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On a new cotorsion pair

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Abstract – In cotorsion theories, the cotorsion pairs $(\mathcal{SF}, \mathcal{MC})$ of strongly flat and Matlis-cotorsion modules, and $(\mathcal{F}, \mathcal{EC})$ of flat and Enochs-cotorsion modules play important roles. We introduce a new cotorsion pair that in general lies properly between these two (in the partial order generally accepted for cotorsion pairs), and discuss its properties over commutative rings. In particular, we characterize the commutative rings over which this is a perfect cotorsion pair. Our results may shed more light on the relation between the two old cotorsion pairs.

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1. Introduction

Throughout, R denotes an arbitrary commutative ring with identity, Q its classical ring of quotients, and K the factor module Q/R. R is called a *Matlis ring* if p.d. $Q \leq 1$. (We shall use the notations "p.d." for projective dimension and "w.d." for weak dimension.) $R^{\times}$ will denote the monoid of regular (i.e. non-zero-divisor) elements of R. For additional definitions see below.

Most of the theorems in this note hold for integral domains, but only a few of them are valid for all commutative rings. Instead of restricting our considerations to domains, we will frequently work over a newly introduced class of rings that is an effective generalization of the class of integral domains: it is the class of commutative rings R whose quotient rings Q are perfect rings (in the sense of

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Bass [1]), i.e. they are orders in commutative perfect rings. Recent publications (see e.g. [9], [5], and [7]) are convincing evidence that these rings deserve more attention, since they share several relevant features of domains that fail in most commutative rings; following [5], we call such rings *subperfect*. (For a rough comparison, let us point out that modules over Q are injective if R is a domain, while they are in general only weak-injective if R is a subperfect ring. Also, it is perhaps worthwhile mentioning that important examples of subperfect rings include all Cohen–Macaulay rings and their non-Noetherian generalizations.)

The cotorsion pairs $(\mathcal{F}, \mathcal{EC})$ (flat and Enochs-cotorsion) and $(\mathcal{SF}, \mathcal{MC})$ (strongly flat and Matlis-cotorsion) are well known and thoroughly investigated. The former pair is *perfect*, i.e. flat covers and Enochs-cotorsion envelopes exist over any ring, while the second pair is in general only *complete*, i.e. modules admit special strongly flat precovers and special Matlis-cotorsion preenvelopes. (See Göbel and Trlifaj [10].) By Fuchs and Salce [9], these two cotorsion pairs are identical if and only if the ring R is almost perfect.

It is easy to create new cotorsion pairs: any class of modules generates (and cogenerates) one, but there is hardly any of interest that is closely related to classical cotorsion pairs and has homological interpretation. We introduce a new cotorsion pair $({}^*\mathcal{F}, {}^*\mathcal{C})$ that lies between $(\mathcal{SF}, \mathcal{MC})$ and $(\mathcal{F}, \mathcal{EC})$ in the partial order defined in the class of cotorsion pairs: $(\mathcal{SF}, \mathcal{MC}) \leq ({}^*\mathcal{F}, {}^*\mathcal{C}) \leq (\mathcal{F}, \mathcal{EC})$, and its right hand class can be defined in terms of Ext. The new pair is generated by the class of pure submodules of strongly flat modules (called here **-flat* modules). The class ${}^*\mathcal{C}$ consists of those modules C for which $\operatorname{Ext}^1_R(Q, C) = 0$ and $\operatorname{Ext}^2_R(F, C) = 0$ for all flat F. We show that this new cotorsion pair is hereditary (Theorem 4.4(a)) and is generated by a set (Lemma 4.2). We also discuss over subperfect rings the cases when the new pair coincides with one of the two previously mentioned pairs: $({}^*\mathcal{F}, {}^*\mathcal{C}) = (\mathcal{SF}, \mathcal{MC})$ if and only if h-divisible torsion modules are Enochs-cotorsion, and $({}^*\mathcal{F}, {}^*\mathcal{C}) = (\mathcal{F}, \mathcal{EC})$ if and only if h-divisible Enochs-cotorsion torsion modules are weak-injective (Theorems 3.5, 4.5) (definitions below). Finally, we show that over a commutative ring, our new cotorsion pair is perfect (i.e. ${}^*\mathcal{F}$-covers and ${}^*\mathcal{C}$-envelopes exist) if and only if the ring is almost perfect; in this case it coincides with the cotorsion pair $(\mathcal{SF}, \mathcal{MC})$ (Theorem 5.3).

We start recalling some definitions needed in this paper. We follow a customary notation: for a non-negative integer n, $\mathcal{F}_n$ will denote the class of R-modules of w.d. $\leq n$. In particular, $\mathcal{F}_0$ is the class of flat modules.

An R-module N is *torsion-free* if $\operatorname{Tor}^R_1(R/Rr, N) = 0$ holds for each $r \in R^\times$. An R-module D is *divisible* if $rD = D$ for all $r \in R^\times$. It is *h-divisible* if every homomorphism $R \to D$ extends to a homomorphism $Q \to D$; or, equivalently,

D is an epimorphic image of a direct sum of copies of Q. $\mathcal{D}$ will denote the class of divisible, and $\mathcal{HD}$ the class of h-divisible modules. It follows that $D \in \mathcal{D}$ if and only if $\operatorname{Ext}^1_R(R/Rr, D) = 0$ for all $r \in R^\times$, and $D \in \mathcal{HD}$ whenever $\operatorname{Ext}^1_R(K, D) = 0$. We have $\mathcal{D} = \mathcal{HD}$ if and only if R is a Matlis ring [6]. Note that the h-divisible torsion-free R-modules M are exactly the Q-modules, thus they satisfy both $\operatorname{Hom}_R(Q, M) \cong M$ and $Q \otimes_R M \cong M$. An R-module M is *h-reduced* if it contains no h-divisible submodule $\neq 0$. M is called *weak-injective* if $\operatorname{Ext}^1_R(A, M) = 0$ for all R-modules A with w.d. $A \leq 1$ (Lee [11]). $\mathcal{WI}$ will denote the class of weak-injective modules. Weak-injective modules are h-divisible: $\mathcal{WI} \subseteq \mathcal{HD}$.

A ring R is *perfect* if its flat modules are projective, or equivalently, the R-modules admit projective covers [1]. Modules over perfect rings are weak-injective. R is *subperfect* if its total ring of quotients is a perfect ring. It is *almost perfect* if it is subperfect and R/Rr is a perfect ring for each $r \in R^\times$. (For almost perfect rings and examples, see [9].)

An R-module M is said to be *Matlis-cotorsion* if $\operatorname{Ext}^1_R(Q, M) = 0$, and *Enochs-cotorsion* if $\operatorname{Ext}^1_R(F, M) = 0$ for all flat R-modules F. The Matlis-cotorsion envelope of an h-reduced torsion-free module M is $\widetilde{M} = \operatorname{Ext}^1_R(K, M)$. A *strongly flat* module S is defined by the property that $\operatorname{Ext}^1_R(S, M) = 0$ holds for all Matlis-cotorsion M. In Göbel and Trlifaj [10] it is shown that S is strongly flat if and only if it is a summand of a module N that fits into a pure-exact sequence $0 \to F \to N \to D \to 0$ where F is a free R-module and D is Q-filtered (actually, in this special case, D is a direct sum of copies of Q). Recall that for a class $\mathcal{C}$ of R-modules, an R-module M is *$\mathcal{C}$-filtered* if M is the union of a well-ordered continuous chain of submodules M_σ ($\sigma < \tau$) (for some ordinal τ) such that the factor modules $M_{\sigma+1}/M_\sigma$ ($\sigma + 1 < \tau$) belong to $\mathcal{C}$.

The *generalized Matlis category equivalence* for commutative rings (see Matlis [15]) establishes an equivalence between the category $\mathcal{M}$ of h-reduced torsion-free Matlis-cotorsion modules M and the category $\mathcal{H}$ of h-divisible torsion modules D. It is implemented by the functors

$$\mathbf{F}\colon M \longmapsto K \otimes_R M \quad (M \in \mathcal{M})$$

and

$$\mathbf{G}\colon D \longmapsto \operatorname{Hom}_R(K, D) \quad (D \in \mathcal{H}).$$

The corresponding modules are related as is shown by the exact sequence

$$0 \longrightarrow M \longrightarrow Q \otimes_R M \cong \operatorname{Hom}_R(Q, D) \longrightarrow D \longrightarrow 0.$$

We will refer to an exact sequence of this kind as a *Matlis sequence*.

For unexplained terminology and facts, we refer to Göbel and Trlifaj [10], Enochs and Jenda [4], and Fuchs and Salce [8].

2. *-Flat modules

We start our discussion with the following lemma, a generalization of Lee [13, Theorem 3.2] that was proved for domains.

Lemma 2.1. *Assume R is a subperfect ring. Then a strongly flat submodule of a projective R-module is projective.*

Proof. Suppose $0 \to S \to P \to C \to 0$ is an exact sequence, where S is strongly flat and P is projective. If M is Matlis-cotorsion, then we have $\mathrm{Ext}^2_R(C, M) \cong \mathrm{Ext}^1_R(S, M) = 0$. Every h-divisible D embeds in a Matlis sequence $0 \to M = \mathrm{Hom}_R(K, D) \to \mathrm{Hom}_R(Q, D) \to D \to 0$ where M is an h-reduced torsion-free Matlis-cotorsion module. Hence we get the induced exact sequence

$$\mathrm{Ext}^1_R(C, \mathrm{Hom}_R(Q, D)) \longrightarrow \mathrm{Ext}^1_R(C, D) \longrightarrow \mathrm{Ext}^2_R(C, M) = 0.$$

Here $\mathrm{Hom}_R(Q, D)$ is weak-injective as it is a Q-module and Q is by hypothesis a perfect ring. Furthermore, w.d. $C \leq 1$ as S is flat, so the first Ext vanishes. Consequently, $\mathrm{Ext}^1_R(C, D) = 0$ holds for all h-divisible D, which implies p.d. $C \leq 1$, i.e. S is projective. □

A few useful facts on strongly flat modules are listed next.

Lemma 2.2. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of modules over a Matlis ring R.*

i. *If B is strongly flat and* p.d. $C \leq 1$, *then A is strongly flat.*
ii. *If A, B are strongly flat, then* p.d. $C \leq 1$.
iii. *If in addition R is subperfect, if A is strongly flat and B is projective, then A is projective, and thus* p.d. $C \leq 1$.

Proof (This was proved for domains in [13, Theorems 3.2 and 4.5]). Both (i) and (ii) follow from the isomorphism $\mathrm{Ext}^1_R(A, M) \cong \mathrm{Ext}^2_R(C, M)$ that holds for Matlis-cotorsion M, while (iii) is a consequence of Lemma 2.1. □

As mentioned above, a main objective here is to study a class of modules that include the strongly flat modules. We call a module A **-flat* if it is a pure submodule of a strongly flat module, i.e. it is embeddable in a pure-exact sequence $0 \to A \to S \to F \to 0$ where S is strongly flat and F is flat. We have the obvious implications

$$\text{strongly flat} \implies *\text{-flat} \implies \text{flat}.$$

The class of *-flat modules is evidently closed under isomorphisms, arbitrary direct sums, and pure submodules. Next, we list a few easy properties of *-flatness.

A. The Matlis-cotorsion envelope of a torsion-free module is *-flat if and only if the module is *-flat. Moreover, a module A is pure in a strongly flat module S if and only if the Matlis-cotorsion envelope $\widetilde{A}$ of A is pure in the Matlis-cotorsion envelope $\widetilde{S}$ of S. This is an immediate consequence of purity.

B. The class of *-flat modules is *resolving*, i.e. if $0 \to A \to B \to C \to 0$ is an exact sequence with *-flat modules B, C, then A is also *-flat.

C. The tensor product of two *-flat modules is *-flat. This follows from the facts that the analogous statement is true for strongly flat modules, and tensoring preserves purity.

Theorem 3.5 below will offer detailed information about the equivalence of *-flatness and strong flatness. For the characterization of rings over which *-flatness coincides with flatness, see Theorem 4.5.

Example 2.3. Let R be an integral domain such that p.d. $Q > 1$. From Theorem 3.5 it is clear that over such a ring R, not all *-flat modules are strongly flat. Also, R has localizations of p.d. 1. These are flat, but fail to be *-flat.

Example 2.4. Bazzoni and Salce [3, Proposition 3.6] If V is a valuation domain, then Theorem 3.5(iii) will show that all *-flat (= torsion-free) V-modules are strongly flat if and only if p.d. $M \leq 1$ for all torsion-free V-modules M. This is the case if and only if V is a Matlis domain of global p.d. ≤ 2.

Example 2.5. A divisible *-flat R-module D is a pure submodule of a projective Q-module. If R is subperfect, then D itself is a projective Q-module. (The same is true for *-flat-filtered modules.)

Example 2.6. An Enochs-cotorsion module M is *-flat if and only if it is strongly flat. In fact, let $0 \to M \to S \to F \to 0$ be a pure-exact sequence where S is strongly flat and F is flat. If M is Enochs-cotorsion, then $\mathrm{Ext}^1_R(F, M) = 0$, which means that the sequence splits. As a summand of S, M is strongly flat.

3. *-Cotorsion modules

We introduce a kind of cotorsion module that lies between Matlis- and Enochs-cotorsions in the sense that it is a special Matlis, but more general than Enochs.

We call an R-module C **-cotorsion* if it satisfies $\mathrm{Ext}^1_R(A, C) = 0$ for all *-flat R-modules A. Evidently, the class of *-cotorsion R-modules is closed under extensions and arbitrary direct products. We have the obvious implications

$$\text{Enochs-cotorsion} \implies \text{*-cotorsion} \implies \textit{Matlis-cotorsion.}$$

These implications are in general irreversible, even for integral domains.

The next lemma characterizes *-cotorsion modules homologically.

Lemma 3.1. *The following conditions for an R-module C are equivalent*:

a. *C is *-cotorsion*;

b. i. *C is Matlis-cotorsion, and*

   ii. $\mathrm{Ext}^1_R(H, C) = 0$ *for all pure submodules H of projective R-modules*;

c. i. *C is Matlis-cotorsion, and*

   ii. $\mathrm{Ext}^2_R(F, C) = 0$ *for all flat R-modules F.*

Proof. (a) $\implies$ (b) is a trivial implication.

(b) $\implies$ (c). Let F be a flat R-module, and consider a presentation $0 \to H \to P \to F \to 0$ where P is projective and H is pure in P. By (b), the first Ext vanishes in the induced exact sequence $\mathrm{Ext}^1_R(H, C) \to \mathrm{Ext}^2_R(F, C) \to \mathrm{Ext}^2_R(P, C) = 0$. Hence we have $\mathrm{Ext}^2_R(F, C) = 0$.

(c) $\implies$ (a). Let A denote a *-flat R-module, so there is a pure-exact sequence $0 \to A \to S \to F \to 0$ where S is strongly flat and F is flat. Assuming (i), in the induced exact sequence $\mathrm{Ext}^1_R(S, C) \to \mathrm{Ext}^1_R(A, C) \to \mathrm{Ext}^2_R(F, C)$ the first Ext is 0, and the third Ext is 0 because of (ii). Therefore we obtain $\mathrm{Ext}^1_R(A, C) = 0$ for all *-flat A, completing the proof. □

Remark 3.2. Observe that (c.ii) in the preceding lemma can be replaced by the condition that $\mathrm{Ext}^k_R(F, C) = 0$ for all $k \geq 2$ and flat R-modules F. Indeed, let again $0 \to H \to P \to F \to 0$ be an exact sequence where P is projective and H is pure in P (and hence also flat). We then have for every $k \geq 2$ the exact sequence

$$\mathrm{Ext}^k_R(H, C) \longrightarrow \mathrm{Ext}^{k+1}_R(F, C) \longrightarrow \mathrm{Ext}^k_R(P, C) = 0$$

whence the induction hypothesis establishes the claim.

We record immediate corollaries.

Corollary 3.3. (i) *A *-cotorsion filtered module is *-cotorsion.*

(ii) *The class of *-cotorsion modules is coresolving, i.e. if* $0 \to A \to B \to C \to 0$ *is an exact sequence where* A, B *are *-cotorsion modules, then* C *is likewise *-cotorsion.*

(iii) *The factor module* M/C *of a Matlis-cotorsion module* M *modulo a *-cotorsion* C *is Matlis-cotorsion.*

(iv) *If* R *is a subperfect ring, then all* Q*-modules* (*i.e. torsion-free divisible* R*-modules*) *are *-cotorsion.*

Proof. Claims (i) and (ii) follow at once from Lemma 3.1(b). (iii) is straightforward. If R is a subperfect ring, then all Q-modules are weak-injective, and hence *-cotorsion in view of the same lemma (actually, they are even Enochs-cotorsion). (iv) is a trivial consequence of the Enochs-cotorsion property of Q-modules whenever Q is a perfect ring. □

Lemma 3.4. *A Matlis-cotorsion* R*-module* C *is *-cotorsion if and only if it is a pure submodule of an Enochs-cotorsion module* E *with Enochs-cotorsion cokernel* E/C.

Proof. Suppose C is *-cotorsion, and let E be its Enochs-cotorsion envelope. For any flat module F, the pure-exact sequence $0 \to C \to E \to E/C \to 0$ induces the exact sequence

$$(1)\quad 0 = \mathrm{Ext}^1_R(F, E) \longrightarrow \mathrm{Ext}^1_R(F, E/C) \longrightarrow \mathrm{Ext}^2_R(F, C) \longrightarrow \mathrm{Ext}^2_R(F, E) = 0.$$

The third Ext is 0 by Lemma 3.1, thus E/C is Enochs-cotorsion.

Conversely, assume that in the pure-exact sequence $0 \to C \to E \to E/C \to 0$ both E and E/C are Enochs-cotorsion. Then from (1) we obtain that C satisfies (c.ii) of Lemma 3.1. □

Our next aim is to find out when Matlis-cotorsion modules are *-cotorsion. Portions of the following lemma are borrowed from [14, Corollary 19.2.7]. (Several implications in the next theorem hold for all commutative rings.)

Theorem 3.5. *If* R *is a subperfect ring, then the following conditions are equivalent*:

i. *Matlis-cotorsion* R*-modules are *-cotorsion*;
ii. h*-reduced Matlis-cotorsion* R*-modules are *-cotorsion*;

iii. *flat R-modules are of* p.d. ≤ 1;

iv. *epimorphic images of Enochs-cotorsion R-modules are Enochs-cotorsion*;

v. *the global Enochs-cotorsion dimension of R is* ≤ 1;

vi. $\mathrm{Ext}^2_R(F, F') = 0$ *for all pairs F, F' of flat R-modules*;

vii. *pure submodules of projective R-modules are strongly flat*;

viii. *pure submodules of projective R-modules are projective*;

ix. **-flat R-modules are strongly flat*;

x. *h-divisible R-modules are Enochs-cotorsion.*

Furthermore, (x) *holds for a ring R only if it is subperfect.*

Proof. Assuming R is a subperfect ring, we prove the equivalence of the stated conditions.

(i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii). Assuming (ii), we show that for a flat module F the equality $\mathrm{Ext}^2_R(F, N) = 0$ holds for all modules N. First, let N be h-reduced and torsion-free. Then for its Matlis-cotorsion envelope $\widetilde{N} = \mathrm{Ext}^1_R(K, N)$ we have the exact sequence $0 \to N \to \widetilde{N} \to \widetilde{N}/N \to 0$, and hence also $\mathrm{Ext}^1_R(F, \widetilde{N}/N) \to \mathrm{Ext}^2_R(F, N) \to \mathrm{Ext}^2_R(F, \widetilde{N})$. Here the first Ext vanishes, since $\widetilde{N}/N$ is a Q-module and hence weak-injective, and the last Ext is 0 in view (ii) and Lemma 3.1.

In the general case, let $0 \to H \to P \to N \to 0$ be an exact sequence for an arbitrary R-module N with projective P. As H, P are torsion-free and h-reduced, and the sequence $0 = \mathrm{Ext}^2_R(F, P) \to \mathrm{Ext}^2_R(F, N) \to \mathrm{Ext}^3_R(F, H)$ is exact, so what has been proved combined with Remark 3.2 completes the proof of (iii).

(iii) $\Longrightarrow$ (iv). By definition, E is Enochs-cotorsion if $\mathrm{Ext}^1_R(F, E) = 0$ for all flat F. Clearly, p.d. $F \leq 1$ implies that the same equality holds if E is replaced by any of its epic images.

(iv) $\Longrightarrow$ (v). Let $0 \to M \to E \to H \to 0$ be the Enochs-cotorsion envelope sequence for an arbitrary R-module M. By (iv), H is Enochs-cotorsion, and hence $\mathrm{Ext}^2_R(F, M) \cong \mathrm{Ext}^1_R(F, H) = 0$ for any flat F. Therefore, (v) follows.

(v) $\Longrightarrow$ (vi). Hypothesis (v) asserts that $\mathrm{Ext}^2_R(F, M) = 0$ holds for all flat modules F and for all R-modules M. Hence (vi) is obvious.

(vi) $\Longrightarrow$ (i). If (vi) holds, then Lemma 3.1(c.ii) is satisfied, so (i) follows.

(iii) $\Longrightarrow$ (vii) is trivial.

(vii) $\Longrightarrow$ (viii). See Lemma 2.1.

(viii) $\Longrightarrow$ (ix). Let F be an arbitrary flat R-module and $0 \to H \to P \to F \to 0$ a presentation of F with projective P. H is then a pure submodule of P, and thus projective by hypothesis. Hence p.d. $F \leq 1$.

(ix) $\Longrightarrow$ (i). Let A be an arbitrary *-flat R-module, and $0 \to A \to S \to F \to 0$ a pure-exact sequence where S is strongly flat. Then F is flat and thus of p.d. ≤ 1 by hypothesis (ix). The claim follows from the induced exact sequence $\mathrm{Ext}^1_R(S, M) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^2_R(F, M) = 0$ where the first Ext is 0 whenever M is Matlis-cotorsion.

(iii) $\Longrightarrow$ (x). By hypothesis, Q is a perfect ring, thus $\oplus Q$ is Enochs-cotorsion both as an R- and as a Q-module. From (iii) we conclude that $\mathrm{Ext}^1_R(F, \oplus Q) = 0$ implies the same equality if $\oplus Q$ is replaced by any of its epic images. Hence (x) follows.

(x) $\Longrightarrow$ (v). Given any M, there is an exact sequence $0 \to M \to D \to D' \to 0$ where D is the injective hull of M and D' is h-divisible. Hence for every module F we have $\mathrm{Ext}^2_R(F, M) \cong \mathrm{Ext}^1_R(F, D')$ where for flat F the right hand Ext vanishes by virtue of (x). Therefore, (v) holds true.

Finally, if (x) is satisfied by a commutative ring R, then all Q-modules M are Enochs-cotorsion R-modules. They are also Enochs-cotorsion as Q-modules, since for Q-modules flatness over R and over Q are equivalent. Moreover, Ext_R can be replaced by Ext_Q. A ring over which all modules are Enochs-cotorsion is necessarily perfect. □

Remark 3.6. Observe that conditions (iii) and (x) imply that a ring satisfying the conditions of the preceding theorem must be a subperfect Matlis ring. But a subperfect Matlis ring need not satisfy these conditions: a counterexample is a valuation domain of global dimension at least 3 with countably generated field of quotients (the p.d. of an uncountably generated torsion-free = flat ideal is ≥ 2).

Example 3.7. Let R be a valuation domain with value group that is the lexicographic extension of the linearly ordered group $\mathbb{R}$ by the linearly ordered $\mathbb{Q}$. Since ideals are countably generated, the global dimension of R is 2. R is a Matlis domain, since its field of quotients is likewise countably generated. The cyclic modules R/Rr $(0 \neq r \in R)$ are *-cotorsion, but they are in general not Enochs-cotorsion. (They are Enochs-cotorsion in case R is an almost maximal valuation domain.)

4. Cotorsion pair generated by the *-flat modules

In this section we focus our attention on the cotorsion pair $({}^*\mathcal{F}, {}^*\mathcal{C})$ that is generated by the class of *-flat modules. By virtue of [10], the class ${}^*\mathcal{F}$ consists of summands of *-flat-filtered modules, and the class ${}^*\mathcal{C}$ consists of the *-cotorsion modules.

Example 4.1. The cotorsion pair $({}^*\mathcal{F}, {}^*\mathcal{C})$ is in general different from the cotorsion pairs $(\mathcal{SF}, \mathcal{MC})$ and $(\mathcal{F}, \mathcal{EC})$. Indeed, in general we have

$$(\mathcal{SF}, \mathcal{MC}) < ({}^*\mathcal{F}, {}^*\mathcal{C}) < (\mathcal{F}, \mathcal{EC}).$$

That $({}^*\mathcal{F}, {}^*\mathcal{C})$ differs in general from $(\mathcal{SF}, \mathcal{MC})$ is clear by Example 2.3, while Example 3.7 and [12, Example 3.8] show that an h-divisible Enochs-cotorsion module is not necessarily weak-injective (cf. also Theorem 4.5 below).

In order to obtain more information about the pair $({}^*\mathcal{F}, {}^*\mathcal{C})$, we verify:

Lemma 4.2. *The cotorsion pair* $({}^*\mathcal{F}, {}^*\mathcal{C})$ *is generated by* Q *and a set of pure submodules of projective* R*-modules.*

Proof. Set $\kappa = \max\{|R|, \aleph_0\}$. Recall that every element in any R-module is contained in a pure submodule of cardinality $\leq \kappa$. If the module is flat, then its pure submodules are also flat, whence it follows that every flat R-module is the union of a continuous well-ordered ascending chain of flat submodules where the factors are flat modules of cardinalities $\leq \kappa$. Therefore, if $\mathfrak{X}$ denotes the set of pairwise non-isomorphic flat R-modules of cardinalities $\leq \kappa$, then $\mathrm{Ext}^2_R(F, M) = 0$ holds for all flat R-modules F provided it holds for all $F \in \mathfrak{X}$. Let $\mathfrak{Y}$ denote the set of pure submodules H, one for each $F \in \mathfrak{X}$, in an exact sequence $0 \to H \to P \to F \to 0$ with some projective P. Then $\mathrm{Ext}^1_R(H, M) \cong \mathrm{Ext}^2_R(F, M)$ implies that condition (c) in Lemma 3.1 is satisfied for M once we know that (ii) holds for all $H \in \mathfrak{Y}$. Thus the set $\mathfrak{Y}$ of *-flat modules along with Q generates the cotorsion pair $({}^*\mathcal{F}, {}^*\mathcal{C})$. □

We refer to [10, Corollary 3.2.3] to argue that the class ${}^*\mathcal{F}$ consists of direct summands of modules Z for which there is an exact sequence $0 \to F \to Z \to G \to 0$ with free F and $\{\mathfrak{Y}, Q\}$-filtered G.

We will need the following lemma.

Lemma 4.3. *Assume* R *is a subperfect ring. In the generalized Matlis category equivalence, the* h*-reduced torsion-free* **-cotorsion modules and the* h*-divisible Enochs-cotorsion torsion modules correspond to each other.*

Proof. Consider the Matlis sequence $0 \to N \to Q \otimes_R N \to K \otimes_R N \to 0$ with a torsion-free Matlis-cotorsion N. For any module F, we obtain the exact sequence

$$\mathrm{Ext}^1_R(F, Q \otimes_R N) \longrightarrow \mathrm{Ext}^1_R(F, K \otimes_R N) \longrightarrow \mathrm{Ext}^2_R(F, N) \longrightarrow \mathrm{Ext}^2_R(F, Q \otimes_R N).$$

If R is a subperfect ring, then Q-modules are weak-injective. It follows that the extremal Exts are 0 whenever F is a flat module. Consequently,

$$\mathrm{Ext}^1_R(F, K \otimes_R N) \cong \mathrm{Ext}^2_R(F, N)$$

for all flat F. This means that $K \otimes_R N$ is Enochs-cotorsion if and only if N is *-cotorsion (Lemma 3.1). □

Before stating the next theorem, we point out for comparison that the cotorsion pair $(\mathcal{F}, \mathcal{EC})$ is hereditary, but $(\mathcal{SF}, \mathcal{MC})$ is not. (Actually, $(\mathcal{SF}, \mathcal{MC})$ is hereditary exactly if R is a Matlis ring; this follows from the ring version of [13, Lemma 4.1].)

Theorem 4.4. (a) *The cotorsion pair* $({}^*\mathcal{F}, {}^*\mathcal{C})$ *is hereditary*: $\mathrm{Ext}^n_R(A, C) = 0$ *for all* $A \in^* \mathcal{F}$, $C \in^* \mathcal{C}$, *and for all integers* $n \geq 1$.

(b) $({}^*\mathcal{F}, {}^*\mathcal{C})$ *is a complete cotorsion pair over every commutative ring* R, *i.e. all* R*-modules admit special* ${}^*\mathcal{F}$*-precovers and special* ${}^*\mathcal{C}$*-preenvelopes.*

Proof. (a) Let C be *-cotorsion. There is a pure-exact sequence $0 \to C \to E \to N \to 0$ with Enochs-cotorsion E and N (Lemma 3.4). By [10, Theorem 4.1.1], the cotorsion pair $(\mathcal{F}, \mathcal{EC})$ is hereditary. Therefore $\mathrm{Ext}^n_R(A, C) = 0$ for each flat module A and for each $n \geq 2$. This equality holds by definition for $A \in^* \mathcal{F}$ and $n = 1$, so $({}^*\mathcal{F}, {}^*\mathcal{C})$ is hereditary.

(b) Since the cotorsion pair $({}^*\mathcal{F}, {}^*\mathcal{C})$ is generated by a set (Lemma 4.2), by [10, Theorem 3.2.1] every module M can be embedded in an exact sequence $0 \to M \to H \to A \to 0$ where $H \in^* \mathcal{C}$ and $A \in^* \mathcal{F}$. This sequence provides a special ${}^*\mathcal{C}$-preenvelope for M. The existence of special ${}^*\mathcal{F}$-precovers follows then from Salce's lemma (e.g. [10, Lemma 2.2.6]). □

We next consider rings over which the cotorsion pair $({}^*\mathcal{F}, {}^*\mathcal{C})$ equals $(\mathcal{F}, \mathcal{EC})$.

Theorem 4.5. *For a subperfect ring* R, *the following are equivalent*:

i. *flat* R*-modules are *-flat*;

ii. **-cotorsion* R*-modules are Enochs-cotorsion*;

iii. h*-reduced torsion-free *-cotorsion* R*-modules are Enochs-cotorsion*;

iv. h*-divisible Enochs-cotorsion torsion* R*-modules are weak-injective.*

Proof. (i) $\iff$ (ii). This is straightforward by observing that $(\mathcal{F}, \mathcal{EC})$ and $(^*\mathcal{F},^* \mathcal{C})$ are cotorsion pairs over commutative rings.

(ii) $\implies$ (iii) is trivial.

(iii) $\implies$ (iv). This implication is a consequence of Lemma 4.3.

(iv) $\implies$ (ii). Form the Matlis sequence $0 \to C \to Q \otimes_R C \to K \otimes_R C \to 0$, starting with an h-reduced torsion-free $*$-cotorsion C. In view of Lemma 4.3, from (iv) we conclude that $K \otimes_R C$ is weak-injective, and therefore [6, Proposition 5.3] shows that C is Enochs-cotorsion. This implication remains true even if C is not torsion-free, because then we can take a flat cover sequence $0 \to E \to F \to C \to 0$ where F is flat and E is Enochs-cotorsion. Then C $*$- cotorsion implies the same for F, and then by what has been proved we have F Enochs-cotorsion. Hence it follows that C is Enochs-cotorsion, so we obtain (ii). □

5. When $(^*\mathcal{F},^* \mathcal{C})$ is a perfect cotorsion pair

It is natural to raise the existence question of covers and envelopes for the cotorsion pair $(^*\mathcal{F},^* \mathcal{C})$. Owing to Theorem 4.4, such special precovers and special preenvelopes exist over any commutative ring. Our Theorem 5.3 below settles the problem of perfectness for the cotorsion pair $(^*\mathcal{F},^* \mathcal{C})$.

Example 5.1. Over any subperfect ring R, every torsion-free divisible R-module D has a $^*\mathcal{F}$-cover. In this case, D is a Q-module, so there exists a Q-projective cover sequence $0 \to C \to P \to D \to 0$ where P is Q-projective. Now P is $*$-flat, and by Lemma 3.3(iii), C is $*$-cotorsion as an R-module, while minimality is obvious.

Our main result on $^*\mathcal{F}$-covers is the next theorem.

Theorem 5.2. *Modules over a commutative ring R admit $^*\mathcal{F}$-covers if and only if R is an almost perfect ring.*

Proof. If R is an almost perfect ring, then flatness and strong flatness are equivalent, and clearly, the same must be true for flatness and $*$-flatness. As flat covers always exist, the claim of sufficiency is evident.

Conversely, assume that $^*\mathcal{F}$-covers exist for R-modules. Let D be any Q-module (i.e. a torsion-free divisible R-module), and $0 \to X \to F \to D \to 0$ an exact sequence with Q-projective F; here X is also a Q-module. As F is $*$-flat as an R-module, in view of the Q-projectivity of F this exact sequence is a

${}^*\mathcal{F}$-precover sequence for D. Then a summand of F is a ${}^*\mathcal{F}$-cover of D (see [10, Lemma 2.1.8]), which means that the ${}^*\mathcal{F}$-cover of D is Q-projective. Therefore, a ${}^*\mathcal{F}$-cover sequence for D may be viewed as a Q-projective cover sequence for the Q-module D. Consequently, Q-modules admit projective covers, i.e. Q is a perfect ring.

Let the top row in the following diagram be a ${}^*\mathcal{F}$-cover sequence for a divisible R-module D (with any $r \in R^\times$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{\ \zeta\ } & A & \xrightarrow{\ \phi\ } & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \| & & \\
0 & \longrightarrow & rA \cap C & \longrightarrow & rA & \xrightarrow{\ \phi\restriction_{rA}\ } & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & A & \xrightarrow{\ \phi\ } & D & \longrightarrow & 0
\end{array}$$

Since D is divisible and hence $A = rA + \zeta C$ for $r \in R^\times$, the middle row is an exact sequence. As $rA \cong A$, rA is *-flat, so there is a map $\alpha : A \to rA$ making the upper right square commute. Using the embedding map $\beta : rA \to A$, we complete the diagram in the obvious way. By commutativity, we have $\phi\beta\alpha = \phi$, whence the cover property of A implies that the endomorphism $\beta\alpha$ of A is an automorphism. It is clear that then β is surjective, and hence A is divisible. We have shown above that R is subperfect, so it follows that A as a divisible *-flat-filtered R-module is Q-projective (cf. Example 2.5).

Let M be an h-reduced torsion-free Matlis-cotorsion R-module, and $0 \to C \to A \to D \to 0$ a ${}^*\mathcal{F}$-cover sequence for the divisible torsion module $D = K \otimes_R M$ (thus A is Q-projective and C is *-cotorsion). This exact sequence induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(K, D) \longrightarrow \mathrm{Ext}^1_R(K, C) \longrightarrow \mathrm{Ext}^1_R(K, A) = 0;$$

the last Ext vanishes as A is weak-injective and w.d. $K = 1$. by the Matlis category equivalence, $\mathrm{Hom}_R(K, D) \cong M$, so M is isomorphic to the h-reduced part $\mathrm{Ext}^1_R(K, C)$ of the *-cotorsion C. The divisible submodule of C is torsion-free, so a Q-module, and hence *-cotorsion, therefore from Lemma 3.3(ii) it follows that M is *-cotorsion. This means that h-reduced torsion-free Matlis-cotorsion R-modules are *-cotorsion, and our subperfect ring R satisfies condition (ii) in Theorem 3.5. Condition (iv) of the same theorem shows that then all *-flat R-modules are strongly flat. Consequently, we have the equality ${}^*\mathcal{F} = \mathcal{SF}$.

If ${}^*\mathcal{F} = \mathcal{SF}$, then every ${}^*\mathcal{F}$-cover is also an $\mathcal{SF}$-cover, thus our hypothesis implies that R-modules admit $\mathcal{SF}$-covers. It only remains to refer to [5, Theorem 3.7] (which says that modules over a commutative ring admit strongly flat covers if and only if the ring is almost perfect) to conclude that R is almost perfect. □

We can now prove a main result.

Theorem 5.3. *The cotorsion pair* $({}^*\mathcal{F}, {}^*\mathcal{C})$ *over a commutative ring is perfect if and only if the ring is almost perfect* (*in which case it is equal to both* $(\mathcal{SF}, \mathcal{MC})$ *and* $(\mathcal{F}, \mathcal{EC})$).

Proof. Only the necessity part requires a proof. By Theorem 5.2, already the existence of ${}^*\mathcal{F}$-covers implies that R is almost perfect. □

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