that any such extension splits up to isogeny. These two remarks allow us to reduce to the abelian case, so that from now on G_1 and G_2 are abelian varieties over k. If

$$\alpha_n : G_1(p^n) \longrightarrow G_2(p^n)$$

denotes the n-th stage of α (a map of p-divisible groups over k), consider the quotient

$$H_n = (G_1 \times G_2)/\mathrm{graph}(\alpha_n)$$

which is also an abelian scheme over k. One then knows that infinitely many H_n's are isomorphic and derives the result (see for example [CF], Ch. V, proof of th. 4.7). One can even adapt that proof to extend theorem 12 to function-fields).

6. PROOF OF THE MAIN RESULT

As before K is a finite extension of $\mathbb{Q}_p$, (with V, π, k, etc.), K^+ an extension, with integers V^+ etc., such that k^+ is a function field in one variable over k. The universal (finite) module of differentials of V^+ is denoted by ω_{V^+}, and let $\omega_{K^+} = \omega_{V^+} \otimes_{V^+} K^+$. Furthermore X should be a geometrically irreducible projective curve over K, of genus $g \geq 2$. A point $x \in X(K^+)$ is called non-constant if it does not lie in $X(\overline{K})$. An equivalent condition is the following:

The pullback (by x) of any global differential form on X defines an element of ω_{K^+}, thus a linear map (the derivative of x) from $\Omega = H^0(X, \omega_{X/K})$ to ω_{K^+}. x is non-constant precisely if this map is not identically zero.

If this is the case, it determines a K^+-point in the projective space $\mathbb{P}(\Omega) = \mathbb{P}^{g-1}$, which is the image of x under the canonical map (embedding if X is not hyperelliptic)

$$X \longrightarrow \mathbb{P}(\Omega).$$

We can recover the derivative from Galois-cohomology, as follows: Let

$$\Delta = \mathrm{Gal}(\overline{K}^+/L) \subset \mathrm{Gal}(\overline{K}^+/K^+)$$

denote the subgroup fixing the compositum L of $\overline{K}$ and the maximal unramified extension of K^+. Note that L contains the maximal tame extension of K^+, so that Δ is a pro-p-group. Then the Galois-cohomology of Δ with values in the p-adic completion of $\overline{V}^+$ is (up to a finite p-power) equal to the tensor product of ω_{V^+} with $W^\wedge(-1)$ (X = integers in L). Furthermore the induced map on fundamental groups $\mathrm{Gal}(\overline{K}^+/K^+) \to \pi_1(X)$ sends Δ into $\pi_1(X \otimes_K \overline{K})$, and induces by pullback a map

$$H^1(X \otimes_K \overline{K}, \overline{K}^\wedge) = \Omega^{\mathrm{dual}} \otimes_K \overline{K}^\wedge \oplus \Omega \otimes_K \overline{K}^\wedge(-1) \longrightarrow \omega_{V^+} \otimes_{K^+} L^\wedge(-1).$$

As it is Galois-invariant it must vanish on the first direct summand $\Omega^{\mathrm{dual}} \otimes_K \overline{K}^\wedge$, and on the second it is the derivative of x. Thus we can recover the derivative of any L-valued point from its induced map on fundamental groups. We also remark that both sides have natural integral structures, and then the p-powers in the denominators of these maps have an upper bound only depending on X, and not even that if X has semistable reduction. Also the association

$$\{\text{maps of extensions } \mathrm{Gal}(\overline{K}^+/K^+) \longrightarrow \pi_1(X)\} \longrightarrow \mathrm{Hom}_K(\Omega, \omega_{V^+} \otimes_{V^+} L^\wedge)$$

is continuous, where on the left the topology is uniform convergence, and on the right the p-adic one. Especially if for a sequence of points $x_n \in X(L)$ the induced maps on fundamental groups converge, and the limit has non-trivial derivative, then the images of the points in $\mathbb{P}(\Omega)$ points converge p-adically to a non-constant limit, and so do the points themselves if X is not hyperelliptic.

13. **Remark.**— If $X \to Y$ is an etale Galois-covering with group G, and if X is hyperelliptic, then so is Y, and G is abelian of exponent 2:

We have to show that any element in G has exponent 2, thus may assume that G is cyclic and thus abelian. The hyperelliptic involution on X is unique, thus commutes with G and induces a hyperelliptic involution on Y. However this involution acts as $-\mathrm{id}$ on the Jacobian of Y and thus on any covering group of an abelian cover. Hence $\mathrm{id} = -\mathrm{id}$ on G.

This remark will be helpful to exclude hyperelliptic curves.

Finally we can formulate and prove the main result:

14. **THEOREM.**— *Suppose K is a finite extension of $\mathbb{Q}_p$, X and Y smooth geometrically irreducible projective curves over K,*

$$\alpha : \pi_1(X) \longrightarrow \pi_1(Y)$$

an open homomorphism of extensions (of $\mathrm{Gal}(\overline{K}/K)$*). Then α is induced from a unique dominant morphism of curves $X \to Y$.*

Proof.— We start with some general remarks. Obviously we may assume that α is surjective. Then α is uniquely determined by its restriction to the geometric fundamental group $\pi_1(X \otimes_K \overline{K})$, as the geometric π_1's have trivial centers. Hence we can always pass to finite extensions of K, and for example assume that X and Y have semistable models. Furthermore passing to (two disjoint) cyclic etale coverings of Y we may assume that Y is not hyperelliptic, and that $g(Y) - 1$ is divisible by p. Also Hodge-Tate theory defines an injective "pullback"

$$\alpha^* : \Omega_Y \longrightarrow \Omega_X$$

(which will be the differential of our map $X \to Y$).

Choose an extension V^+ as before, and a non-constant point $x \in X(K^+)$ such that the pullback of some differential on Y does not vanish in x. If $X^+ = X \otimes_K K^+$ and Y^+ denote the base-changes, then the point x defines a section

$$\mathrm{Gal}(\overline{K}^+/\overline{K}) \longrightarrow \pi_1(X^+) = \text{fibered product of } \pi_1(X) \text{ and } \mathrm{Gal}(\overline{K}^+/K^+),$$

and its composition with α a section s into $\pi_1(Y^+)$. If we project further to $\pi_1(J_Y^{(1)})^{(p)}$ the corresponding section is geometric up to torsion. It thus follows that Y^+ admits a line-bundle of degree prime to p. Of course this already follows from the existence of a semistable model.

But any characteristic open subgroup $H \subset \pi_1(Y \otimes_K \overline{K})$ extends to an open subgroup $H \cdot s(\mathrm{Gal}(\overline{K}^+/K))$ of $\pi_1(Y^+)$, which is the fundamental group of a cover $Y^+(H)$ of Y^+. Then s defines a section $s_{ab}(H)$ for $J^{(1)}$ which is limit of sections geometric up to torsion :

If $X^+(H)$ denotes the covering of X^+ corresponding to the subgroup $\alpha^{-1}(H) \cdot s$ $(\mathrm{Gal}(\overline{K}^+/K^+))$ of $\pi_1(X^+)$, then x lifts to a rational point $x(H) \in X^+(H)(K^+)$ (which in turn defines s). Furthermore everything is definable over a finite extension of K, thus the assertion by proposition 11.

Especially each $Y(H)$ admits a line-bundle $\mathcal{L}(H)$ of degree prime to p, and thus contains a rational point in an extension field of K^+ of degree prime to p which is contained in L. Thus a point $y^*(H) \in Y^+(H)(L)$, and a corresponding section (well defined up to H-conjugation)

$$s^*(H) : \mathrm{Gal}(\overline{L}/L) \longrightarrow H \cdot s(\mathrm{Gal}(\overline{K}^+/K^+)) \subset \pi_1(Y^+)$$

lifting s. As H decreases the $s^*(H)$ must converge to s, and thus the Y-projections of the $y^*(H)$ converge to a point $y \in Y(L^\wedge)$ (considering the induced map on Tate-modules, which determines the points as Y is not hyperelliptic). Also y will take values in K^+. Similarly for fixed H we consider the $Y(H)$-projections of all $y^*(H')$ (for $H' \subset H$), which have a limit $y(H)$. These $y(H) \in Y(H)(L^\wedge)$ form a compatible projective system of K^+-points. Considering induced maps on Tate-modules one sees that on $\mathrm{Gal}(\overline{K}^+/K^+)$ they induce the section s (as the geometric fundamental group has trivial center), and are non-constant. Especially Y is equal to the smallest K-scheme containing the point $y(H) \in \mathbb{P}^{g-1}(K^+)$.

By Hodge-Tate theory the map α induces an injective

$$\alpha^* : H^0(Y, \omega_Y) \longrightarrow H^0(X, \omega_X)$$

which maps the linear form given by evaluation on x to a multiple for the corresponding form for y. For any element f of the symmetric algebra of $H^0(Y, \omega_Y)$ which vanishes on Y and thus on $y \in Y(K^+)$, its pullback $\alpha^*(f)$ thus vanishes on x and also X. That is α^* defines a map (first birational, but then regular) of (canonically embedded) curves $X \to Y$.

By the same reasoning we lift to a compatible system of pointed maps from $X^+(H)$ to $Y^+(H)$. On fundamental groups the induced transformation α' : $\pi_1(X^+) \to \pi_1(Y^+)$ thus has the property that it sends $\alpha^{-1}(H)$ into H, for all characteristic open $H \subset \pi_1(Y \otimes_K \overline{K})$. It is thus equal to the composition of α with an endomorphism of $\pi_1(Y)$ which respects all such H, and which induces (Hodge-Tate) the identity on global differentials of $Y(H)$. As it also induces the identity on the covering group (over $\overline{K}^+$) of $Y^+(H) \to Y$ it is the identity on $\pi_1(Y \otimes_K \overline{K})$ and thus on all of $\pi_1(Y)$.

15. *Remark.*— a) The proof needs only the maximal pro-p-quotient of the geometric fundamental group, that is any open homomorphism of the extensions of $\mathrm{Gal}(\overline{K}/K)$ of these groups comes from a dominant morphism. Even quotients by suitable commutator-subgroups suffice (but not in the descending central series).

¿From this one derives again the previous result, by applying it to all coverings like $X(H)$.

b) One can treat open hyperbolic curves: Suppose U is such a curve, with compactification X. Replace the Jacobian J_X by the generalised Jacobian which classifies line-bundles on X with a trivialisation on the divisor at infinity $X - U$, and ω_X by differentials with simple poles along this divisor. Then for two curves U_1, U_2 any open homomorphism (of extensions) between $\pi_1(U)$'s comes first from a map between compactifications X. As etale coverings of the open curve U_2 (allowed to be ramified along the boundary) pull back to such coverings of U_1, this maps U_1 to U_2.

c) For complete curves (but not for affines) one may replace the condition "α is open" by "α has non-zero degree", that is the induced map

$$\alpha^* : H^2(Y \otimes_K \overline{K}, \mathbb{Q}_p) = H^2(\pi_1(Y \otimes_K \overline{K}, \mathbb{Q}_p)) \longrightarrow H^2(X \otimes_K \overline{K}, \mathbb{Q}_p)$$

is an isomorphism. This property is inherited by all finite coverings.

d) The result over K implies it over any subfield. Namely suppose one has found a K-rational point of the Hom-scheme $\mathbf{Hom}(X, Y)$. If the induced map on Tate-modules is defined over this subfield, then so is the map. Especially this applies to number fields.

e) The theorem extends to finitely generated extensions of $\mathbb{Q}_p$.

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