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S E M I N A I R E

D ' A N A L Y S E F O N C T I O N N E L L E

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ON OPERATORS FIXING COPIES OF c_o AND ℓ_∞

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In this seminar, we report on a part of a joint work with W.B. Johnson and T. Figiel [1] concerning the structure of non-weakly compact operators on Banach lattices. First, we recall the following two fundamental theorems.

Theorem (A) : (A. Pełczynski [4]) . A non-weakly compact operator from a C(K)-space into any Banach space must preserve a copy of c_0 ; that is there exists a subspace of C(K), isomorphic to c_0, on which T acts as an isomorphism.

Theorem (B) : (H. Rosenthal [5]). If K is a σ-Stonian compact space, then every non-weakly compact operator from C(K) into any Banach space must preserve a copy of ℓ_∞ .

Our goal is to see to which extent, one can replace C(K) in theorems (A) and (B) by a larger class of Banach spaces.

§ I. NON WEAKLY COMPACT OPERATORS :

The existence of the James space [2] eliminates the possibility of replacing C(K) in theorem (A) by any Banach space not containing a subspace isomorphic to ℓ_1, since c_0 and ℓ_1 do not embed in this space and yet it is not reflexive. However, the result does hold for the idendity operator acting on a Banach lattice since if the latter is not reflexive, then it must contain a sublattice isomorphic either to ℓ_1 or c_0 [3]. A natural problem is then to check if the result holds for any operator or equivalently if whether in theorem (A), C(K) can be replaced by any Banach lattice not containg ℓ_1.

Surprisingly, Pełczynski's theorem does not extend even to this case as we show in the following counterexample.

Example (1) : For every p , $1 \leq p < \infty$, there exists a Banach lattice

X_p and a lattice homomorphism T_p from X_p onto c_o so that

(i) T_p is strictly singular for each p, $1 \leq p < \infty$

(ii) X_p contains no subspace isomorphic to ℓ_1 for p, $1 < p < \infty$.

We first give the idea. Let c be the space of converging sequences and set $X = \ell_1(c)$; that is the space of doubly-indexed sequences $a = (a_{i,j})$, where $i = 1,2,\ldots$; $j = 1,2,\ldots, \omega$ such that

$$\lim_{j\to\infty} a_{i,j} = a_{i,\omega} \text{ for } i = 1,2,\ldots$$

and

$$\|a\|_X = \sum_{i=1}^{\infty} \sup_j |a_{i,j}| < \infty$$

Define the norm one operator $T : X \longrightarrow c_o$ by

$$Ta = (a_{i,\omega})_{i=1}^{\infty} \quad .$$

Clearly, T is weakly compact and X contains lots of sublattices isomorphic to ℓ_1. However, we can turn T into a non-weakly compact operator by adding to the unit ball of X vectors (f_n) for which (Tf_n) is not weakly compact in c_o and taking for the new unit ball in X the absolute convex solid hull of the old unit ball and the f_n's, in order to get a normed lattice. The completion of the resulting space probably still contains ℓ_1 complementably, but we can kill them by taking the p-convexification of the space for some $1 < p < \infty$.

Letting X and T be defined as above we define $f_n \in X$ by

$$(f_n)_{i,j} = \begin{cases} 1 & , \text{ if } i \leq n \leq j \\ 0 & , \text{ otherwise} \end{cases} \quad .$$

Clearly

$$Tf_n = \sum_{i=1}^{n} e_i$$

where $(e_i)_{i=1}^{\infty}$ is the unit vector basis for c_o.

Let X_o be the dense sublattice of X consisting of those vectors $a = (a_{i,j})$ whose rows are eventually zero ; i.e., for some n, $a_{i,j} = 0$ for all $i \geq n$ and all $j = 1,2,\ldots,\omega$.

Let $\|\cdot\|_1$ be the greatest lattice norm on X_o such that

$$\|f_n\|_1 \leq 1 \quad , \quad \|x\|_1 \leq \|x\|$$

for $n = 1,2,...$ and all $x \in X_o$. That is, $\|\cdot\|_1$ is the gauge of the closed absolutely convex solid hull of the unit ball of X_o and the sequence (f_n). Thus $\|x\|_1 < 1$ if and only if there are $g \in X_o^+$ and eventually zero sequence $s_1, s_2, ...$ in $\mathbb{R}^+$ so that

$$|x| \leq g + \sum_{i=1}^{\infty} s_i f_i \text{ and}$$

$$\|g\|_X + \sum_{i=1}^{\infty} s_i < 1.$$

Let $(X_1, \|\cdot\|_1)$ be the completion of $(X_o, \|\ \|_1)$ and for $1 < p < \infty$, let $(X_p, \|\cdot\|_p)$ be the completion of the p-convexification of $(X_o, \|\cdot\|_1)$; that is, for $x \in X_o$,

$$\|x\|_p = \|\ |x|^p\|_1^{1/p} .$$

(See chapter 1.e in [3] for a discussion of p-convexity.)

We claim that $\|T\|_p = 1$ for every $1 \leq p < \infty$; i.e., T has norm one as an operator from $(X_o, \|\cdot\|_p)$ into c_o. This claim is a consequence of the observation that for each i and j, the coordinatewise evaluation functional on X_o defined by $a \longrightarrow a_{i,j}$ has $\|\cdot\|_p$-norm one. (For p=1 this is clear, because $|f_n| \leq 1$ for each n, the general case then follows from the definition of $\|\cdot\|_p$.)

Since X_o is dense in X_p, T extends to a norm one operator, T_p, from X_p into c_o. Note also that T_p is a lattice homomorphism and for every choice of signs $\pm$ and $n = 1,2,...$; there is $g \in X_o$, $|g| \leq f_n$, so that $Tg = \sum_{i=1} \pm e_i$ which shows that T_p is a quotient map.

In the sequel, we shall say that a sequence $(x_n)_{n=1}^{\infty}$ in X_p is a <u>special c_o-sequence</u> if there exist $K < \infty$ and integers $i_1 < i_2 < ...$ such that for every $n = 1,2,...,$

$$x_n \geq 0 \quad , \quad \|x_n\|_p = 1$$

$$(x_n)_{i,j} = 0 \text{ if } i \neq i_n$$

$$\left\| \sum_{k=1}^{n} x_k \right\|_p < K.$$

Note that if $1 \leq i < \infty$ and $x \in X_o$ with

$$x_{\ell,j} = 0 \text{ for } \ell \neq i,$$

then

$$\|x\|_X = \sup_j |x_{i,j}| \;;$$

consequently,

$$\|x\|_p = \sup_j |x_{i,j}|$$

for $p = 1$ and hence for all $1 \leq p < \infty$. In particular, all the terms of a special c_o-sequence lie in X_o.

We now show that X_1 contains no special c_o-sequence.

If such a sequence $(x_n)_{n=1}^{\infty}$ exists in X_1, pick for each n an index $j_n < \omega$ so that

$$(x_n)_{i_n,j_n} \geq 1/2 \sup_j (x_n)_{i_n,j} = 1/2\|x_n\|_1 = 1/2 .$$

By passing to a subsequence, we may assume that $i_{n+1} > j_n$ for each n.

Given an integer N, find $g \in X_o^+$ and $(s_i)_{i=1}^{\infty} \subseteq \mathbb{R}^+$ so that

$$\sum_{n=1}^{N} x_n \leq g + \sum_{i=1}^{\infty} s_i f_i ,$$

$$\|g\|_X + \sum_{i=1}^{\infty} s_i < \left\| \sum_{n=1}^{N} x_n \right\|_1 + 1.$$

Evaluating both sides of the first inequality at (i_n,j_n), we get

$$1/2 \leq (g)_{i_n,j_n} + \sum_{i=i_n}^{j_n} s_i \text{ for } n = 1,2,\ldots,N.$$

It follows that

$$N/2 \leq \sum_{n=1}^{N} (g)_{i_n,j_n} + \sum_{n=1}^{N} \sum_{i=i_n}^{j_n} s_i \leq$$

$$\leq \|g\|_X + \sum_{i=1}^{\infty} s_i < \| \sum_{n=1}^{N} x_n \|_1 + 1$$

which for large N contradicts the inequality

$$\| \sum_{n=1}^{N} x_n \|_1 < K.$$

To prove (i), suppose that $T_p : X_p \longrightarrow c_o$ is an isomorphism on an infinite dimensional subspace E of X_p which we way assume is isomorphic to c_o. Let $(z_n)_{n=1}^{\infty}$ be a normalized basis for E which is K-equivalent to the unit vector basis of c_o ; since X_o is dense in X_p, we can assume that each z_n lies in X_o .

Since

$$\|T_p z_n\| = \max_i |(z_n)_{i,\omega}| \text{ and}$$

$$\lim_{n\to\infty} (z_n)_{i,\omega} = 0 \text{ for each } i \in \mathbb{N} ,$$

we can find a sequence $i_1 < i_2 < \cdots$ and $\delta > 0$ such that for all n,

$$|(z_n)_{i_n,\omega}| > \delta .$$

Define the band projection $P_n : X_p \longrightarrow X_p$ by

$$(P_n x)_{i,j} = \begin{cases} x_{i,j} , & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

By the diagonal principle (cf. p.20 in [2]) it follows that the disjoint sequence $(P_{i_n} z_n)_{n=1}^{\infty}$ is K/δ-equivalent to the unit vector basis of c_o. Consequently,

$$y_n = \| P_{i_n} z_n \|_p^{-1} | P_{i_n} z_n |$$

is a special c_o-sequence in X_p and hence the sequence $x_n = y_n^p$ is a special c_o-sequence in X_1, which is a contradiction.

To prove (ii), note that if E is a subspace of X_p isomorphic to ℓ_1, and if $S_m X_p = \sum_{i=1}^{m} P_i X_p$ determines the natural Schauder decomposition of X_p, then $S_m|_E$ cannot be an isomorphism for any m because $S_m X_p$ is isomorphic to c_o. Thus there exists a normalized sequence $(x_n)_{n=1}^{\infty}$ in E which is equivalent to the unit vector basis for ℓ_1 and a disjoint sequence $(y_n)_{n=1}^{\infty}$ in X_o so that

$$\lim_{n\to\infty} \|x_n - y_n\|_p = 0 .$$

It follows that the sublattice of X_p generated by $(y_n)_{n=1}^{\infty}$ is isomorphic to ℓ_1, which is impossible for $p > 1$ because X_p is p-convex. □

§ II. <u>OPERATORS WHOSE ADJOINT ARE NOT WEAK*-SEQUENTIALLY COMPACT</u> :

To study the extensions of theorem (B), we note first that if K is σ-Stomian, then C(K) is a Grothendieck space, that is the weak-star sequential convergence in its dual coincide with the weak convergence. The problem then reduces to the study of the structure of operators whose adjoints are not weak-star sequentially compact and whose domain is a Banach lattice which contains no complemented copy of ℓ_1. The first theorem reduces the problem to C(K)-spaces, where much is known.

Given any u in the positive cone L^+ of a Banach lattice L, denote by L_u the (not necessarily closed) ideal generated by u. The canonical injection from L_u into L is denoted by j_u or just j if there is no ambiguity. If we put the natural norm on L_u, defined by

$$\|x\|_u = \inf \{\lambda > 0 : |x| \leq \lambda u\}$$

then $(L_u, \|\cdot\|_u)$ is an abstract M-space with unit u and hence is isometrically isomorphic to a C(K) space by Kakutani's Theorem. The operator $j_u : (L_u, \|\cdot\|_u) \longrightarrow L$ obviously has norm $\|u\|$.

<u>Theorem 2</u> : Let L be a Banach lattice which does not contain a copy of ℓ_1 as a sublattice and let T be an operator from L into a Banach space X such that $T^* \mathrm{Ball}(X^*)$ is not weak* sequentially compact. Then there exists $u \in L^+$ so that $(Tj_u)^* \mathrm{Ball}(X^*)$ is not weak* sequen-

tially compact.

To prove the theorem we will need a few lemmas. Given an infinite subset of $\mathbb{N}$, denote by $[M]$ the set of all infinite subsets of M. Given a Banach space L and a bounded sequence (f_n) in L^*, we define for $x \in L$ and $M \in [\mathbb{N}]$

$$\alpha_M(x) = \limsup_{m \in M} f_m(x) - \liminf_{m \in M} f_m(x) .$$

Note that

$$\alpha_M(x) \leq 2 \sup_{m \in M} \|f_m\| \, \|x\|$$

and there exists $P \in [M]$ so that

$$\left|\lim_{p \in P} f_p(x)\right| \geq 1/2 \, \alpha_M(x) .$$

Given $A \subseteq L$, define

$$\alpha_M(A) = \sup\{\alpha_M(x) : x \geq 0, \|x\| \leq 1, x \in A\}$$

$$\beta_M(A) = \inf\{\alpha_P(A) : P \in [M]\} .$$

<u>Lemma (3)</u> : Let L be a Banach space and let (f_n) be a bounded sequence in L^*. If $A \subseteq \text{Ball}(L)$ and $M \in [\mathbb{N}]$, then either $\beta_P(A) > 0$ for some $P \in [M]$ or there exists $P \in [M]$ such that $(f_p)_{p \in P}$ converges pointwise on A.

<u>Proof</u> : If $\beta_P(A) = 0$ for all $P \in [M]$, we can recursively define infinite sets $M \supseteq P_1 \supseteq P_2 \supseteq ..$ so that $\alpha_{P_n}(A) < \frac{1}{n}$. If P is a diagonal sequence with respect to the P_n's, then $\alpha_P(A) = 0$; i.e., $(f_p)_{p \in P}$ converges on A.

From lemma (3) it follows that if L is a Banach lattice and $(f_n) \subseteq \text{Ball}(L^*)$ has no weak* convergent subsequence, then we may assume, by passing to a subsequence of (f_n) that $\beta_{\mathbb{N}}(L^+) > 0$.

To prove Theorem 2, we fix a sequence $(f_n) \subseteq T^*\text{Ball}(X^*)$ with $\sup_n \|f_n\| \leq 1$ so that $\beta_{\mathbb{N}}(L^+) > 0$. We assume that $\beta_M(L_x) = 0$ for all $x \in L^+$ and $M \in [\mathbb{N}]$ since this is the case if $(j_x^* f_m)_{m \in M}$ has a subsequence which converges weak* in L_x^*. The conclusion that this set-up implies that L must contain a disjoint positive sequence equivalent to the unit vector basis of ℓ_1 is an immediate consequence of the next two lemmas. Lemma (4), produces an "almost disjoint" sequence in $\text{Ball}(L^+)$ which, by Lemma (5), has a subsequence which is a small perturbation of a disjoint ℓ_1 sequence.

Lemma (4) : Suppose that L is a Banach lattice, $(f_n) \subseteq \text{Ball}(L^*)$, $\alpha_{\mathbb{N}}(L^+) > \delta > 0$, $\beta_M(L_x) = 0$ for all $M \in \{\mathbb{N}\}$ and $x \in L^+$, and $\varepsilon_n \downarrow 0$. Then there exists $f \in$ weak* closure (f_n) and $(y_n) \subseteq \text{Ball}(L^+)$ so that for each $n = 1,2,\ldots,$

(i) $$\left\| \left(\sum_{i=1}^{n-1} y_i \right) \wedge y_n \right\| < \varepsilon_n$$

(ii) $$|f(y_n)| \geq \delta/2 .$$

Proof : By induction we construct a sequence $(y_n) \subseteq \text{Ball}(L^+)$ and $(M_n) \subseteq [\mathbb{N}]$ to satisfy for each $n = 1,2,\ldots$ condition (i) and

(iii) $$M_{n+1} \subseteq M_n$$

(iv) $$|f_m(y_n)| > \delta/2 \text{ for all } m \in M_n .$$

Having done this, we simply let f be any element of $\text{Ball}(L^*)$ which is a weak* cluster point of $(f_k)_{k \in M_n}$ for each $n = 1,2,\ldots$.

Choosing $y_1 \in \text{Ball}(L^+)$ so that $\alpha_{\mathbb{N}}(y_1) > \delta$, we have that

$$\limsup_{m \in \mathbb{N}} |f_m(y_1)| > \delta/2$$

so that we can choose $M_1 \in [\mathbb{N}]$ to satisfy (iv) for $n = 1$.

Having defined $(M_n)_{n=1}^N$ and $(y_n)_{n=1}^N$ to satisfy (i), (iii), and (iv) for $n \leq N$, we pick $M \in [M_N]$ so that

$$\alpha_M([0, \sum_{i=1}^N y_i]) = 0$$

and choose $z \in \text{Ball}(L^+)$ so that $\alpha_M(z) > \delta$. Define

$$y_{N+1} = z - z \wedge \left(\varepsilon_{N+1}^{-1} \sum_{i=1}^{N} y_k\right) .$$

Since

$$\alpha_M \left(z \wedge \varepsilon_{N+1}^{-1} \sum_{i=1}^{N} y_i \right) = 0.$$

we have that

$$\alpha_M(y_{N+1}) = \alpha_M(z) > \delta .$$

Thus we can choose $M_{N+1} \in [M]$ so that for all $m \in M_{N+1}$,

$$|f_m(y_{N+1})| > \delta/2 .$$

To check (i), just note that if $z, x \in L^+$ and $\lambda \in \mathbb{R}^+$, then

$$(z - z \wedge \lambda x) \wedge x = (z - \lambda x)^+ \wedge x \le \lambda^{-1} z. \qquad \square$$

Lemma (5) : Suppose that L is a Banach lattice, $f \in \mathrm{Ball}(L^*)$, $(y_n) \subseteq \mathrm{Ball}(L^+)$, and $0 < \delta < \delta + \varepsilon$. Suppose that for each $n = 1,2,\ldots,$ $f(y_n) \ge \delta + \varepsilon$ and $\lim_{k\to\infty} \|(\sum_{i=1}^{n} y_i) \wedge y_k\| = 0$. Then there is a subsequence $(y_{n(i)})$ of (y_n) and a disjoint sequence (x_i) in L^+ with $x_i \le y_{n(i)}$ so that for each $i = 1,2,\ldots$

$$\|y_{n(i)} - x_i\| < 4^{-i+1}\varepsilon .$$

Consequently, $|f(x_i)| > \delta$ for each $i = 1,2,\ldots,$ and hence (x_i) is $1/\delta$-equivalent to the unit vector basis for ℓ_1 and $[x_i]$ is $1/\delta$-complemented in L.

Proof : Assume, by passing to a subsequence of (y_n), that for $n = 1,2,\ldots$

(a) $\|y_{n+1} \wedge \sum_{i=1}^{n} y_i\| < 4^{-n}\varepsilon$

We define by recursion a double sequence $(y_{n,k})_{n=1}^{\infty}{}_{k=n}^{\infty} \subseteq \mathrm{Ball}(L^+)$ to satisfy

(b) $(y_{n,k})_{n=1}^{k}$ is disjoint for $k = 1,2,\ldots$

(c) $y_{n,k+1} \le y_{n,k} \le y_n$ for $1 \le n \le k$.

(d) $\|y_n - y_{n,n}\| < 4^{-n}\varepsilon$ for $n = 1,2,\ldots$.

(e) $\|y_{n,k} - y_{n,k+1}\| < 4^{-k}\varepsilon$ for $1 \le n \le k$.

Once this is done, we can in view of (e) set

$$x_n = \lim_{k\to\infty} y_{n,k} ;$$

from (b) and (c) we have that $(x_n)_{n=1}^{\infty}$ is disjoint and $0 \le x_n \le y_n$ for each $n = 1,2,\ldots$. From (d) and (e) we infer that

$$\|y_n - x_n\| < 4^{-n+1}\varepsilon.$$

We turn now to the construction of the $y_{n,k}$'s. Set $y_{1,1} = y_1$. Suppose that $(y_{n,k})_{n=1}^{N}{}_{k=n}^{N}$ has been defined. Let

$$y_{N+1,N+1} = y_{N+1} - y_{N+1} \wedge \left(\sum_{k=1}^{N} y_{N,k} \right)$$

and, for $1 \le n \le N + 1$, set

$$y_{n,N+1} = y_{n,N} - y_{n,N} \wedge y_{N+1} .$$

We leave the verification of (b) - (e) to the reader. □

By applying Theorem (B) we obtain the following two corollaries of Theorem (2).

<u>Corollary (6)</u> : Let L be a σ-complete Banach lattice which does not contain a copy of ℓ_1 as a sublattice. If T is an operator from X into some Banach space Y and T* Ball(Y*) is not weak* sequentially compact, then T preserves a copy of ℓ_∞ .

<u>Proof</u> : By Theorem (2) there is $u \in L^+$ so that $(Tj_u)^*$Ball (Y*) is not weak* sequentially compact and hence not weakly compact. When L_u is represented as C(K) space, K is σ-Stonian because L is σ-complete. Therefore, by Theorem (B) Tj_u, hence also T, preserves a copy of ℓ_∞ . □

<u>Corollary</u> (7) : If L is a σ-complete Grothendieck Banach lattice, then every non-weakly compact operator from L into any Banach space preserves a copy of ℓ_∞ .

Proof : A Grothendieck space cannot contain ℓ_1 (or any other non-reflexive separable space) as a complemented subspace, and non-weakly compact operators from a Grothendieck space have adjoints which are not weak* sequentially compact, and hence Corollary (6) can be applied to any non-weakly compact operator from a σ-complete Grothendieck Banach lattice. □

Problem : It is still unknown whether every non-weakly compact operator from a Grothendieck space into any Banach space preserves a copy of ℓ_∞ .

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