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Density of indecomposable locally finite groups

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Dedicated to my friend, László Fuchs

ABSTRACT – We prove that for any locally finite group there is an extension of the same cardinality which is indecomposable for almost all regular cardinals smaller than its cardinality. Note that a group G is called *θ-indecomposable* when for every increasing sequence $\langle G_i : i < \theta \rangle$ of subgroups with union G there is $i < \theta$ such that $G = G_i$.

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1. Introduction

We are interested here in the class $\mathbf{K}_{\mathrm{lf}}$ of locally finite groups; the subject naturally uses finite group theory and infinite combinatorics, see e.g. the book by Kegel and Wehfritz [6].

Wehrfritz asked about the categoricity of the class $\mathbf{K}_{\mathrm{exlf}}$ of existentially closed, locally finite groups (exlf) in any cardinality $\lambda > \aleph_0$. This was answered by Macintyre and Shelah [8] who proved that for every $\lambda > \aleph_0$ there are 2^λ non-isomorphic members of $\mathbf{K}^{\mathrm{exlf}}_\lambda$. This was disappointing in some sense: for $\aleph_0$ the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

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A natural and frequent question on a class of structures is the existence of rigid members, i.e. ones with no non-trivial automorphism. Now, any exlf group $G \in \mathbf{K}_{\text{exlf}}$ has non-trivial automorphisms—the inner automorphisms (recalling that it has a trivial center). So the natural question is about complete members where a group is called *complete* if and only if it has no non-inner automorphism.

Concerning the existence of a complete, locally finite existentially closed group of cardinality λ, Hickin [5] proved that it exists for $\aleph_1$ (and more generally, he even finds a family of $2^{\aleph_1}$ such groups which are pairwise far apart in the sense that no uncountable group is embeddable in two of them). Thomas [24] assumed G.C.H. and built one in every successor cardinal (and moreover he showed that it has no Abelian or just solvable subgroup of the same cardinality). Related results can be found in Macintyre [7], Giorgetta and Shelah [3], and Shelah and Ziegler [23], which investigate the class of groups $\mathbf{K}_{G_*}$; recall that if G_* is a countable existentially closed group then $\mathbf{K}_{G_*}$ is the class of groups such that every finitely generated subgroup is embeddable into G_*.

On the existence and non-existence of universal members see Grossberg and Shelah [4].

The paper [22] investigates the group of permutation of the natural numbers, and asks: what can be the set of regular cardinals θ such that the group is θ-indecomposable (denoted there $\theta \in \mathrm{CF}(G)$); the result is that essentially there are some so-called "pcf restrictions" and those essentially are all the restrictions (for more details on pcf theory see [13]).

Lately [15] has appeared which connects to stability theory, in particular though the class $\mathbf{K}_{\text{exlf}}$ is very unstable it has many definable complete quantifier free types. One application was to use this to build canonical extensions of a locally finite group which are existentially closed and of the same cardinality. Another application was to build so-called "complete extensions" in many cardinals.

Here we deal more specifically with the density of so-called "θ-indecomposable extensions" of the same cardinality, working simultaneously for almost all relevant regular cardinals θ obtaining essentially best possible results. Observe that for a regular cardinal θ, a group G of cardinality λ is trivially θ-indecomposable if $\theta > \lambda$ and is not so if $\theta = \lambda$ or just θ is equal to the cofinality of λ. Those are almost the only restrictions. The problematic case is when $\theta \neq \mathrm{cf}(\mu) < \mu, \mu^+ = \lambda$ but other cases causing difficulties are dealt with in Theorem 3.5 and Claim 3.7.

We prove that essentially for every locally finite group G there is a locally finite group H extending G of the same cardinality which is κ-indecomposable for every regular $\kappa \neq \mathrm{cf}(|G|)$; it is possible in some cases to handle the situation where $\kappa \neq \mathrm{cf}(\mu)$ when $\mathrm{cf}(\mu) < \mu, \mu^+ = \lambda$.

In addition to being of self interest, this helps in [21], in proving that for μ strong limit singular of cofinality $\aleph_0$, there is a universal locally finite group of cardinality μ if and only if there is a canonical such group. The results apply to many other classes (in general for so-called "abstract elementary classes") which have enough indecomposable members.

The result here also helps in [19], in proving results of the form "any locally finite group of cardinality $\lambda > \aleph_0$ can be extended to a complete one of the same cardinality (not just its successor)."

The current work and [21] were originally part of [19] and were separated by requests. In 2019, the existence of θ-indecomposables in λ (see 3.5) was considerably improved after Corson and Shelah [1] dealt with indecomposable groups (while here we are dealing with locally finite groups). The improvement was that earlier it was for many rather than all cardinals. The aim of [1] was to prove the existence of strongly bounded groups.

It is fitting that this work is dedicated to László: he has been the father of modern Abelian group theory and much more. In 1973 his book [2] made me start working in group theory (in particular, on the Whitehead problem (in [10], [11], and the old better versions of the general compactness theorem in [17]).

2. Notations and preliminaries

The following started in Todorcevic [26] and will be used in the proof of Theorem 3.5.

LEMMA 2.1. *The following hold*:

1. $\mu^+ \to [\mu^+]^2_{\lambda^+}$ *except possibly when* $\lambda = \mu^+$, μ *singular limit of* (*possibly weakly*) *inaccessibles*;
2. *if* $\lambda > \aleph_0$ *is regular, then* $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \aleph_0)$;
3. $\aleph_1 \nrightarrow (\aleph_1; \aleph_1)^2_{\aleph_1}$.

PROOF. Part (1) follows from Todorcevic [26] and [12, 3.1,3.3(3)] while (2) follows from [20, Chapter III] (see also the history and the definition there) and (3) follows by Moore [9]. □

We now recall and assign appropriate notation to some classes of groups that will be considered in the sequel; note that we use the notation $\mathrm{sb}(A, G)$ for the subgroup of a group G generated by a set of elements A of G. We write $\mathbf{K}_{\mathrm{lf}}$ for the class of locally finite groups and $\mathbf{K}^{\mathrm{lf}}_{\lambda}$ will denote the class of $G \in \mathbf{K}_{\mathrm{lf}}$ which are of cardinality λ; $\mathbf{K}_{\rm exlf}$ will denote the class of locally finite existentially closed groups, that is, the class of locally finite groups G, such that for every pair of finite groups $H_1 \subseteq H_2$ and embedding f_1 of H_1 into G there is an embedding f_2 of H_2 into G extending f_1; then $\mathbf{K}^{\rm exlf}_\lambda$ will be the class of $G \in \mathbf{K}_{\rm exlf}$ of cardinality λ.

Convention 2.2. We will use the following conventions and notations.

1. $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$ denote an a.e.c., see [18]. with $K_\mathfrak{k}$ being a class of structures and $\leq_\mathfrak{k}$ a partial order on it (the reader can ignore this or use $\leq_\mathfrak{k}$ being a sub-structure).
2. A major case is $\mathfrak{k}$ being a universal class.

Finally, throughout G, H, K will denote groups, usually locally finite while δ denotes a limit ordinal; k, ℓ, m, n natural numbers; $i, j, \alpha, \beta, \gamma$ ordinals and $\lambda, \mu, \kappa, \theta$ cardinals.

3. Indecomposability

In this section we show the density of indecomposable locally finite groups, moreover for any $\lambda > \aleph_0$ and locally finite group G of cardinality λ there is an extension H of the same cardinality which is θ-indecomposable for almost all regular cardinals θ, noting that for $\theta > \lambda$ this trivially holds and for $\theta = {\rm cf}(\lambda)$ it trivially fails. The only additional exclusion is that for λ a successor of singular, we may exclude its cofinality. This is proved in Theorem 3.5(3)(b); before this in Proposition 3.4 we showed how to use a colouring $\mathbf{c}\colon [\lambda]^2 \to \lambda$ to build a group extension. Lastly in Claim 3.7 we justify the excluded cardinal.

Definition 3.1. We introduce the following terminology.

1. M is *θ-decomposable* or $\theta \in {\rm CF}(M)$ when θ is regular and if $\langle M_i : i < \theta\rangle$ is $\subseteq$-increasing with union M, then $M = M_i$ for some i.
2. M is *Θ-indecomposable* when it is θ-indecomposable for every $\theta \in \Theta$.
3. M is *$(\neq \theta)$-indecomposable* when θ is regular and if $\sigma = {\rm cf}(\sigma) \neq \theta$ then M is σ-indecomposable.
4. $\mathbf{c}\colon [\lambda]^2 \to S$ is *θ-indecomposable* provided that if $\langle u_i : i < \theta\rangle$ is $\subseteq$-increasing with union λ then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u_i\}$ for some $i < \theta$; similarly for the other variants.
5. If we replace $\subseteq$ by $\leq_\mathfrak{k}$ where $\mathfrak{k}$ is an a.e.c., then we write "$(\theta - \mathfrak{k})$-indecomposable" or $\theta \in {\rm CF}_\mathfrak{k}(M)$.

Note that a group G may be indecomposable as a group or as a semi-group; the default choice is semi-group; but note that for locally finite groups the two are the same.

Definition 3.2. We say that G is *θ-indecomposable inside* G^+ if the following hold:

a. $\theta = \mathrm{cf}(\theta)$;

b. $G \subseteq G^+$;

c. if $\langle G_i : i \leq \theta \rangle$ is $\subseteq$-increasing continuous and $G \subseteq G_\theta = G^+$, then for some $i < \theta$ we have $G \subseteq G_i$.

The point of the above definition of indecomposable is the following Observation 3.3.

Using cases of indecomposability (see Theorem 3.5), help elsewhere to prove density of complete members of $\mathbf{K}^{\mathrm{lf}}_\lambda$ and to improve characterizations of the existence of universal members in e.g. cardinality $\beth_\omega$.

Below recall that δ is here a limit ordinal.

Observation 3.3. We observe the following.

1. Assume $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing with union M, each M_{i+1} is $(\theta - \mathfrak{k})$-indecomposable or just each M_{2i+1} is $(\theta - \mathfrak{k})$-indecomposable in M_{2i+2}. If $\mathrm{cf}(\delta) \neq \theta$, then M is $(\theta - \mathfrak{k})$-indecomposable.

2. If for $\ell = 1, 2$ the sequence $\langle M^\ell_i : i < \theta \rangle$ is $\leq_{\mathfrak{k}}$-increasing and $\bigcup_i M^1_i = M = \bigcup_i M^2_i$ and each M^1_i is $(\theta - \mathfrak{k})$-indecomposable or just M^1_{2i+1} is θ-indecomposable inside M^1_{2i+2} for $i < \theta$, then
$$\bigwedge_{i<\theta} \bigvee_{j<\theta} M^1_i \leq_{\mathfrak{k}} M^2_j.$$

3. If for $\ell = 1, 2$ the sequence $\langle M^\ell_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and each M^ℓ_{i+1} is $(\theta - \mathfrak{k})$-indecomposable or just M^ℓ_{2i+1} is θ-indecomposable in M^ℓ_{2i+2} for $i < \delta$ and $M^1_\delta = M^2_\delta$ and $\theta = \mathrm{cf}(\delta) > \aleph_0$, then $\{i < \delta : M^1_i = M^2_i\}$ is a club of δ.

4. If M is a Jonsson algebra of cardinality λ, then M is $(\neq \mathrm{cf}(\lambda))$-indecomposable.

5. Assume J is a directed partial order, $\langle M_s : s \in J \rangle$ is $\subseteq$-increasing and $J_* := \{s \in J : M_s \text{ is } (\theta - \mathfrak{k})\text{-indecomposable}\}$ is cofinal in J. Then $\bigcup_{s \in J} M_s$ is $(\theta - \mathfrak{k})$-indecomposable provided that

(∗) if $\bigcup_{i<\theta} J_i \subseteq J$ is cofinal in J and $\langle J_i : i < \theta\rangle$ is $\subseteq$-increasing, then for some i, J_i is cofinal in J or at least $\bigcup_{s\in J_i} M_s = \bigcup_{s\in J} M_s$.

6. Assume G is a model (e.g. a group), $\alpha_* < \theta = \mathrm{cf}(\theta)$, $G_\alpha \subseteq G \subseteq H$ for $\alpha < \alpha_*$ and $\cup\{G_\alpha : \alpha < \alpha_*\}$ generate G. If each G_α is θ-indecomposable inside H then G is θ-indecomposable inside H.
7. G is θ-indecomposable if and only if G is θ-indecomposable inside G.
8. If $G_1 \subseteq G_2 \subseteq H_2 \subseteq H_1$ and G_2 is θ-indecomposable inside H_2 then G_1 is θ-indecomposable inside H_1.

Proof. The claims are easily proved but for the convenience of the reader we elaborate on (5). Towards contradiction let $\langle N_i : i < \theta\rangle$ be $\subseteq$-increasing with union $\bigcup_{s\in J} M_s$. For each $s \in J_*$ there is $i(s) < \theta$ such that $N_{i(s)} \supseteq M_s$. Let $J_j = \{i(s) : s \in J_* \text{ and } i(s) \le j\}$ for $i < \theta$. Clearly $\langle J_i : i < \theta\rangle$ is as required in the assumption of (∗), hence for some $i < \theta$ we have $\bigcup_{s\in J} M_s = \bigcup_{s\in J_i} M_s$, so necessarily $N_i \supseteq \bigcup_{s\in J} M_s$, and thus equality holds. □

We now turn to the class $\mathbf{K}_{\mathrm{lf}}$.

Proposition 3.4. *The following hold.*

1. *Assume I is a linear order and $\mathbf{c} : [I]^2 \to \mathcal{U}$ is θ-indecomposable (hence onto $\mathcal{U}$, see Definition 3.1(4)), $G_1 \in \mathbf{K}_{\mathrm{lf}}$ and $a_i \in G_1 (i \in \mathcal{U})$ are*[1] *pairwise commuting and each of order 2 (or 1). Then there is G_2 such that*
   a. *$G_2 \in \mathbf{K}_{\mathrm{lf}}$ extends G_1;*
   b. *G_2 is generated by $G_1 \cup \bar{b}$ where $\bar{b} = \langle b_s : s \in I\rangle$;*
   c. *b_s has order 2 for $s \in I$;*
   d. *if $s_1 \neq s_2$ are from I, then $a_{\mathbf{c}\{s_1,s_2\}} \in \mathrm{sb}(\{b_{s_1}, b_{s_2}\})$ and, moreover,*
   $$a_{\mathbf{c}\{s_1,s_2\}} = [b_{s_1}, b_{s_2}];$$
   e. *$G_1 \subseteq_{\mathfrak{S}} G_2$, for $\mathfrak{S} = \Omega[K_{\mathrm{lf}}]$ (used only in [19], we can use much smaller $\mathfrak{S}$, see [15, Definitions 0.9 and 1.4 and Claim 1.16]);*
   f. *$\mathrm{sb}(\{a_i : i \in \mathcal{U}\}, G_1)$ (the subgroup of G_1 generated by $\{a_i : i \in \mathcal{U}\}$) is θ-indecomposable inside G_2; see Definition 3.2.*

[1] The demand "the a_i's commute in G_1" is used in the proof of $(*)_8$, and the demand "a_{β_i} has order 2 (or 1)" is used in the proof of $(*)_7$.

2. *Assume* $G_1 \in \mathbf{K}_{\mathrm{lf}}$ *and* I *a linear order which is the disjoint union of* $\langle I_\alpha : \alpha < \alpha_* \rangle, u_\alpha \subseteq \mathrm{Ord}$ *and* $\mathbf{c}_\alpha : [I_\alpha]^2 \to u_\alpha$ *is* θ_α*-indecomposable for* $\alpha < \alpha_*, \langle u_\alpha : \alpha < \alpha_* \rangle$ *is a sequence of pairwise disjoint sets with union* $\mathcal{U}$ *and* $0 \notin \mathcal{U}$ *and* $a_\varepsilon \in G_1$ *for* $\varepsilon \in \mathcal{U}$ *and* a_ε, a_ζ *commute for* $\varepsilon, \zeta \in u_\alpha, \alpha < \alpha_*$ *and each* a_ε *has order* 2 (*or* 1)*, and we let* $a_0 = e$*. Let* $\mathbf{c} : [I]^2 \to \mathcal{U} \cup \{0\}$ *extend each* $\mathbf{c}_\alpha$ *and be* 0 *otherwise. Then there are* G_2 *and* $\langle d_\alpha : \alpha < \alpha_* \rangle$ *such that*:

- (a), (c), *and* (e) *above hold*;
- b′. G_2 *is generated by* $G_1 \cup \bar{b} \cup \bar{d}$;

  d′. $\langle d_\alpha : \alpha < \alpha_* \rangle$ *are pairwise commuting of order* 2;

  f′. *if* $\alpha < \alpha_*$ *then* $\mathrm{sb}(\{a_\varepsilon : \varepsilon \in u_\alpha\}, G_2)$ *is* θ_α*-indecomposable inside* G_2.

3. *In parts* (1) *and* (2)*,*

   a. *the cardinality of* G_2 *is* $|G_1| + |I|$ (*or both are finite*);

   b. *if we omit the assumption "*$\mathbf{c}$ *is* θ*-indecomposable," then clauses* (a)–(e) *of part* (1) *holds*;

   c. *e.g. in part* (1) *if* σ *is a regular cardinal and* $\mathbf{c}$ *is* σ*-indecomposable, then the subgroup* $(\{a_i : i \in \mathfrak{U}\}, G_1)$ *is* σ*-indecomposable in* G_2.

Proof. (1) Let

$(*)_1$ $\mathcal{X} = \{(u, a) : u \subseteq I$ is finite and $a \in G_1\}$.

We shall choose below members $h_c, h_s \in \mathrm{Sym}(\mathcal{X})$ for $c \in G_1, s \in I$, as follows. First,

$(*)_2$ for $c \in G_1$ we choose $h_c \in \mathrm{Sym}(\mathcal{X})$ as follows: for $u \in [I]^{<\aleph_0}$ and $a \in G_1$ let

$$h_c(u, a) = (u, ac^{-1}).$$

Now clearly,

$(*)_3$ a. $h_c \in \mathrm{Sym}(\mathcal{X})$ for $c \in G_1$;

b. the mapping $c \mapsto h_c$ is an embedding of G_1 into $\mathrm{Sym}(\mathcal{X})$;

c. so, without loss of generality, this embedding is the identity.

Next,

$(*)_4$ for $t \in I$ we define $h_t: \mathcal{X} \to \mathcal{X}$ by defining $h_t(u, a)$ by induction on $|u|$ for $(u, a) \in \mathcal{X}$ as follows:

a. if $u = \emptyset$, then $h_t(u, a) = (\{t\}, a)$;

b. if $u = \{s\}$, then $h_t(u, a)$ is defined as follows:

(α) if $t <_I s$, then $h_t(u, a) = (\{t, s\}, a)$;

(β) if $t = s$, then $h_t(u, a) = (\emptyset, a)$;

(γ) if $s <_I t$, then $h_t(u, a) = (\{s, t\}, d)$ where

$$d = a a_{\mathbf{c}\{s,t\}};$$

c. if $s_1 < \cdots < s_n$ list $u \in [I]^n$ and $k \in \{0, \ldots, n\}$ and $t \in (s_k, s_{k+1})_I$, where we stipulate $s_0 = -\infty$, $s_{n+1} = +\infty$, then

$$h_t(u, a) = (u \cup \{t\}, a a_{\mathbf{c}\{s_1,t\}} \ldots a_{\mathbf{c}\{s_k,t\}});$$

d. if $s_1 < \cdots < s_n$ list $u \in [I]^n$ and $k \in \{0, \ldots, n-1\}$ and $t = s_{k+1}$ then[2]

$$h_t(u, a) = (u \backslash \{t\}, a a^{-1}_{\mathbf{c}\{s_k,t\}}, \ldots, a^{-1}_{\mathbf{c}\{s_2,t\}} a^{-1}_{\mathbf{c}\{s_1,t\}}).$$

Note that

$(*)_5$ a. $(*)_4$(b)(α) is the same as $(*)_4$(c) for $n = 1, k = 0$;

b. $(*)_4$(b)(β) is the same as $(*)_4$(d) for $n = 1, k = 0$;

c. $(*)_4$(b)(γ) is the same as $(*)_4$(c) for $n = 1, k = 1$;

$(*)_6$ a. h_a, h_s are permutations of $\mathcal{X}$;

b. let G_3 be the subgroup of $\mathrm{Sym}(\mathcal{X})$ generated by

$$Y = \{h_a, h_s: a \in G_1, s \in I\};$$

c. the group G_3 is locally finite.

◀ Why? Clause (a), just check and clause (b) is a definition. For clause (c), let Z be a finite subset of Y, without loss of generality for some finite subgroup H of G_1 and finite subset J of I the set Z is included in the set $\{h_a, h_s: a \in H, s \in J\}$. Without loss of generality $\{\mathbf{c}\{s,t\}: s \neq t,\ s,t \in J\} \subseteq H$. It suffice to prove that for every pair $(u, a) \in \mathcal{X}$ the closure of $\{(u, a)\}$ under $\{h_d, h_s: d \in H, s \in J\}$ is not just finite but has at most $2^{|J|} \times |H \times H|$ elements. Now this closure is obviously included in the set $\{((u \setminus v) \cup w, c): v \subseteq J \cap u, w \subseteq J \setminus u, c \in (HaH)\}$ which satisfies the inequality. ▶

[2] The a_m^{-1} and inverting the order are more natural but immaterial as long as we are assuming the "of order 2" and "pairwise commuting," but those are now used in fewer points.

Now clearly

$(*)_7$ if $t \in I$ then $h_t \in \mathrm{Sym}(\mathcal{X})$ has order 2.

◄ It is enough to prove $h_t(h_t(u,a)) = (u,a)$. We divide to cases according to "by which clause of $(*)_4$ is $h_t(u,a)$ defined."

- If the definition is by $(*)_4$(a) then $h_t(\emptyset, a) = (\{t\}, a)$ and by $(*)_4$(b)(β), then

$$h_t h_t(\emptyset, a) = h_t(\{t\}, a) = (\emptyset, a).$$

- If the definition is by $(*)_4$(b)(β), then the proof is similar.
- If the definition is by $(*)_4$(b)(γ), then, recalling $(*)_4$(d),

$$\begin{aligned} h_t(h_t(u,a)) &= h_t(h_t(\{s\}, a)) = h_t(\{s,t\}, a a_{\mathbf{c}\{s,t\}}) \\ &= (\{s\}, a a_{\mathbf{c}\{s,t\}} a^{-1}_{\mathbf{c}\{s,t\}}) = (u,a). \end{aligned}$$

- If the definition is by $(*)$(b)(α), then the proof is similar.
- If the definition is by $(*)_4$(c), then recall $(*)_4$(d) and compute similarly to the two previous cases, recalling $\langle a_{\mathbf{c}\{s,t\}} : s \in I \rangle$ are pairwise commuting of order 2 (or 1).
- If the definition is by $(*)_4$(d), then this is just like the last case.

So $(*)_7$ holds indeed. ►

$(*)_8$ $[h_s, h_t] = h_{a_i}$ in G_3 where $i = \mathbf{c}\{s,t\}$

◄ Why? We have to check by cases; here we use "the a_i's are pairwise commuting in G_1 for $i \in \mathcal{U}$". Without loss of generality $s <_I t$; we shall now checked four representative cases (the point is that for (u,c), the members of $u \setminus \{s,t\}$ have little influence).

First,

$(*)_{8.1}$ how is $(\emptyset, c)$ mapped?

a.	$h_s^{-1} h_t^{-1} h_s h_t(\emptyset, c)$	(by $(*)_4$(a))
b.	$= h_s^{-1} h_t^{-1} h_s(\{t\}, c)$	(by $(*)_4$(b)(α))
c.	$= h_s^{-1} h_t^{-1}(\{s,t\}, c)$	(by $(*)_4$(b)(γ))
d.	$= h_s^{-1}(\{s\}, c a^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(b)(β))
e.	$= (\emptyset, c a^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_2$)
f.	$= h_{\mathbf{c}\{s,t\}}(\emptyset, c)$.	

Second,

$(*)_{8.2}$ how is $(\{s\}, c)$ mapped?

a.	$h_s^{-1}h_t^{-1}h_sh_t(\{s\}, c)$	(by $(*)_4$(b)(γ))
b.	$= h_s^{-1}h_t^{-1}h_s(\{s,t\}, ca_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(d) with $(s_1, s_2) = (s,t), k = 0$)
c.	$= h_s^{-1}h_t^{-1}(\{t\}, ca_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(b)(β))
d.	$= h_s^{-1}(\emptyset, ca_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(a))
e.	$= (\{s\}, ca_{\mathbf{c}\{s,t\}})$	(by "every a_i has order 2")
f.	$= (\{s\}, ca^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_2$)
g.	$= h_{\mathbf{c}\{s,t\}}(\{s\}, c)$.	

Third,

$(*)_{8.3}$ how is $(\{t\}, c)$ mapped?

a.	$h_s^{-1}h_t^{-1}h_sh_t(\{t\}, c)$	(by $(*)_4$(b)(β))
b.	$= h_s^{-1}h_t^{-1}h_s(\emptyset, c)$	(by $(*)_4$(a))
c.	$= h_s^{-1}h_t^{-1}(\{s\}, c)$	(by $(*)_4(d)$ with $(s_1, s_2) = (s,t), k = 1$)
d.	$= h_s^{-1}(\{s,t\}, ca_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(d))
e.	$= (\{t\}, ca_{\mathbf{c}\{s,t\}})$	(by $(*)_2$ and "every a_i has order 2")
f.	$= h_{\mathbf{c}\{s,t\}}(\{t\}, c)$.	

Fourth and lastly,

$(*)_{8.4}$ how is $(\{s,t\}, c)$ mapped?

a.	$h_s^{-1}h_t^{-1}h_sh_t(\{s,t\}, c)$	(by $(*)_4(d)$ with $(s_1.s_2) = (s,t), k = 1$)
b.	$= h_s^{-1}h_t^{-1}h_s(\{s\}, ca^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(b)(β))
c.	$= h_s^{-1}h_t^{-1}(\emptyset, ca^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(b)(β))
d.	$= h_s^{-1}(\{t\}, ca^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_4$(c) with $(s_1, s_2) = (s,t), k = 0$)
e.	$= (\{s,t\}, ca^{-1}_{\mathbf{c}\{s,t\}})$	(by $(*)_2$)
f.	$= h_{\mathbf{c}\{s,t\}}(\{s,t\}, c)$. ►	

$(*)_9$ $G_2 = G_3$ is the subgroup of $\mathrm{Sym}(\mathcal{X})$ generated by $\{h_a, h_s : a \in G_1, s \in I\}$ recalling that we have identify $c \in G$ with h_c we have $G_1 \subseteq G_2$.

◀ Why? By $(*)_{10}$(b) and $(*)_3$(b). ▶

$(*)_{10}$ $\mathrm{sb}(\{a_i : i \in S\}, G_1)$ is θ-indecomposable inside G_2.

◀ Why? Because the function $\mathbf{c}$ is θ-indecomposable by an assumption of the proposition and by $(*)_8$. ▶

Together we are done proving part (1).

(2) First,

$(*)_{11}$ we can find a pair $(G_2, \bar{d})$ such that (this G_2 is not the final one):

a. $G_2 \in \mathbf{K}^{\mathrm{lf}}_{\lambda}$;
b. $\bar{d} = \langle d_\alpha : \alpha < \alpha_* \rangle$;
$\bar{d}$ is a sequence of members of G_2, pairwise commuting each of order 2, and letting d_u be the product $\langle d_\alpha : \alpha \in u \rangle$ for finite $u \subseteq \alpha_*$ we have $d_u = e$ if and only if $u = \emptyset$;
c. the group G_2 extend G_1 and is generated by $G_1 \cup \langle d_\alpha < \alpha < \alpha_* \rangle$;
d. the sequence $\langle d_u G_1 d_u : u \in [\alpha_*]^{<\aleph_0} \rangle$ is a sequence of pairwise commuting subgroups, with the intersection of any two being $\{e\}$;
e. (follows) $G_1 \leq_{\mathfrak{S}} G_2$, see clause (e) of Proposition 3.4(1).

◀ Why? Let $\mathcal{X} = [\alpha_*]^{<\aleph_0} \times G_1$. For $c \in G_1$ we define the permutation h_c of $\mathcal{X}$ by $h_c(u, s) = (u, ca)$ if $u = \emptyset$ and $h_c(u, a) = (u, a)$ otherwise. Next for $\alpha < \alpha_*$ we define h_α, a permutation of $\mathcal{X}$ by: $h_\alpha((u, a)) = (u \Delta \{\alpha\}, a)$ where Δ is the symmetric difference.

Easy to check. ▶

Now let $a'_i = d_\alpha^{-1} a_i d_\alpha$ for $i \in u_\alpha$; so clearly they are pairwise commuting, each of order 2. So we can apply part (1) with $G_2, \langle a'_i : i \in \mathcal{U} \rangle, \mathbf{c} : [I]^2 \to \mathcal{U} \cup \{0\}$ here standing for $G_1, \langle a_i : i \in \mathcal{U} \rangle, \mathbf{c} : [I]^2 \to \mathcal{U}$ there. We get $G_3, \langle b^2_s : s \in I \rangle$.

Let $\bar{b} = \bar{b}^2$ and we shall show that the triple $(G_2, \bar{b}, \bar{d})$ is as require, this suffice.

Clauses (a)–(e) are obvious. As for clause (f), fix $\alpha < \alpha_*$, and let $\langle G_{2,i} : i < \theta \rangle$ be an increasing sequence of subgroups of G_2 with union G_2. Recalling that $\mathbf{c}_\alpha = \mathbf{c} \restriction [I_\alpha]^2$, as in the proof of part (1) for some $i < \theta_\alpha$ the set $\{a'_s : s \in I_\alpha\}$ is included in $G_{2,i}$. Without loss of generality $d_\alpha \in G_{2,i}$ hence for every $s \in I_\alpha$ we have $a_\alpha = d_\alpha a'_s d_\alpha^{-1} \in G_{2,i}$ so we are done.

(3) By the proofs of parts (1) and (2). □

Our main result is the following Theorem 3.5, in particular part (3).

THEOREM 3.5. *The following hold.*

1. *If* $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$, *then for some* $G_2 \in \mathbf{K}^{\mathrm{lf}}_{\lambda}$ *extending* G_1 *and* $a^\ell_\alpha \in G_2$ *for* $\ell \in \{1,2\}, \alpha < \lambda$,

   $\oplus$ a. $\mathrm{sb}(\{a^\ell_\alpha : \ell \in \{1,2\}, \alpha < \lambda\}, G_2)$ *includes* G_1;

   b. *if* $\ell \in \{1,2\}$ *then* $\langle a^\ell_\alpha : \alpha < \lambda\rangle$ *is a sequence of pairwise distinct commuting elements of order 2 of* G_2;

   c. G_2 *is generated by* $\{a^\ell_\alpha : \alpha < \lambda, \ell \in \{1,2\}\}$;

   d. $G_1 \leq_{\mathfrak{S}} G_2$, *like clause* (e) *of Proposition* 3.4(1).

2. *If* $\lambda \geq \mu$ *and* $\mathbf{c}: [\lambda]^2 \to \mu$ *is* θ*-indecomposable and* $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\mu}$, *then there is* $G_2 \in \mathbf{K}^{\mathrm{lf}}_{\lambda}$ *extending* G_1 *such that* G_1 *is* θ*-indecomposable inside* G_2 *and* $G_1 \leq_{\mathfrak{S}} G_2$, *like clause* (e) *of Proposition* 3.4(1).

3. *If* $\lambda \geq \aleph_1$ *and* $\Theta = \{\mathrm{cf}(\lambda)\}$ *except that* $\Theta = \{\mathrm{cf}(\lambda), \partial\}$ *when* $(\mathrm{c}_{\lambda,\partial})$ *below holds, then* (a) *and* (b) *holds, where*

   a. *some* $\mathbf{c}: [\lambda]^2 \to \lambda$ *is* θ*-indecomposable when* $\theta = \mathrm{cf}(\theta) \notin \Theta$;

   b. *for every* $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ *there is an extension* $G_2 \in \mathbf{K}^{\mathrm{exlf}}_{\lambda}$ *which is* θ*-indecomposable for every regular* $\theta \notin \Theta$ (*and* $G_1 \leq_{\mathfrak{S}} G_2$, *see clause* (e) *of Proposition* 3.4(1));

   $\mathrm{c}_{\lambda,\partial}$. *for some* $\mu, \lambda = \mu^+, \mu > \partial = \mathrm{cf}(\mu)$ *and*

$$\mu = \sup\{\theta < \mu : \theta \text{is a regular Jonsson cardinal}\}.$$

REMARK 3.6. Given $\lambda \geq \aleph_1$ the demand $(\mathrm{c})_{\lambda.\partial}$ determines ∂ and implies $\lambda > \aleph_\omega$.

We intend to sharpen $(c)_{\lambda,\partial}$ in [19].

PROOF OF THEOREM 3.5. (1) Without loss of generality, the group G_1 is generated by its set of elements of order 2 (see [6] or [15], for clause (d) of Proposition 3.4(1) only the later). Let $\bar{a} = \langle a_i : i < \lambda\rangle$ list the elements of G_1 of order 2, possibly with repetition.

Let $\alpha_* = \lambda, I = \lambda \times \{1,2\}$ lexicographically ordered, $I_\alpha = \{\alpha\} \times \{1,2\}$, $a'_{1+\alpha} = a_\alpha, u_\alpha = \{1+\alpha\}, \mathcal{U} = \{1+\alpha : \alpha < \alpha_*\}, \mathbf{c}_\alpha\{(\alpha,1),(\alpha,2)\} = 1+\alpha$ and apply Proposition 3.4(2) getting G_2 and $\langle b_s : s \in I\rangle$, $\langle d_\alpha : \alpha < \alpha_*\rangle$. Letting $a^\ell_\alpha = b_{(\alpha,\ell)}$ for $\alpha < \lambda, \ell \in \{1,2\}$ we are done.

(2) Let $G'_1 = G_1$, by part (1) with μ here for λ there is $G'_2 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ extending G'_1 with $\langle a^\ell_\alpha : \ell \in \{1,2\}, i < \lambda\rangle$ as there.

Now $(G'_2, \langle a^1_i : i < \mu\rangle)$ satisfies the assumptions in 3.4(1) hence there is $G'_3 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ extending G'_2 such that $H_1 = \mathrm{sb}(\{a^1_i : i < \lambda\}), G'_2)$ is θ-indecomposable in G'_3. Similarly there is $G'_4 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ extending G'_3 such that $H_2 = \mathrm{sb}(\{a^2_i : i < \lambda\}), G'_2)$ is θ-indecomposable inside G'_4. Now $H = \mathrm{sb}(H_1 \cup H_2, G'_2)$ include G'_1 and recalling the previous sentences, by Observation 3.3(6), it is θ-indecomposable inside G'_4 but $G_1 = G'_1 \subseteq H$ hence by Observation 3.3(8) also G_1 is θ-indecomposable inside G'_5, so letting $G_2 = G'_4$ we are done.

(3) To prove the last part

$(*)_1$ it suffices to prove clause (a).

◀ Why? So we are given $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$. Let $\Theta' = \{\theta \leq \lambda : \theta = \mathrm{cf}(\Theta)\} \setminus \Theta$ and let $\sigma = \mathrm{cf}(\lambda)$ so it is a regular cardinal $\leq \lambda$. Let $\partial = |\Theta'|$ so it is a cardinal $\leq \lambda$ and let $\langle \theta_\varepsilon : \varepsilon < \partial\rangle$ list Θ'. We choose $G_{2,i}$ by induction on $i \leq \partial\sigma$ ($\partial\sigma$ is ordinal product) such that:

$(*)_{1.1}$ a. $G_{2,i} \in \mathbf{K}^{\mathrm{exlf}}_\lambda$;

b. $\langle G_{2,j} : j \leq i\rangle$ is increasing continuous;

c. $G_{2,0}$ extends G_1;

d. if $i = \delta j + \varepsilon, \varepsilon < \partial$ then $G_{2,i}$ is θ_ε-indecomposable inside $G_{2,i+1}$;

e. $G_i \leq_{\mathfrak{S}} G_{i+1}\mathbf{K}^{\mathrm{excl}}_\lambda$, see clause (e) of Proposition 3.4(1).

We can carry the induction, e.g. for $i = \partial j + \varepsilon + 1$ by Theorem 3.5(2); well the $\in \mathbf{K}^{\mathrm{exlf}}_\lambda$ holds by [15] (recalling Observation 3.3(8)). By Observation 3.3, $G_2 := G_{2,\partial\sigma}$ is as required. ▶

We shall now prove clause (a) by induction on λ.

Case 1: $\lambda = \theta^+, \theta$ regular. Recall 2.1(1).

Case 2: λ a limit cardinal and $\lambda > \theta$. Let $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ be an increasing sequence of regular cardinals with limit λ, now let

$(*)_2$ a. $\mathbf{c}_{i+1} : [\lambda_i^{++}]^2 \to \lambda_i^{++}$;

b. $\langle \mathbf{c}_j : j \leq i\rangle$ is $\subseteq$-increasing;

c. $\mathbf{c}_i$ is θ-indecomposable for θ regular but $\neq \lambda_i^{++}$.

Arriving to i use Case 1 knowing that $\mathbf{c}_i \restriction \left[\bigcup_{j<i} \lambda_j^{++}\right]^2$ does not matter.

Now $\mathbf{c} = \bigcup\{\mathbf{c}_i : i < \mathrm{cf}(\lambda)\}$ is as required by Observation 3.3(8) and Observation 3.3(5).

CASE 3: $\lambda = \mu^+, \mu > \kappa = \mathrm{cf}(\mu) \neq \theta$. Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of cardinals with limit μ, each a successor of regular. Let $\mathbf{c}_i : [\lambda_i]^2 \to \lambda_i$ witness $\lambda_i \nrightarrow [\lambda_i]^2_{\lambda_i}$ and let $\lambda_{<i} = \bigcup\{\lambda_j : j < i\}$.

For $\alpha < \lambda$ let f_α be a one-to-one function from $\mu(1+\alpha)$ onto μ. Now define $\mathbf{c} : [\lambda]^2 \to \lambda$ such that

$(*)_3$ a. if $\alpha \neq \beta$ belongs to the interval $[\mu(1+\varepsilon) + \lambda_{<i}, \mu(1+\varepsilon) + \lambda_i)$, then $\mathbf{c}\{\alpha, \beta\} = f_\varepsilon^{-1}(\mathbf{c}_i\{\alpha - \mu(1+\varepsilon), \beta - \mu(i+\varepsilon)\})$;

b. if not, then $\mathbf{c}\{\alpha, \beta\} = 0$.

Then,

$(*)_4$ it suffices to prove $\mathbf{c}$ witness the desired conclusion.

So let θ be regular and $\notin \Theta$ and $\theta < \lambda$, so $\theta < \mu$, say $\theta < \lambda_{i(*)}$ for $i(*) < \kappa$.

$(*)_5$ Let $h : \lambda \to \theta$. We should prove that for some $\varepsilon < \theta$,

$$\{\mathbf{c}\{\alpha, \beta\} : h(\alpha), h(\beta) < \varepsilon\} = \lambda.$$

Now for each $\gamma < \lambda$ and $i < \kappa$, we define a function $h_{\gamma,i} : \lambda_i \to \theta$ by

$(*)_6$ $h_{\gamma,i}(\alpha) = h((1+\gamma)\mu + \alpha)$ for $\alpha < \lambda_i$.

By the choice of $\mathbf{c}_i$,

$(*)_7$ for $\gamma < \lambda$, $i < \kappa$ there is $\varepsilon_{\gamma,i} < \theta$ such that

$$\{\mathbf{c}_i\{\alpha, \beta\} : \alpha, \beta < \lambda \text{ and } h_{\gamma,i}(\alpha), h_{\gamma,i}(\beta) < \varepsilon_{\gamma,i}\} = \lambda_i.$$

◄ Why $\varepsilon_{\gamma,i}$ exists? By the choice of $\mathbf{c}_i$. ►

$(*)_8$ For each $\gamma < \lambda$, there exists $\varepsilon_\gamma < \theta$ such that $\kappa = \sup\{i < \kappa : \varepsilon_{\gamma,i} \leq \varepsilon_\gamma\}$.

◄ Why? Because κ, θ are regular cardinals and $\kappa \neq \theta$. ►

$(*)_9$ there is $\varepsilon < \theta$ such that $\lambda = \sup\{\gamma < \lambda : \varepsilon_\gamma \leq \varepsilon\}$.

◄ Why ε exists? Because λ is a regular cardinal $> \theta$. ►

Now by the choices of the f_γ's and of $\mathbf{c}$ we can finish.

CASE 4: $\lambda = \mu^+, \mu > \kappa = \mathrm{cf}(\mu) = \theta$ but μ not a limit of Jonsson cardinals. Let $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta, \delta$ divisible by μ for transparency$\}$ and let $\bar{C}$ be such that

$\boxplus_1$ a. $\bar{C} = \langle C_\delta : \delta \in S \rangle$;

b. α. C_δ is a club of δ;

β. C_δ is of order type κ if $\kappa > \aleph_0$ and μ if $\kappa = \aleph_0$;

γ. $0 \in C_\delta$;

δ. each $\alpha \in C_\delta \backslash \{0\}$ is a limit ordinal;

c. if E is a club λ, then, for some $\delta \in S \cap E$,

- for every $\sigma < \mu$ we have $\mu = \sup\{\alpha \in \mathrm{nacc}(C_\delta) : \mathrm{cf}(\alpha) > \sigma$ and $\alpha \in C$; moreover, $\alpha = \sup(E \cap \alpha)\}$.

◄ Why such $\bar{C}$ exists? See [13, Chapter III,§1]. ►

$\boxplus_2$ choose

a. $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle, e_\alpha$ a club of α of order type $\mathrm{cf}(\alpha)$;

b. $\mathbf{c}_\partial : [\partial]^{<\aleph_0} \to \partial$ witness $\partial \nrightarrow [\partial]^{<\aleph_0}_\partial$ for ∂ a regular non-Jonsson cardinal from (∂_*, μ) for some $\partial_* \in [\theta, \mu]$;

c. $\bar{f} = \langle f_\alpha : \alpha \in [\mu, \lambda) \rangle$, f_α is a function from μ onto α.

Now a major point is the choice of $\mathbf{c} : [\lambda]^2 \to \lambda$:

$\boxplus_3$ we choose $\mathbf{c} : [\lambda]^2 \to \lambda$ such that (A) $\implies$ (B), where

A. a. $\delta_2 \in S$ and $\delta_1 \in S \cap \delta_2$;

b. $\beta = \min\{\beta : \delta_1 < \beta \in C_{\delta_2}\}$ so necessarily $\beta \in \mathrm{nacc}(C_2)$;

c. $\mathrm{cf}(\beta) > \partial_*$;

d. $u = \{\gamma \in e_\beta :$ for some $\alpha \in C_{\delta_1}, \gamma = \mathrm{suc}_{e_\beta}(\alpha)\}$;

e. $\mathrm{otp}(u)$ is $\zeta + n, \zeta$ is zero or a limit ordinal;

f. $\gamma_0 < \cdots < \gamma_{n-1}$ list the last n members of u;

g. $\partial = \mathrm{cf}(\beta)$;

B. $\mathbf{c}(\{\delta_1, \delta_2\}) = f_{\delta_2}(\mathbf{c}_\partial(\{\mathrm{otp}(e_\beta \cap \gamma_\ell) : \ell < n\}))$.

Now

$\boxplus_4$ there is indeed $\mathbf{c}$ as in $\boxplus_3$.

◄ Why? The point is proving that for any $\delta_1 < \delta_2$ from S, at most one case of (A) of $\boxplus_3$ holds, i.e. there is at most one sequence pair$(\beta, \langle \gamma_\ell : \ell < n \rangle)$ as there. But this is obvious from the way $\boxplus_3(A)$ is stated. ►

So it suffices to prove

$\boxplus_5$ $\mathbf{c}$ is θ-indecomposable, moreover it witnesses $\lambda \nrightarrow [\lambda]^2_\lambda$,

and

$\boxplus_6$ for $h: \lambda \to \theta$,

$$(\exists \zeta < \theta)[\lambda = \{\mathbf{c}\{\alpha, \beta\}: \alpha \neq \beta < \lambda \text{ and } h(\alpha), h(\beta) < \zeta\}].$$

First,

$\boxplus_{6.1}$ a. let $\chi = [2^\lambda]^+ : <^*_\chi$ be a well ordering of $\mathcal{H}(\chi)$;

b. $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$ is $\prec$-increasing continuous;

c. $M_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ and M_α has cardinality $\leq \mu$ for $\alpha < \lambda$;

d. $\mathbf{c}, \bar{e}, \overline{C}$ and h belong to M_0, hence to M_α for $\alpha < \lambda$;

e. $\bar{M} \restriction (\alpha + 1) \in M_{\alpha+1}$.

Next,

$\boxplus_{6.2}$ a. let $E_1 = \{\alpha < \lambda : M_\alpha \cap \lambda = \alpha\}$;

b. let $E_2 = \{\delta \in E_2 : \text{otp}(E_1 \cap \delta) = \delta\}$.

Now,

$\boxplus_7$ there is δ_2 such that

a. $\delta_2 \in E_2 \cap S$;

b. for every $\sigma < \mu$,

$$\delta_2 = \sup(A_\sigma), \quad \text{where } A_\sigma = \{\alpha \in \text{nacc}(C_{\delta_2}) : \alpha \in E_2 \text{ and } \text{cf}(\alpha) > \sigma\}.$$

The rest is as in [14]. □

It is an obvious question if we can eliminate the exceptional θ in Theorem 3.5(3)(b). By the following claim we cannot, at least as long as the following famous open problem is unresolved (it is whether every successor of singular cardinality is a Jonsson algebra.)

Claim 3.7. *We claim the following*:

1. *if* $\lambda = \mu^+$, μ *singular and* λ *is a Jonsson cardinal, then every* $G \in \mathbf{K}^{\mathrm{lf}}_\lambda$ *is* $\mathrm{cf}(\mu)$*-decomposable*;
2. *moreover this holds for every model* M *with universe* λ *and vocabulary of cardinality* $< \mu$.

Proof. Easy and it will not be used. □

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