

Extensions of uniserial modules

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ABSTRACT – Let R be any ring and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of R -modules which does not split with A and C uniserial. Then either B is indecomposable or B has a decomposition of the form $B = B_1 \oplus B_2$ where B_1 and B_2 are indecomposable and at least one of them is uniserial.

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1. Introduction

Let us start by recording the third open problem which has been asked in [1, Problem 3, p. 411]. Let A and C be uniserial modules. Give a method for deciding if there are exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with the property that B is also uniserial.

To see that B is not uniserial in general, it suffices to consider the direct sum of two simple modules. We can consider also the Dynkin diagram A_3 with subspace

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orientation $\bullet_1 \rightarrow \bullet_2 \rightarrow \bullet_3$ and the Auslander–Reiten sequence

$$0 \longrightarrow \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus 2 \longrightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \longrightarrow 0.$$

This will also serve as an example to our Remark 2.2. On the other hand if we consider the two other orientations

$$\bullet_2 \longleftarrow \bullet_1 \longrightarrow \bullet_3 \quad \text{and} \quad \bullet_1 \longrightarrow \bullet_3 \longleftarrow \bullet_2$$

we see that B may be the indecomposable module $\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$ or the indecomposable module $\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}$.

Moreover, given an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of finitely generated modules over a left artinian ring with A and C uniserial, we know from [3, Proposition 1.1.2] or [4, Theorem 6.1] that B is either indecomposable or the direct sum of two uniserial modules. On the other hand, the well known Four Terms in the Middle Theorem of Bautista and Brenner [2] (which is generalized in [7] and [8]) states that if A is of finite representation type, then the middle term of an almost split sequence in the category of finitely generated left A -modules has at most four indecomposable summands, and the number four occurs only in the case where one indecomposable summand is projective-injective. Exact sequences of the form $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with A and C uniserial modules have been object of recent studies. See, for instance [5].

Inspired by the above motivations, as a first result of our paper (Theorem 2.1), we characterize the middle term of an exact sequence of R -modules (over any ring R) which does not split and has uniserial end terms. In this way, we may view this result as the analogue of the “Four terms in the middle theorem” which is described above, by replacing four by two, projective-injective by uniserial and Auslander–Reiten sequences by non split exact sequences. We also note (Proposition 2.5) that $\mathbb{Z} \cup \{+\infty, -\infty\}$, $\mathbb{N} \cup \{+\infty\}$, and $(\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$ are the lattices of the submodules of uniserial modules A and C such that there is a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with B is indecomposable. Finally, we construct (Example 2.6 and Example 2.7) uniserial modules A and C such that there exist non split exact sequences $0 \rightarrow A \rightarrow X \rightarrow C \rightarrow 0$, where either X is uniserial or X is an indecomposable module with a decomposable socle and/or a decomposable top. However (Corollary 2.10) subquotients of the extensions of two uniserial modules are again extensions of two uniserial modules. On the other hand extensions of uniserial modules are not necessarily closed under extensions (Example 2.11).

A module M is said to be *uniserial* if the lattice of its submodules is a chain. As usual we say that a module M is *uniform* if the intersection of two nonzero submodules of M is different from zero. A module M is called a *hollow* (or *couniform*) module if $M = M_1 + M_2$ implies $M_1 = M$ or $M_2 = M$.

Let x be a vertex of a quiver Q . Then $S(x)$ will denote the simple representation corresponding to the vertex x . On the other hand, $P(x)$ (resp. $I(x)$) will denote the indecomposable projective (resp. injective) representation corresponding to the vertex x . Sometimes, for short, $S(x)$ is replaced by x . As in [9], pictures of the form

$$\begin{array}{ccccccc} 1 & 2 & 1 & 1\ 2 & 1\ 3\ 5\ 7 & & \\ 2\ ' & 2\ ' & 2\ 3\ ' & 3\ ' & 2\ 4\ 6\ 8\ ' & \cdots & \end{array}$$

denote the composition series of indecomposable modules. Our convention for the composition of paths p, q in the path algebra is as in [1], namely qp stands for q after p whenever the concatenation is defined. For more background on quivers we refer to [1] and [9].

Throughout this paper all modules will be unitary left R -modules over any ring R .

2. Results

We start with the main result of this paper.

THEOREM 2.1. *Let R be any ring and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of R -modules which does not split with A and C uniserial. Then either B is indecomposable or B has a decomposition of the form $B = B_1 \oplus B_2$ with the following properties:*

- i. B_2 is uniserial;
- ii. B_1 is indecomposable;
- iii. there is an exact sequence of the form $0 \rightarrow U \rightarrow B_1 \rightarrow C \rightarrow 0$ with U uniserial;
- iv. there is an exact sequence of the form $0 \rightarrow A \rightarrow B_1 \rightarrow W \rightarrow 0$ with W uniserial;
- v. A and C are not simple.

PROOF. Suppose $B = B_1 \oplus B_2$ with $B_1 \neq 0$ and $B_2 \neq 0$. Then there exist morphisms $f_i: A \rightarrow B_i$ and $g_i: B_i \rightarrow C$ such that $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = (g_1 \ g_2)$.

Since A and C are uniserial, $\text{Ker } f_1 \cap \text{Ker } f_2 = 0$ and $\text{Im } g_1 + \text{Im } g_2 = C$, we have

1. $\text{Ker } f_i = 0$ for some i and $\text{Im } g_j = C$ for some j .

Suppose $\text{Ker } f_1 = 0$. Since f_1 is not an isomorphism, it follows that

2. f_1 is injective but not surjective.

Consequently we deduce from [1, Corollary 5.7, page 22] that

3. g_2 is injective but not surjective.

Putting (1) and (3) together, we conclude that

4. g_1 is surjective but not injective.

Applying again [1, Corollary 5.7, page 22], we deduce from (4) that

5. f_2 is surjective but not injective.

This implies that $B_2 = f_2(A)$ is uniserial. Hence (i) holds. Assume by contradiction that (ii) does not hold. Then we have $B_1 = X \oplus Y$ with $X \neq 0$ and $Y \neq 0$. Consequently $f_1 = \begin{pmatrix} h \\ l \end{pmatrix}$ with $h: A \rightarrow X$ and $l: A \rightarrow Y$. Since $\text{Ker } h \cap \text{Ker } l = 0$ and A is uniserial, we have $\text{Ker } h = 0$ or $\text{Ker } l = 0$. Assume $\text{Ker } h = 0$. Then we have $B = B'_1 \oplus B'_2$ with $B'_1 = X$, $B'_2 = Y \oplus B_2$ and $f = \begin{pmatrix} h \\ l + f_2 \end{pmatrix}$ with h is injective. Hence the first part of the proof shows that $B'_2 = Y \oplus B_2$ is uniserial. This contradiction proves that B_1 is indecomposable, and so (ii) holds. Moreover, by [1, Proposition 5.6, page 22], we have

6. $\text{Ker } g_1 \cong \text{Ker } f_2$ and $\text{CoKer } f_1 \cong \text{CoKer } g_2$.

Since $\text{Ker } f_2 \leq A$, we deduce from (6) that $\text{Ker } g_1$ is a uniserial module. Hence the sequence $0 \rightarrow \text{Ker } g_1 \hookrightarrow B_1 \xrightarrow{g_1} C \rightarrow 0$ satisfies (iii). Finally, since $\text{CoKer } g_2 = C/g_2(B_2)$, we deduce from (6) that $\text{CoKer } f_1$ is a uniserial module. Consequently the sequence $0 \rightarrow A \xrightarrow{f_1} B_1 \rightarrow \text{CoKer } f_1 \rightarrow 0$ satisfies (iv). Since $B_2 \neq 0$, we deduce from (5) and (3) that $0 \subsetneq \text{Ker } f_2 \subsetneq A$ and $0 \subsetneq \text{Im } g_2 \subsetneq C$. Hence (v) holds. The proof is complete. \square

REMARK 2.2. In Theorem 2.1, B_2 may be simple. To see it, it suffices to consider the Auslander–Reiten sequence

$$0 \longrightarrow \begin{matrix} 2 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 2 \oplus 2 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 2 \end{matrix} \longrightarrow 0.$$

Let A, B_1, B_2, C be finitely generated modules as in Theorem 2.1, defined over a left artinian ring R . As we recall in the introduction, this implies ([3, Proposition 1.1.2] or [4, Theorem 6.1]) that also the summand B_1 of B is uniserial. Hence B is the direct sum of two uniserial modules.

With the notation and the hypotheses of Theorem 2.1 we show that A and C can have any finite length > 1 .

EXAMPLE 2.3. Let r, s, m be positive integers with $r > m$ and $s > m$. Then there are a K -algebra R and an exact sequence of R -modules $0 \rightarrow A \rightarrow B_1 \oplus B_2 \rightarrow C \rightarrow 0$ such that A, B_1, B_2, C are uniserial and r, s, m are the lengths of A, C, B_2 , respectively.

CONSTRUCTION. Let $n = r + s - m$ and let R be the K -algebra given by the quiver $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$. Then the R -modules

$$\begin{aligned}
 A &= P(s - m + 1) = \begin{matrix} s - m + 1 \\ \vdots \\ n \end{matrix}, \\
 B_1 &= P(1) = \begin{matrix} 1 \\ \vdots \\ n \end{matrix}, \\
 B_2 &= P(s - m + 1)/P(s + 1) = \begin{matrix} s - m + 1 \\ \vdots \\ s \end{matrix}, \\
 C &= I(s) = P(1)/P(s + 1) = \begin{matrix} 1 \\ \vdots \\ s \end{matrix}
 \end{aligned}$$

have all the desired properties.

In the sequel $L(M)$ denotes the lattice of the submodules of a module M . Moreover vertices denote the elements of a basis of a module M defined over a K -algebra, while arrows describe obvious linear maps.

LEMMA 2.4. *There exist uniserial modules V, U, W with the following properties:*

- a. $L(V) \simeq \mathbb{N} \cup \{+\infty\}$;
- b. $L(U) \simeq (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$;
- c. $L(W) \simeq \mathbb{Z} \cup \{+\infty, -\infty\}$.

PROOF. First of all $\cdots \rightarrow \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$ describes a module V satisfying (a), and defined over the subalgebra of $\text{End}_K V$ generated by the linear map f . Next, $\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \xrightarrow{f_3} \bullet \rightarrow \cdots$ describes a module U satisfying (b), and defined over the subalgebra of $\text{End}_K U$ generated by the linear maps f_n with $n \geq 1$. Finally, as in [6, Example 2.12], $\cdots \rightarrow \bullet \xrightarrow{f_{-1}} \bullet \xrightarrow{f_0} \bullet \xrightarrow{f_1} \bullet \rightarrow \cdots$ describes a module W satisfying (c), and defined over the subalgebra of $\text{End}_K W$ generated by the maps f_i with $i \in \mathbb{Z}$. \square

With all the previous notation and conventions, we can now prove the following result.

PROPOSITION 2.5. *There exist K -algebras R and R -modules A and B with $A \subseteq B$ such that A and B/A are uniserial, B is indecomposable and one of the following conditions hold:*

1. $L(A) \simeq L(B/A) \simeq \mathbb{N} \cup \{+\infty\}$;
2. $L(A) \simeq L(B/A) \simeq (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$;
3. $L(A) \simeq (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$ and $L(B/A) \simeq \mathbb{N} \cup \{+\infty\}$;
4. $L(A) \simeq \mathbb{Z} \cup \{+\infty, -\infty\}$ and $L(B/A) \simeq \mathbb{N} \cup \{+\infty\}$;
5. $L(A) \simeq (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$ and $L(B/A) \simeq \mathbb{Z} \cup \{+\infty, -\infty\}$;
6. $L(A) \simeq L(B/A) \simeq \mathbb{Z} \cup \{+\infty, -\infty\}$;
7. $L(A) \simeq \mathbb{Z} \cup \{+\infty, -\infty\}$ and $L(B/A) \simeq (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$;
8. $L(A) \simeq \mathbb{N} \cup \{+\infty\}$ and $L(B/A) \simeq \mathbb{Z} \cup \{+\infty, -\infty\}$;
9. $L(A) \simeq \mathbb{N} \cup \{+\infty\}$ and $L(B/A) \simeq (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$.

PROOF. (1) Let B be the module described by the picture

$$\cdots \rightarrow \bullet \xrightarrow{f} \bullet \xrightarrow{f} \underset{v}{\bullet} \xleftarrow{g} \bullet \xleftarrow{g} \bullet \leftarrow \cdots$$

and let R be the subalgebra of $\text{End}_K B$ generated by the linear maps f and g . Next let A be the submodule of B generated by v and by all the vectors on the left of v . Then B is uniform and (1) follows from condition (a) of Lemma 2.4.

(2) The picture

$$\cdots \leftarrow \bullet \xleftarrow{f_2} \underset{v}{\bullet} \xleftarrow{f_1} \bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \rightarrow \cdots$$

describes a module B defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps $f_1, g_1, f_2, g_2, \dots$. Let A be the submodule Rv . Then B is a hollow module and (2) follows from condition (b) of Lemma 2.4.

(3) The picture

$$\dots \longrightarrow \bullet \xrightarrow{f} \bullet \xrightarrow{f} \underset{v}{\bullet} \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \longrightarrow \dots$$

describes a module B defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps f, g_1, g_2, \dots . Let A be the submodule Rv . Then B is uniform and (3) follows from conditions (a) and (b) of Lemma 2.4.

(4) Let B be the module described by the picture

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & & & \downarrow & & & \\ & & & \bullet & & & \\ & & & \downarrow g & & & \\ & & & \bullet & & & \\ & & & \downarrow g & & & \\ \dots & \longrightarrow & \underset{v_{-1}}{\bullet} & \xrightarrow{f_{-1}} & \underset{v_0}{\bullet} & \xrightarrow{f_0} & \underset{v_1}{\bullet} & \xrightarrow{f_1} & \bullet & \xrightarrow{f_2} & \underset{v_3}{\bullet} & \longrightarrow \dots \end{array}$$

defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps g and f_i with $i \in \mathbb{Z}$. Let A be the submodule generated by all the vectors v_i with $i \in \mathbb{Z}$. If X and Y are two nonzero submodules of B , then we clearly have $Rv_n \subseteq X \cap Y$ for n large enough. Hence B is uniform and (4) follows from conditions (a) and (c) of Lemma 2.4.

(5) Let B be the R -module described by the picture

$$\begin{array}{ccccccc} \dots & \longrightarrow & \bullet & \xrightarrow{g_3} & \bullet & \xrightarrow{g_2} & \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_0} & \bullet & \xrightarrow{g_{-1}} & \bullet & \longrightarrow \dots \\ & & \underset{w_3}{\bullet} & & \underset{w_2}{\bullet} & & \underset{w_1}{\bullet} & & \underset{w_0}{\bullet} & & \underset{w_{-1}}{\bullet} & & \underset{w_{-2}}{\bullet} & \\ & & & & & & & & \downarrow f_0 & & & & & \\ & & & & & & & & \underset{v}{\bullet} & & & & & \\ & & & & & & & & \downarrow f_1 & & & & & \\ & & & & & & & & \bullet & & & & & \\ & & & & & & & & \downarrow & & & & & \\ & & & & & & & & \vdots & & & & & \end{array}$$

defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps f_n and g_i with $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. Let $A = Rv$ and assume $B = X + Y$ with $X \neq 0$ and

$Y \neq 0$. Without loss of generality we may assume that X contains a vector x of the form $x = w_n + x'$, where $n \geq 1$ and x' belongs to the submodule generated by w_{n-1} . Then we clearly have $(g_1 \circ g_2 \circ \dots \circ g_n)(x) = w_0$. Since Rw_0 is an essential submodule of B , we conclude that $X \cap Y \neq 0$. Hence B is indecomposable and (5) follows from conditions (b) and (c) of Lemma 2.4.

(6) The fully commutative quiver

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_0} & \bullet & \xrightarrow{g_{-1}} & \bullet & \longrightarrow & \dots \\
 & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \\
 \dots & \longrightarrow & \bullet & \xrightarrow{f_1} & \bullet & \xrightarrow{f_0} & \bullet & \xrightarrow{f_{-1}} & \bullet & \longrightarrow & \dots
 \end{array}$$

describes a module B defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps $h, f_0, g_0, f_1, g_1, f_{-1}, g_{-1}, \dots$. Let A be the submodule of B generated by the vectors belonging to the lower copy of A_∞^∞ , containing the arrows f_i with $i \in \mathbb{Z}$. Then the intersection of two nonzero submodules of B contains a nonzero submodule of A . Hence B is uniform and (6) follows from condition (c) of Lemma 2.4.

(7) Let B be the submodule described by the following fully commutative quiver

$$\begin{array}{ccccccc}
 & & \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_2} & \bullet & \xrightarrow{g_3} & \bullet & \longrightarrow & \dots \\
 & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \\
 \dots & \xrightarrow{f_{-1}} & \bullet & \xrightarrow{f_0} & \bullet & \xrightarrow{f_1} & \bullet & \xrightarrow{f_2} & \bullet & \xrightarrow{f_3} & \bullet & \longrightarrow & \dots
 \end{array}$$

and defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps f_i, g_n, h with $i \in \mathbb{Z}$ and $n \geq 1$. Let A be the submodule of B generated by the vectors belonging to the copy of A_∞^∞ , containing the arrows f_i with $i \in \mathbb{Z}$. Since B is a submodule of a uniform module used to prove (6), it follows that also in this case B is uniform. Hence (7) follows from conditions (b) and (c) of Lemma 2.4.

(8) Let B be the module described by the picture

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \bullet & \xrightarrow{g_3} & \bullet & \xrightarrow{g_2} & \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_0} & \bullet & \xrightarrow{g_{-1}} & \bullet & \longrightarrow & \dots \\
 & & w_3 & & w_2 & & w_1 & & w_0 & & w_{-1} & & w_{-2} & & \\
 & & & & & & & & \downarrow h & & & & & & \\
 \dots & \longrightarrow & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & & & & & & \\
 & & u_3 & & u_2 & & u_1 & & u_0 & & & & & &
 \end{array}$$

and defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps f, g_i, h with $i \in \mathbb{Z}$. Let A be the submodule of B generated by the vectors u_0, u_1, u_2, \dots .

Suppose $B = X + Y$ with $X \neq 0$ and $Y \neq 0$. Without loss of generality we may assume that X contains a vector x of the form $x = w_n + x'$, where $n \geq 1$ and x' belongs to the subspace generated by $\{w_i, u_m \mid i < n \text{ and } m \geq 0\}$. Since Rw_0 is an essential submodule of B and $(g_1 \circ \dots \circ g_n)(x) = w_0$, we conclude that $X \cap Y \neq 0$. Hence B is indecomposable and (8) follows from conditions (a) and (c) of Lemma 2.4.

(9) Let B be the module described by the quiver

$$\begin{array}{ccccccc}
 & & & & \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_2} & \bullet & \xrightarrow{g_3} & \bullet & \longrightarrow & \dots \\
 & & & & v & & u_1 & & u_2 & & u_3 & & \\
 & & & & \downarrow h & & & & & & & & \\
 \dots & \longrightarrow & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & & & & \\
 & & u_{-3} & & u_{-2} & & u_{-1} & & u_0 & & & &
 \end{array}$$

and defined over the subalgebra R of $\text{End}_K B$ generated by the linear maps f, g_n, h with $n \geq 1$. Let A be the submodule generated by $u_0, u_{-1}, u_{-2}, \dots$. Assume $B = X + Y$ with $X \neq 0$ and $Y \neq 0$. Without loss of generality we may assume that X contains a vector x of the form $x = v + x'$, where x' belongs to the submodule generated by the u_i with $i \in \mathbb{Z}$. Then X contains u_0, u_1, u_2, \dots . Since these elements generate an essential submodule of B , we have $X \cap Y \neq 0$. Hence B is indecomposable and (9) follows from conditions (a) and (b) of Lemma 2.4. \square

It suffices to consider modules over finite dimensional algebras to see that the indecomposable extensions of two uniserial modules may be quite different even in very special cases.

EXAMPLE 2.6. There exist K -algebras R and indecomposable R -modules A, B, U with the following properties:

- i. $A \subseteq B$, $A \subseteq U$, and B/A is isomorphic to U/A ;
- ii. B is not uniserial;
- iii. U is uniserial;
- iv. there exist a simple module S , an indecomposable projective (resp. injective) module P (resp. I) and two exact sequences of the form $0 \rightarrow S \rightarrow P \rightarrow B \rightarrow 0$ and $0 \rightarrow S \rightarrow P \rightarrow U \rightarrow 0$ (resp. $0 \rightarrow B \rightarrow I \rightarrow S \rightarrow 0$ and $0 \rightarrow U \rightarrow I \rightarrow S \rightarrow 0$).

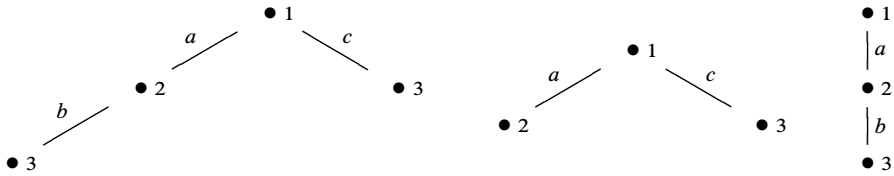
CONSTRUCTION. Let R be the algebra given by the extended Dynkin diagram \tilde{A}_3 with orientation

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{a} & \bullet & \xrightarrow{b} & \bullet \\
 1 & & 2 & & 3 \\
 & \searrow & & \nearrow & \\
 & & c & &
 \end{array}$$

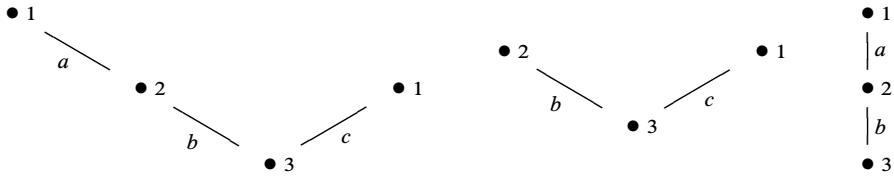
Assume first

$$A = S \simeq P(3), \quad B = P(1)/baP(1), \quad U = P(1)/cP(1), \quad P = P(1).$$

Then the modules P , B , and U are described by the following pictures respectively:



Since $baP(1) \simeq cP(1) \simeq S$, conditions (i)–(iv) clearly hold. Assume now $A \simeq P(2)$, $S \simeq I(1)$, $I = I(3)$ and let B and U be the maximal submodules of I satisfying $baB = 0$ and $cU = 0$. Then I , B and U are described by the following pictures respectively:

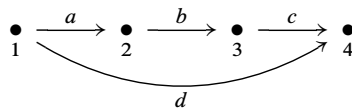


Since $I/B \simeq I/U \simeq S$, also in this case conditions (i)–(iv) hold.

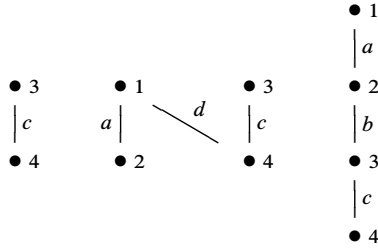
EXAMPLE 2.7. There exist a K -algebra R and indecomposable R -modules A, B, U with the following properties:

- i. $A \subseteq B$, $A \subseteq U$, and B/A is isomorphic to U/A ;
- ii. U is uniserial;
- iii. $\text{Soc } B$ and $B/\text{Soc } B$ are the direct sum of two simple modules.

CONSTRUCTION. Let R be the algebra given by the extended Dynkin diagram \tilde{A}_4 with orientation



Let A , B and U be the modules described by the following pictures respectively:



Then conditions (i)–(iii) clearly hold.

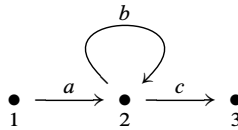
COROLLARY 2.8. *There exist K -algebras R and uniserial R -modules A , and C with the following properties:*

- if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a non split exact sequence, then B is indecomposable;*
- if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a non split exact sequence, then B is decomposable;*
- neither (a) nor (b) holds.*

PROOF. (a) If either A or C is simple, this follows from condition (v) of Theorem 2.1.

(b) Let R , A and C be as in Example 2.3. Then $\dim A = r$, $\dim C = s$, and $\dim M < r + s$ for every indecomposable R -module M . Hence B is decomposable.

(c) Let R be the algebra given by the quiver



with relation $b^2 = 0$. Let $A = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ and let $C = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$. Then both modules

$$\begin{smallmatrix} 1 \\ 2 \\ 2 \\ 3 \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} 1 \\ 2 \oplus 2 \\ 3 \end{smallmatrix}$$

are possible choices for B . □

In order to obtain new extensions of uniserial modules, we may use the following theorem.

THEOREM 2.9. *Let B be the extension of two uniserial modules, and let X be a submodule of B . Then X and B/X are extensions of uniserial modules.*

PROOF. Let A be a submodule of B such that A and B/A are uniserial. Next let $A_1 = A \cap X$ and let $A_2 = \frac{A+X}{X}$. Then we have $A_1 \subseteq A$ and $\frac{X}{A_1} \simeq \frac{X+A}{A} \leq \frac{B}{A}$. Hence X has the desired property. On the other hand we have $A_2 \simeq \frac{A}{A \cap X}$ and $\frac{B/X}{A_2} \simeq \frac{B}{A+X}$. Hence also B/X is the extension of two uniserial modules. \square

We can now state a positive result on subquotients.

COROLLARY 2.10. *Let B be the extensions of two uniserial modules and let C be a subquotient of B . The following facts hold:*

- a. *C is the extension of two uniserial modules;*
- b. *either C is indecomposable or $C = C_1 \oplus C_2$ with C_1 indecomposable and C_2 uniserial.*

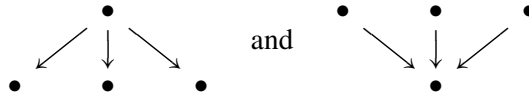
PROOF. (a) This follows from two applications of Theorem 2.9.

(b) This follows from condition (a) and Theorem 2.1. \square

The next example points out a negative result on extensions.

EXAMPLE 2.11. There exist an extension B of two uniserial modules and a non split exact sequence $0 \rightarrow S \rightarrow E \rightarrow B \rightarrow 0$ (resp., $0 \rightarrow B \rightarrow E \rightarrow S \rightarrow 0$) such that S is simple and E is not the extension of two uniserial modules.

CONSTRUCTION. Over the algebras given by the Dynkin diagram D_4 with orientations



we have non split exact sequences of the following form:

$$(1) \quad 0 \longrightarrow 2 \longrightarrow \begin{matrix} 1 \\ 2 \ 3 \ 4 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 3 \ 4 \end{matrix} \longrightarrow 0,$$

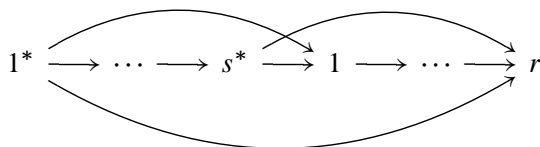
$$(2) \quad 0 \longrightarrow \begin{matrix} 1 \ 2 \\ 4 \end{matrix} \longrightarrow \begin{matrix} 1 \ 2 \ 3 \\ 4 \end{matrix} \longrightarrow 3 \longrightarrow 0.$$

Since ${}_3^1 4$ and ${}_4^1 2$ are extensions of two uniserial modules, the conclusion follows from the sequences (1) and (2) and from condition (b) of Corollary 2.10.

We may roughly speaking say that some extensions of two uniserial modules A and C of finite length are “strings” obtained by “glueing together” the socle or the top of A with the socle or the top of C .

EXAMPLE 2.12. For any $r, s \geq 1$ there exist a K -algebra R and uniserial R -modules A and C of length r and s such that there are extensions of A and C of Loewy length $r + s$, $r + s - 2$, $\max\{r + 1, s\}$, and $\max\{r, s + 1\}$ respectively.

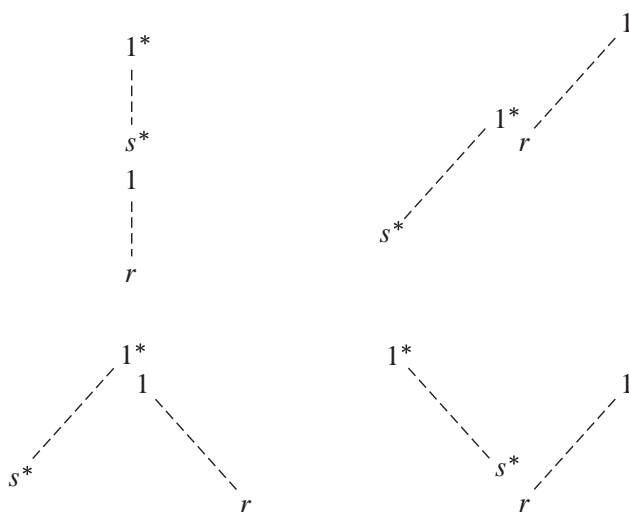
CONSTRUCTION. Let R be the algebra given by the quiver



Next let

$$A = \begin{array}{c} 1 \\ \vdots \\ r \end{array} \quad \text{and} \quad C = \begin{array}{c} 1^* \\ \vdots \\ s^* \end{array}.$$

By glueing together $i \in \{1^*, s^*\}$ and $j \in \{1, r\}$ in all possible ways, we obtain four indecomposable extensions with the desired properties described by the following pictures:



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