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On the rationality of period integrals and special value formulas in the compact case

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Abstract – We study rationality properties of period integrals that appear in the Gan–Gross–Prasad conjectures in the compact case using Gross' theory of algebraic modular forms. In situations where the refined Gan–Gross–Prasad are known, our rationality result for period can be interpreted as a special value formula for automorphic L-functions which proves automorphic versions of Deligne's conjecture on rationality of periods. Moreover, this special value formula is well suited to p-adic interpolation, as illustrated in [10].

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Contents

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1. Introduction

Suppose that $\eta\colon \mathbf{H} \subset \mathbf{G}$ is an inclusion of algebraic subgroups (over $\mathbb{Q}$, for simplicity, in this introduction) such that $\mathbf{H}(\mathbb{R})$ and $\mathbf{G}(\mathbb{R})$ are connected and such that $\mathbf{G}(\mathbb{R})/\mathbf{S_G}(\mathbb{R})$ is compact, where $\mathbf{S_G} \subset \mathbf{Z_G}$ (resp. $\mathbf{S_H} \subset \mathbf{Z_H}$) is the maximal split torus in the center $\mathbf{Z_G}$ of $\mathbf{G}$ (resp. $\mathbf{Z_H}$ of $\mathbf{H}$). Let $\mathbb{A}$ be the adele ring (over $\mathbb{Q}$) and suppose that $\omega\colon \mathbf{S_H}(\mathbb{A}) \to \mathbb{C}^\times$ and $\omega^\eta\colon \mathbf{H}(\mathbb{A}) \to \mathbb{C}^\times$ are two continuous characters that are trivial on $\mathbf{S_H}(\mathbb{Q})$ and, respectively, $\mathbf{H}(\mathbb{Q})$, that ω is unitary and that they are related by $\omega^\eta_{|\mathbf{S_H}(\mathbb{A})} = \omega_{|\mathbf{S_H}(\mathbb{A})}$. In this paper, we study the rationality properties of period integrals of the form

$$I_\eta(f) := \int_{[\mathbf{H}(\mathbb{A})]_{\mathbf{S_H}}} f(\eta(x))\omega^{-\eta}(x)d\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{S_H}}}(x), \tag{1}$$

where $f \in L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{Q}), \omega)$, $[\mathbf{H}(\mathbb{A})]_{\mathbf{S_H}} := \mathbf{S_H}(\mathbb{R})\backslash\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})$ and the measure is normalized in a suitable way (as explained after (8)). We set

$$m_{\mathbf{S_H}\backslash\mathbf{H},\infty} := \mu_{\mathbf{S_H}\backslash\mathbf{H},\infty}(\mathbf{S_H}(\mathbb{R})\backslash\mathbf{H}(\mathbb{R})).$$

Roughly speaking, we prove that, if we restrict I_η to a suitable subspace of "algebraic automorphic forms," the assignment $f \mapsto I_\eta(f)$ turns out to be defined over a Galois splitting field E of $\mathbf{G}$ with respect to suitable rational structures. This is the content of Theorem 7.6, expressing $m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta$ as a morphism

$$m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta = J_\eta\colon M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)} \longrightarrow \mathbf{A}^1_{/E(\omega_f)}, \tag{2}$$

where ω_0 is an appropriate twist of ω_f and $M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)}$ is a suitable space of Gross' style algebraic modular forms, as that we are going to explain.

Writing $A(\mathbf{G}(\mathbb{A}), \omega) \subset L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{Q}), \omega)$ for the dense submodule of finite vectors, we may write

$$A(\mathbf{G}(\mathbb{A}), \omega) = \bigoplus_{\pi^u_\infty} A(\mathbf{G}(\mathbb{A}), \omega)[\pi^u_\infty],$$

where π^u_∞ runs over all the unitary irreducible representations of $\mathbf{G}(\mathbb{R})$ with central character ω^{-1}_∞. Let us suppose, for simplicity, that $\mathbf{G}(\mathbb{R})$ is compact. In this case, the Borel–Weil theorem implies the existence of (canonical) rational models ρ of π^u_∞ over E. Then, for every $E(\omega_f)$-algebra R, we can consider Gross' style algebraic modular forms $M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)}(R)$ that are defined over R and we have

$$M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)}(R) \simeq R \otimes_{E(\omega_f)} M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)}$$

(see (27), Definition 3.3 and Proposition 4.1 (3) for the above identification). The $\mathbb{C}$-points of the source of (2) are identified, by means of an adelic Peter–Weyl theorem (see Proposition 6.1), with

$$M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)}(\mathbb{C}) \simeq A(\mathbf{G}(\mathbb{A}), \omega)[\pi^u_\infty].$$

When $\mathbf{G}(\mathbb{R})$ may be non-compact (as in our application to the interpolation problem, see below), i.e. $\mathbf{S}_{\mathbf{G}}(\mathbb{R}) \neq \{1\}$, it is important to take into account possible twists π_∞ of π^u_∞. Assuming that π_∞ is even (and parallel when $\mathbb{Q}$ is replaced by a more general totally real number field - see Definition 7.4), we prove that $m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta$, when restricted to $M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)}(\mathbb{C}) \subset L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{Q}), \omega)$, equals the $\mathbb{C}$-points of a morphism of functors $J_\eta \colon M[\mathbf{G}, \rho, \omega_0]_{/E(\omega_f)} \to \mathbf{A}^1_{/E(\omega_f)}$. This special value formula $m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta = J_\eta$ is our rationality statement.

Remark 1.1. According to [14, Proposition 2.2], E is a CM-field with totally real field E'. In particular, when π_∞ is a real representation, (2) descend to $E'(\omega_f)$ (see Theorem 7.6).

The interest in this kind of integrals is motivated by the fact that this formalism presents itself in many central value formulas, such as the special value formulas for the Rankin L-functions (see [31]), the special value formulas for the triple product L-functions (see [17, §11] and [20]) and, more generally, in the Gan–Gross–Prasad conjectures of [7, §24]. Motivated by these conjectures, we expect these period integrals I_η to be frequently related to special values formulas: hence our rationality result is consistent with Deligne period conjectures and could be viewed as a generalization of some of the rationality results of [17, §11] to a broader context (given their formula). For example (under our compactness assumptions), it applies to the more general triple product L-functions considered in [20] and to the (mainly conjectural) formulas appearing in the refined Gan–Gross–Prasad conjectures of [21], [16] and [26], as discussed below. In order to place our result in this broader context, let us describe the inclusion of classical algebraic groups $\eta \colon \mathbf{H} \subset \mathbf{G}$ that appears in the Gan–Gross–Prasad conjectures temporarily removing our compactness assumption in $\mathbf{G}$.

Let $K/\mathbb{Q}$ be a Galois field extension with Galois group $G_{K/\mathbb{Q}} = \{1, c\}$ (where we may have $c = 1$), let V be a finite dimensional vector space over K and suppose that $\langle -, - \rangle \colon V \otimes_{\mathbb{Q}} V \to K$ is a non-degenerate, c-sesquilinear form on V, which is ε-symmetric for $\varepsilon \in \{\pm 1\} \subset K^\times$. These data define an algebraic group $\mathbf{G}(V, \langle -, - \rangle)$ over $\mathbb{Q}$ and we set $\mathbf{G}_V := \mathbf{G}(V, \langle -, - \rangle)^\circ$. Take $W \subset V$ which is non-degenerate for $\langle -, - \rangle$, i.e. such that $V = W \oplus W^\perp$; then $\mathbf{G}(W, \langle -, - \rangle_{|W}) \subset \mathbf{G}(V, \langle -, - \rangle)$ is embedded as the subgroup of transformations acting as the identity on $W^\perp$. Suppose that $W^\perp$ is a split space and $(-1)^{\dim(W^\perp)} = -\varepsilon$. Explicitly, when $\varepsilon = 1$ (resp. $\varepsilon = -1$) this means that $\dim(W^\perp) = 2r + 1$ (resp. $\dim(W^\perp) = 2r$)

and $W^\perp$ is the orthogonal direct sum $W^\perp = X \oplus X^\vee \oplus L$ (resp. $W^\perp = X \oplus X^\vee$), with X isotropic, $X^\vee$ isotropic and dual to X and L a non-isotropic line. Setting $\mathbf{G} := \mathbf{G}_V \times \mathbf{G}_W$, one may define $\mathbf{H} \subset \mathbf{G}$ as follows. Let $\mathbf{P}_X \subset \mathbf{G}_V$ be the parabolic subgroup which stabilizes a complete flag of isotropic subspaces in X; since $\mathbf{G}_W$ fixes both X and $X^\vee$, we have that it is contained in a Levi subgroup $\mathbf{L}$ of $\mathbf{P}_X$ and acts by conjugation on the unipotent radical of $\mathbf{N}$ of $\mathbf{P}_X = \mathbf{N} \rtimes \mathbf{L}$. Setting $\mathbf{H} := \mathbf{N} \rtimes \mathbf{G}_W$, the inclusion $\mathbf{H} \subset \mathbf{G}$ is defined to be the product of the inclusion $\mathbf{H} \subset \mathbf{P}_X \subset \mathbf{G}_V$ and the projection $\mathbf{H} \to \mathbf{G}_W$. Let $\pi = \pi_V \boxtimes \pi_W$ be an irreducible cuspidal representation of $\mathbf{G}$, with π_V (resp. π_W) an irreducible cuspidal representation of $\mathbf{G}_V$ (resp. $\mathbf{G}_W$), and suppose that π_V and π_W are almost locally generic (see [26, After Remark 2.4]). It is defined in [7, §12 and §23] a unitary automorphic representation ν of $\mathbf{H}$ and it is proved that

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbf{H}(\mathbb{Q}_v)}(\pi_{V,v} \otimes \pi_{W,v} \otimes \bar{\nu}_v, \mathbb{C})) \leq 1 \tag{3}$$

in [23] and [7] (the proof reduces to the $r = 0$ case handled in [1], [30], and [32]). Indeed, up to changing $\mathbf{G}$ by a pure inner form $\mathbf{G}'_\pi$, the equality should be achieved: this rule should be governed by symplectic local root numbers. We suppose from now these root numbers place ourselves on the right group $\mathbf{G} = \mathbf{G}'_\pi$. Then the Gan–Gross–Prasad conjecture exhibits a close relationship between the period integral (1) with ω^η replaced by ν and the central values of the automorphic L-functions associated to symplectic representations of the L-group of $\mathbf{G}$: these two quantities should vanish or not at the same time (see [33] for a proof in the unitary case). Indeed, when $\varepsilon = 1$, $\nu\colon \mathbf{H}(\mathbb{A}) \to \mathbb{C}^\times$ is a generic automorphic character and a refinement of the Gan–Gross–Prasad conjecture has been proposed in [26], generalizing the $r = 0$ orthogonal and unitary cases discussed in [21] and, respectively, [16]. After a suitable normalization (see [26, Remark 2.6]), it takes the form of the formula (5) below where $I_\eta(f)$ is replaced by $\int_{\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})} f(\eta(x))\nu^{-1}(x)d\mu_{\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})}(x)$. We remark that, when $r = 0$, the refined conjectures of [26] is known in the orthogonal case when $\dim(V) = 3$ or 4 (by [31] and [17] or [20], as explained in [21]) and 5 when π_V is a theta lift (see [8]). When $r = 0$ it is known in the unitary case when $\dim(V) = 2$ or 3 (see [16]) and in general, up to a non-zero factor $c(\pi_{V,\infty} \otimes \pi_{W,\infty}) \in \mathbb{C}^\times$ which only depends on the archimedean components of $\pi_V \otimes \pi_W$ under suitable assumptions (see [34]). See also [26] for two examples in the $r = 1$ and orthogonal case.

Remark 1.2. Suppose that $\mathbf{G} := \mathbf{G}_V \times \mathbf{G}_W$ as above. Then $\mathbf{G}(\mathbb{R})/\mathbf{S}_{\mathbf{G}}(\mathbb{R})$ is compact if and only if $\mathbf{G}(\mathbb{R})$ is compact and this means that $\varepsilon = 1$ and $\langle -, - \rangle$ should be definite: then $\mathbf{G}_V(\mathbb{R})$ and $\mathbf{G}_W(\mathbb{R})$ are either $\simeq$ $\mathbf{SO}(n)$ or $\simeq$ $\mathbf{U}(n)$ according to whenever c is trivial or not and $r = 0$ (there is no isotropic vector by compactness of $\mathbf{G}_V(\mathbb{R})$). In fact, in this case $\mathbf{S_H} \subset \mathbf{S_G} = \{1\}$ is trivial and there is no condition placed on the character ω^η appearing in (1), that could be taken $\omega^\eta = \nu$.

Suppose from now on that

$$\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W,\ \mathbf{G} = \mathbf{G}'_\pi \text{ and } \mathbf{G}(\mathbb{R}) \text{ is compact.} \tag{4}$$

It follows from Remark 1.2 and the discussion before it that we are in the setting of the refined Gan–Gross–Prasad conjecture (because $\varepsilon = 1$) and that, taking $\omega^\eta = \nu$ in (1), the conjecture predicts that

$$\left|I_\eta(f)\right|^2 = \frac{1}{2^\beta} \frac{\Delta_{\mathbf{G}_V} L(1/2, \pi_V \boxtimes \pi_W)}{L(1, \pi_V, \mathrm{Ad}) L(1, \pi_W, \mathrm{Ad})} \prod_v \alpha_v(f_v) \tag{5}$$

for $f = \otimes_v f_v \in \pi_V \boxtimes \pi_W$. Here α_v are appropriately regularized integral of matrix coefficients which should be non-zero on $\pi_{V,v} \otimes \pi_{W,v} \otimes \nu_v^{-1}$ because $\mathbf{G} = \mathbf{G}'_\pi$ (see [26, Conjecture 2.5 (2)]), $\Delta_{\mathbf{G}_V}$ is a product of abelian L-values (attached to dual Gross motives) and β is an integer. This explains the relationship between our investigation and the Gan–Gross–Prasad conjecture. In particular, we obtain the following result as a direct application of (2), where $E(\pi)$ (resp. $E'(\pi)$) denotes the field generated over $E(\omega_f)$ (resp. $E'(\omega_f)$) by the eigenvalues of π.

Theorem A. *If* (4) *hold, the period integrals $m_{\mathbf{H},\infty}^{-1} I_\eta$ with $\omega^\eta = \nu$ considered in* [7, §24] *are defined over $E(\omega_f)$. Hence, depending on the validity of* (5)*, also the right hand side is defined over E. Explicitly, this means that a test vector f can be chosen so that $m_{\mathbf{H},\infty}^{-1} I_\eta(f) \in E(\pi)$.*

Let us write ω_i for the weight of $\wedge^i V_\mathbb{R}$ or its restriction to $\mathbf{SU}_V(\mathbb{R})$ in the Hermitian case and, in the orthogonal case, let α and β be the weights of the half-spin representations. Then $\pi_{V,\infty}$ (resp. $\pi_{V,\infty|\mathbf{SU}_V(\mathbb{R})}$) is classified by its dominant weight, that can be written in the form $\lambda_{\pi_{V,\infty}} = \sum_{i=1}^{\dim(V)/2} n_i\omega_i + n_\alpha\alpha + n_\beta\beta$ (orthogonal case) or $\lambda_{V,\infty|\mathbf{SU}_V(\mathbb{R})} = \sum_{i=1}^{\dim(V)-1} n_i\omega_i$ (unitary case).

Theorem B. *Suppose that* (4) *hold. If we are in the orthogonal case, suppose that either* $\dim(V) \equiv 1, 3, 4 \mod(4)$ *or that* $\dim(V) \equiv 2 \mod(4)$ *and $n_\alpha = n_\beta$. If we are in the unitary case, suppose that $n_i = n_{\dim(V)-1-i}$ for every i and that $\pi_{V,\infty}$ is trivial when restricted to the center. Then the period integrals $m_{\mathbf{H},\infty}^{-1} I_\eta$ with $\omega^\eta = \nu$ considered in* [7, §24] *are defined over $E'(\omega_f)$, which is totally real when $\omega_f^2 = 1$. Hence a test vector can be chosen so that $m_{\mathbf{H},\infty}^{-1} I_\eta(f) \in E'(\pi)$ and, when $\omega_f^2 = 1$ and the eigenvalues of π are real, $|I_\eta(f)|^2 = I_\eta(f)^2$. (For example, one can take $E' = \mathbb{Q}$ in the setting of triple product L-functions, see* [17, §11] *and Theorem* 8.2*).*

Proof of Theorem B. Indeed, the above assumptions implies that $\pi_{V,\infty}$ is either real or quaternionic (see [6, Propositions 26.4, 26.6 and 26.7]), so that the same property is enjoyed by $\pi^{\vee}_{V,\infty|\mathbf{H}(\mathbb{R})}$. It follows from (3) with $v = \infty$ that $\pi_{W,\infty} \otimes \nu_{\infty}^{-1}$ appears with multiplicity one in $\pi^{\vee}_{V,\infty|\mathbf{H}(\mathbb{R})}$; hence the morphism c such that $c^2 = \pm 1$ which gives the real or quaternionic structure on $\pi_{V,\infty}$ induces the same kind of structure on $\pi_{W,\infty} \otimes \nu_{\infty}^{-1}$. Since $\overline{\nu_{\infty}} = \nu_{\infty}$, we know that ν_{∞}^{-1} is real; it follows that $\pi_{W,\infty}$ is real and we deduce that $\pi_{V,\infty} \boxtimes \pi_{W,\infty}$ is also real as a representation of $\mathbf{G}(\mathbb{R})$. Now the claim follows from Remark 1.1 □

Another application of Theorem 7.6 is the following result (to be proved in §7.2), which assumes the validity of (5), [26, Conjecture 2.3] and, hence, it is a theorem in the cases discussed before Remark 1.2 above. In order to state the result, let $E(\pi, \mathrm{Ad})$ be the field generated over $E(\pi)$ by the coefficients of the polynomials $L_v(1, \pi_V, \mathrm{Ad})$ and the values $\Delta_{\mathbf{G}_V, v}$ at finite primes v and then set $E^*(\pi) := E(\pi, \mathrm{Ad})\overline{E(\pi, \mathrm{Ad})}$. If we are given $a, b \in \mathbb{C}$, we write $a \sim b$ to mean that $b \neq 0$ and $\frac{a}{b} \in E^*(\pi)$. Hence, we have fixed $E(\pi, \mathrm{Ad}) \subset \mathbb{C}$ and write $x \mapsto \bar{x}$ for the induced complex conjugation on x. Recall that, in the general unitary case, eq. (5) is only known up to $c(\pi_{V,\infty} \otimes \pi_{W,\infty}) \in \mathbb{C}^{\times}$ in order to justify the assumptions we are going to make.

Theorem C. *Suppose that* (4) *hold and that the refined Gan–Gross–Prasad conjecture holds, possibly with* (5) *satisfied up to some non-zero constant* $c(\pi_{V,\infty} \otimes \pi_{W,\infty}) \in \mathbb{C}^{\times}$ *which only depends on the archimedean components of* $\pi_V \otimes \pi_W$. *Then*

$$\frac{L(1/2, \pi_V \boxtimes \pi_W)}{L_{\infty}(1/2, \pi_V \boxtimes \pi_W)} \sim c(\pi_{V,\infty} \otimes \pi_{W,\infty}) \frac{\Delta_{\mathbf{G}_V,\infty}}{\Delta_{\mathbf{G}_V}} \frac{L(1, \pi_V, \mathrm{Ad}) L(1, \pi_W, \mathrm{Ad})}{L_{\infty}(1, \pi_V, \mathrm{Ad}) L_{\infty}(1, \pi_W, \mathrm{Ad})}. \tag{6}$$

The above result provides evidences to conjectures of Deligne and Shimura. We refer the reader to [15] for the relations with Deligne's conjectures and to [13] for a proof of a similar result in the unitary case (see also the remark at the end of this introduction).

The period integral (1) is first studied in Theorem 7.2, while Theorem 7.6 provides conditions for its applicability. As explained above, the result is (2) which gives, when (5) is known, explicit special value formulas. We exemplify this fact in §8, where we specialize our result to the case of triple product L-functions and Rankin L-functions, thus getting, respectively, an explicit Harris–Kudla–Ichino and explicit Waldspurger formula. The former yields a generalization and simplification (of the proof) of the special value formula [4, Theorem 5.7] which removes the squarefree level assumptions there (see Theorem 8.2). The latter removes the assumption that the conductors of the modular form and the character should be coprime which appears in the Hatcher-Hui formulas proved in [18] and [19] (see (44)). Our generalization is due to the fact that, rather than focusing ourselves on the L-functions, we focus ourselves on making explicit (1) regarded as a functional, without trying to include the theta correspondence and the test vector in the special value formula itself. In other words, justified by (5), the problem of studying the special values of complex L-functions is split in two parts: relate special values to period integrals (which is (5)) and then study the period integrals themselves getting the "almost algebraic" expression of Theorem 7.2 (which is a considerably more modest task). The pay-off of this approach is that, although less explicit in the computation of local constants (because we do not specify a test vector), it works in greater generality (removing, for example, all the level assumptions) and it is still suitable for p-adic interpolation. It suffices indeed to apply this philosophy in the p-adic realm: p-adic L-functions arise from p-adic variation of $J_\eta = m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta$ regarded as a functional. Indeed, motivated by the above general rationality results, we expect that these periods could be frequently p-adically interpolated and we hope this formalism could be useful in order to address this issue. Although a further general investigation in this direction requires a better understanding of the Hecke operators at p of the Ash-Stevens distribution modules appearing in the definition of a p-adic family (see [2]), an application of our results in the case of triple product p-adic L-functions yields triple product p-adic L-functions which interpolate in the balanced region: this is the content of [10]. Also, as an application of our explicit Waldspurger formula, it is possible to generalize the construction of the p-adic L-functions considered in [3] (see [11] for details).

Remark 1.3. We end the introduction with few remarks.

(1) In this paper, we prove the rationality results discussed above in the case where $\mathbb{Q}$ is replaced, more generally, by a totally real field F. The (refined) Gan–Gross–Prasad conjectures and the formulas (5) have been formulated/proved in this more general setting. Let us remark once again that (6) depends on the validity of (5) up to $c(\pi_{V,\infty}\otimes\pi_{W,\infty}) \in \mathbb{C}^\times$, which is known in the orthogonal cases discussed before Remark 1.2 with $c(\pi_{V,\infty}\otimes\pi_{W,\infty}) = 1$, but the general unitary case requires the assumptions [34, §1.2, RH(I) and RH(II)] and the local factor $c(\pi_{V,\infty} \otimes \pi_{W,\infty})$ is not known.

(2) The rationality issues of (1) should not involve the condition $\mathbf{G} = \mathbf{G}'_\pi$ in the context of Gan–Gross–Prasad conjectures $\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W$; indeed one should expect that $I_\eta(f) = 0$ when $f \in \pi$ but $\mathbf{G} \neq \mathbf{G}'_\pi$ (for example, when $r = 0$ and $\mathrm{Hom}_{\mathbf{H}(F_\infty)}(\pi_{V,\infty} \otimes \pi_{W,\infty} \otimes \nu_\infty^{-1}, \mathbb{C}) = 0$, this can be easily checked) and, hence, I_η should be rational also on this portion of the space of automorphic forms. This is indeed the case when $\mathbf{G}(\mathbb{R})$ is compact (because, of course, the first rationality statement of Theorem A does not depend on $\mathbf{G} = \mathbf{G}'_\pi$).

(3) After a first version of our paper appeared, we have been warned that our Theorem C has been obtained in some cases also in [13]. More precisely, if we specialize it to the case $F = \mathbb{Q}$ and to the unitary case and make the assumptions of [34, §1.2, RH(I) and RH(II)] ensuring that (5) is in force (up to $c(\pi_{V,\infty} \otimes \pi_{W,\infty})$), our Theorem C is [13, Theorem 1.5]. Our compactness assumption $\mathbf{G} = \mathbf{G}'_\pi$ corresponds in loc.cit. to the fact that the representations under consideration should arise as the base change from automorphic representation of a definite unitary group. Our method is different and much more elementary than Grobner-Harris's strategy, this latter being based on Rankin-Selberg L-functions for $\mathbf{GL}_{n/K} \times \mathbf{GL}_{n-1/K}$ over an imaginary quadratic field $K/\mathbb{Q}$, Zhang's proof of refined Gan–Gross–Prasad conjecture for unitary groups (see [34]), rational structures arising both from Whittaker models and from the cohomology of the relevant Shimura varieties and base change results from definite unitary groups over $\mathbb{Q}$ to general linear groups over K. Rather, we construct and study the needed global and local rational structures to prove Theorem C without abandoning our group $\mathbf{G}$ over a totally real field F and we deduce Theorem C from Theorem A. Of course, in order to have (5) in force in the unitary case (up to $c(\pi_{V,\infty} \otimes \pi_{W,\infty})$), as remarked we need to appeal to [34] and, hence, definitely the same kind of assumptions of [13, Theorem 1.5] are needed.

2. Automorphic forms and the period integrals

In this section, we precisely define the period integrals central to our study. Let $\mathbf{G}$ be a reductive algebraic group over a field F with adele ring $\mathbb{A} = \mathbb{A}_f \times F_\infty$ and let $\mathbf{Z}_\mathbf{G}$ be its center. Let

$$\Delta_\mathbf{G}\colon \mathbf{G}(F) \longrightarrow \mathbf{G}(\mathbb{A}) \quad \text{and} \quad \Delta_{\mathbf{G},f}\colon \mathbf{G}(F) \longrightarrow \mathbf{G}(\mathbb{A}_f)$$

be the diagonal embedding and, for a closed, algebraic subgroup $\mathbf{Z}$ of $\mathbf{Z}_\mathbf{G}$, set

$$[\mathbf{G}(\mathbb{A})]_\mathbf{Z} := \mathbf{Z}(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})/\mathbf{G}(F) \quad \text{and} \quad [\mathbf{G}(\mathbb{A}_f)]_\mathbf{Z} := \mathbf{Z}(\mathbb{A}_f)\backslash \mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F).$$

Let

$$\mathbf{PG_Z} = \mathbf{G}/\mathbf{Z}.$$

We make the following assumptions on the pair $(\mathbf{G}, \mathbf{Z})$:

(A1) $\mathbf{PG_Z}(F_\infty)$ is compact;

(A2) $\Delta_{\mathbf{G},f}$ embeds $\mathbf{G}(F)$ as a discrete subgroup of $\mathbf{G}(\mathbb{A}_f)$;

(A3) $\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F)$ is compact.

Remark 2.1. Let $\mathbf{S_G}$ be the maximal split torus in the center of $\mathbf{G}$. Applying [14, Proposition 1.4] after restricting scalars from F to $\mathbb{Q}$ shows that (A2) implies (A2) and (A3). When $\mathbf{Z} = \mathbf{S_G}$ and $F = \mathbb{Q}$, it is proved in [14, Proposition 1.4] that (A1) is indeed equivalent to (A2) (and to (A3)). We also remark that (A1) (for any $\mathbf{Z}$) implies that $\mathbf{G} = \mathbf{Z_G}$ or F is totally real.

Thanks to our assumptions (A2) and (A3), the results of the following §3.1 applies. In particular, we may normalize the non-zero left $\mathbf{G}(\mathbb{A}_f)$-invariant Radon measures $\mu_{\mathbf{G}(\mathbb{A}_f)}$ on $\mathbf{G}(\mathbb{A}_f)$, $\mu_{\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F)}$ on $\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F)$ and $\mu_{[\mathbf{G}(\mathbb{A}_f)]_\mathbf{Z}}$ on $[\mathbf{G}(\mathbb{A}_f)]_\mathbf{Z}$ so that $\mu_{\mathbf{G}(\mathbb{A}_f)}(K) \in \mathbb{Q}$ for some (and hence every) $K \in \mathcal{K}$, $\mu_{\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F)}$ satisfies (11) and restricts to $\mu_{[\mathbf{G}(\mathbb{A}_f)]_\mathbf{Z}}$ on $\mathbf{Z}(\mathbb{A}_f)$-invariant functions. Furthermore, it easily follows from (A1) and (A2) that we may normalize the left $\mathbf{G}(\mathbb{A})$-invariant (resp. $\mathbf{G}(F_\infty)$-invariant) non-zero Radon measure $\mu_{[\mathbf{G}(\mathbb{A})]_\mathbf{Z}}$ (resp. $\mu_{\mathbf{Z}\backslash\mathbf{G},\infty}$) on $[\mathbf{G}(\mathbb{A})]_\mathbf{Z}$ (resp. $\mathbf{Z}(F_\infty)\backslash\mathbf{G}(F_\infty)$) so that the following formula is satisfied:

$$\int_{[\mathbf{G}(\mathbb{A})]_\mathbf{Z}} f(x)d\mu_{[\mathbf{G}(\mathbb{A})]_\mathbf{Z}}(x) = \int_{[\mathbf{G}(\mathbb{A}_f)]_\mathbf{Z}} \Big(\int_{\mathbf{Z}(F_\infty)\backslash\mathbf{G}(F_\infty)} f(x_f x_\infty) d\mu_{\mathbf{Z}\backslash\mathbf{G},\infty}(x_\infty)\Big) d\mu_{[\mathbf{G}(\mathbb{A}_f)]_\mathbf{Z}}(x_f). \tag{7}$$

For the remainder of this paper, we suppose that we are given two pairs $(\mathbf{H}, \mathbf{Z^H})$ and $(\mathbf{G}, \mathbf{Z}) = (\mathbf{G}, \mathbf{Z^G})$ as above and a morphism of algebraic groups

$$\eta\colon \mathbf{H} \longrightarrow \mathbf{G} \tag{8}$$

such that $\eta(\mathbf{Z^H}) \subset \mathbf{Z^G}$. We assume (A1), (A2) and (A3) for the pairs $(\mathbf{H}, \mathbf{Z^H})$ and $(\mathbf{G}, \mathbf{Z^G})$. In addition, we impose the above-mentioned normalizations to the measures obtained from the couple $(\mathbf{H}, \mathbf{Z^H})$. We use the abbreviations $[\mathbf{K}(\mathbb{A})] := [\mathbf{K}(\mathbb{A})]_{\mathbf{Z^K}}$, $[\mathbf{K}(\mathbb{A}_f)] := [\mathbf{K}(\mathbb{A}_f)]_{\mathbf{Z^K}}$ and $\mu_{\mathbf{Z^K}\backslash\mathbf{K},\infty} = \mu_{\mathbf{K},\infty}$ for $\mathbf{K} \in \{\mathbf{H}, \mathbf{G}\}$. We define $m^{-1}_{\mathbf{Z^H}\backslash\mathbf{H},\infty} := \mu_{\mathbf{Z^H}\backslash\mathbf{H},\infty}(\mathbf{PH_{Z^H}}(F_\infty))$.

Fix once and for all a *continuous* and *unitary* character $\omega\colon \frac{\mathbf{Z^G}(\mathbb{A})}{\mathbf{Z^G}(F)} \to \mathbb{C}^\times$. Let $S(\mathbf{G}(\mathbb{A}), \omega)$ be the space of functions $f\colon \mathbf{G}(\mathbb{A}) \to \mathbb{C}$ such that

$$f(zx) = \omega(z)f(x) \quad \text{for every } z \in \mathbf{Z^G}(\mathbb{A}),$$

endowed with the $(\mathbf{G}(\mathbb{A}), \mathbf{G}(\mathbb{A}))$-action defined by the rule

$$(\gamma\varphi u)(x) := \varphi(ux\gamma) \quad \text{for every } \gamma \in \mathbf{G}(\mathbb{A}) \text{ and } u \in \mathbf{G}(\mathbb{A}).$$

We write $L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega)$ for the right Hilbert space $\mathbf{G}(\mathbb{A})$-representation of L^2-automorphic forms with right $\mathbf{G}(\mathbb{A})$-invariant scalar product

$$\langle f_1, f_2\rangle := \int_{[\mathbf{G}(\mathbb{A})]} f_1(x)\overline{f_2}(x)\mu_{[\mathbf{G}(\mathbb{A})]}(x).$$

We recall that, if $S^{\infty\text{-fin}}(\mathbf{G}(\mathbb{A}), \omega)$ denotes the subspace of right $\mathbf{G}(F_\infty)$-finite vectors in $S(\mathbf{G}(\mathbb{A}), \omega)$ and $S^{\infty\text{-fin}}(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega) := S^{\infty\text{-fin}}(\mathbf{G}(\mathbb{A}), \omega)^{(\mathbf{G}(F),1)}$, then

$$A(\mathbf{G}(\mathbb{A}), \omega) := \bigcup_{K\in\mathcal{K}} S^{\infty\text{-fin}}(\mathbf{G}(\mathbb{A}), \omega)^{(\mathbf{G}(F),K)} = (S^{\infty\text{-fin}}(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega))^{\mathcal{K}}$$

is a right $\mathbf{G}(\mathbb{A}_f)$-submodule of $L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega)$, which is known to be dense in it. Indeed, due to the compactness of $\mathbf{PG_Z}(F_\infty)$, it is even a right $\mathbf{G}(\mathbb{A})$-submodule of it. In particular, if $\pi_\infty^u \in \mathrm{Irr}^u(\mathbf{G}(F_\infty), \omega_\infty^{-1})$, it makes sense to talk about the π_∞^u-isotypic component $A(\mathbf{G}(\mathbb{A}), \omega)[\pi_\infty^u]$ of $A(\mathbf{G}(\mathbb{A}), \omega)$.

We suppose that there is a character $\omega^\eta\colon \mathbf{H}(\mathbb{A}) \to \mathbb{C}^\times$ such that ω^η is trivial on $\mathbf{H}(F)$ and $\omega^\eta_{|\mathbf{Z^H}(\mathbb{A})} = \omega \circ \eta_{\mathbb{A}|\mathbf{Z^H}(\mathbb{A})}$. We write $\omega^{-\eta} := (\omega^\eta)^{-1}$.

Definition 2.2 (Global period integral). Define the *global period integral*

$$I_\eta\colon L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega) \longrightarrow \mathbb{C}$$

by the rule

$$I_\eta(f) := \int_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z^H}}} f(\eta(x))\omega^{-\eta}(x)d\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z^H}}}(x). \tag{9}$$

It is easily seen to be well defined and to satisfy the following $\mathbf{H}(F_\infty)$-equivariance property:

$$I_\eta(f\eta(h)) = \omega^\eta(h)I_\eta(f) \quad \text{for every } h \in \mathbf{H}(\mathbb{A}). \tag{10}$$

Our goal is to study the rationality properties of (9). As discussed in the introduction, in establishing rationality it is natural to work with one $\mathbf{G}(F_\infty)$-isotypic component of (the subspace of automorphic forms in) $L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega)$ at a time; this component can be conveniently described by means of Gross *algebraic modular forms* (see [14]). In §3 we develop a formalism of vector valued modular forms in the sense of Gross and define formal period integrals J. Using this language, we can define analytic and algebraic formal integrals J_{F_∞} and $J_{\mathbb{C}}$. Roughly, the rationality property of (9) is proved showing that

$$m^{-1}_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty} I_\eta \overset{(1)}{\simeq} J_{F_\infty} \overset{(2)}{\simeq} J_{\mathbb{C}}.$$

More precisely, we first prove (2) (Proposition 5.6) and then (1) (Theorem 7.2) assuming that the isotypic component we are working with has a rational model and is well behaved under twists; then we provide quite general conditions for these assumptions being satisfied in §7.1. These formal period integrals will also prove useful in our subsequent study of p-adic analogues of (9).

3. The formalism of profinite groups

3.1 – *Vector valued modular forms and the formal period integral*

In this section, we consider a data of the form

$$(\Gamma, G_f, Z_f) = (\Gamma, G_f, Z_f^G)$$

subject to the following assumptions. We suppose that G_f is a locally profinite unimodular group, let $\Gamma \subset G_f$ be a discrete subgroup such that G_f/Γ is compact and let $Z_f \subset Z_{G_f}$ be a closed subgroup. We write $\mathcal{K} = \mathcal{K}(G_f)$ to denote the set of its open and compact subgroups of G_f. We may normalize the Haar measure μ_{G_f} on G_f in such a way that $\mu_{G_f}(K) \in \mathbb{Q}$ for some (hence, every) $K \in \mathcal{K}$. Let $\mu_{G_f/\Gamma}$ be a nonzero, left G_f-invariant Radon measure on G_f/Γ, normalized so that

$$\int_{G_f} f(g)d\mu_{G_f}(g) = \int_{G_f/\Gamma} \sum f(g\gamma)d\mu_{G_f}(g); \tag{11}$$

Its existence is guaranteed by triviality of the module of μ_{G_f} on Γ (the discreteness of Γ is used here) and the unimodularity of G_f. It is unique up to nonzero scalar multiple. By compactness of G_f/Γ there is a nonzero, left G_f-invariant Radon measure $\mu_{Z_f\backslash G_f/\Gamma}$ on $Z_f\backslash G_f/\Gamma$, also unique up to nonzero scalar multiple, such that $\mu_{Z_f\backslash G_f/\Gamma}$ and $\mu_{G_f/\Gamma}$ agree on $C(Z_f\backslash G_f/\Gamma)$. (We view $C(Z_f\backslash G_f/\Gamma)$ as a subspace of $C(G_f/\Gamma)$ in the obvious way.)

Let G_∞ be a group and let $\Gamma \to G_\infty$ be a group homomorphism, so that $\Gamma \subset G_f \times G_\infty =: G$. If $g \in G$, we write $g_f \in G_f$ and $g_\infty \in G_\infty$ for its components. Let (V, ρ) be a right representation of G_∞ with coefficients in some commutative ring R. When ρ is understood, we simply write vg_∞ for $v\rho(g_\infty)$. Let $\omega_0\colon Z_f \longrightarrow R^\times$ be a character.

Definition 3.1. Define $S(G_f,\rho)$ to be the space of maps $\varphi\colon G_f\to V$ endowed with the (G,G_f)-action given by

$$(g\varphi u)(x) := \varphi(uxg_f)\rho(g_\infty^{-1}), \quad \text{where } g\in G \text{ and } u\in G_f.$$

Let

$$S(G_f,\rho,\omega_0)=\{\varphi\in S(G_f,\rho)\colon \varphi z=\omega_0(z)\varphi \text{ for all } z\in Z_f\}.$$

Then $S(G_f,\rho,\omega_0)$ is a (G,G_f)-submodule of $S(G_f,\rho)$. We also write

$$S(G_f/\Gamma,\rho_{/\Gamma},\omega_0) := S(G_f,\rho,\omega_0)^{(\Gamma,1)}.$$

Remark 3.2. The following facts is easily verified.

(1) If $S(G_f,\rho,\omega_0)^{(1,\mathcal{K})}\neq 0$ and V is R-torsion free, then $\omega_0(Z_f\cap K)=1$ for some $K\in\mathcal{K}$. It follows that if $R^\times$ is given any topology with the "no small subgroups" property (i.e., there is an open neighbourhood of 1 in $R^\times$ whose only compact subgroup is $\{1\}$), then ω_0 is continuous.

(2) If $S(G_f,\rho,\omega_0)^{(\Gamma,1)}\neq 0$, V is R-torsion free and (V,ρ) has central character ω_ρ, then $\omega_0(\gamma)=\omega_\rho(\gamma)$ for every $\gamma\in Z\cap Z_{G_\infty}\cap\Gamma\subset Z_\Gamma$. In particular, if $Z\cap\Gamma=Z_\Gamma$ and $Z_{G_\infty}\cap\Gamma=Z_\Gamma$ then $\omega_0(\gamma)=\omega_\rho(\gamma)$ for all $\gamma\in Z_\Gamma$.

Definition 3.3 (Vector-valued modular forms). Define the space of *ρ-valued modular forms on G_f* by

$$M(G_f,\rho)=M_\Gamma(G_f,\rho) := S(G_f,\rho)^{(\Gamma,\mathcal{K})}$$

and the subspace of *ρ-valued modular forms on G_f with character ω_0* by

$$M(G_f,\rho,\omega_0)=M_\Gamma(G_f,\rho,\omega_0) := S(G_f,\rho,\omega_0)^{(\Gamma,\mathcal{K})}.$$

Observe that

$$M_\Gamma(G_f,\mathbb{C})=C(G_f/\Gamma)^{\mathcal{K}} \quad\text{and}\quad M_\Gamma(G_f,\mathbb{C},1)=C(Z\backslash G_f/\Gamma)^{\mathcal{K}}.$$

The following remark is easily checked.

Remark 3.4. Suppose that $\chi_0\colon G_f\to R^\times$ is a character with the property that $\chi_0(K)=1$ for some $K\in\mathcal{K}$ and that $\chi_\infty\colon G_\infty\to R^\times$ is a character with the property that $\chi_{0|\Gamma}=\chi_{\infty|\Gamma}$.

(1) If $\varphi\in M(G_f,\rho,\omega_0)$, then the rule $(\chi_0\varphi)(x) := \chi_0(x)\varphi(x)$ defines an element $\chi_0\varphi\in M(G_f,\rho(\chi_\infty),\chi_{0|Z}\omega_0)$.

(2) We have $\chi_0\in M(G_f,R(\chi_\infty),\chi_{0|Z})$.

The formation of these spaces satisfies obvious functoriality properties. If $\psi\colon \rho \to \rho'$ is a morphism of representations of Γ, then we get

$$\psi_*\colon M(G_f, \rho, \omega_0) \longrightarrow M(G_f, \rho', \omega_0) \tag{12}$$

by the rule $\psi_*(\varphi) := \psi \circ \varphi$. In the opposite direction, suppose that we are given another triple (Δ, H_f, H) satisfying the assumptions that was done on (Γ, G_f, G_∞).

Definition 3.5. A period morphism $\eta\colon (\Delta, H_f, H_\infty, Z_f^H) \to (\Gamma, G_f, G_\infty, Z_f^G)$ is a couple $\eta = (\eta_f, \eta_\infty)$ of group morphisms $\eta_f\colon H_f \to G_f$ and $\eta_\infty\colon H_\infty \to G_\infty$ both mapping Δ to Γ and such that η_f is continuous and maps Z_f^H to Z_f^G.

Writing $\eta_\infty^*(\rho)$ for the H_∞-representation obtained by restriction from η_∞ and setting $\eta_f^*(\omega_0) = \omega_0 \circ \eta_{f|Z_f^H}$, we get

$$\eta^* = (\eta_f, \eta_\infty)^*\colon M_\Gamma(G_f, \rho, \omega_0) \longrightarrow M_\Delta(H_f, \eta_\infty^*(\rho), \eta_f^*(\omega_0)). \tag{13}$$

The following simple fact will be needed later: its proof relies in the finiteness of the double cosets $K\backslash G_f/\Gamma$ for every $K \in \mathcal{K}$ and is left to the reader.

Lemma 3.6. *The following facts hold.*

(1) *Suppose that we are given a family $\{(V_i, \rho_i)\}_{i\in I}$ of right G_∞-representations. Then there is a G_f-equivariant identification*

$$M(G_f, \textstyle\bigoplus_{i\in I} \rho_i, \omega_0) = \textstyle\bigoplus_{i\in I} M(G_f, \rho_i, \omega_0).$$

(2) *Suppose that (V, ρ) is a right G_∞-representation and that we are given a morphism of (unitary) rings $R \to R'$. If R' is R-flat or V is R-free then there is a G_f-equivariant identification*

$$M(G_f, R' \otimes_R \rho, \omega_0) = R' \otimes_R M(G_f, \rho, \omega_0).$$

3.1.1 – Trace maps

For $x \in G_f$ and $K \in \mathcal{K}$, define $\Gamma_K(x) = \Gamma \cap x^{-1}Kx$. Being discrete (as Γ is) and compact (as K is), the set $\Gamma_K(x)$ is finite. For each $K \in \mathcal{K}$ and each set $R_K \subset G_f$ of representatives of $K\backslash G_f/\Gamma$, define

$$T_K = T_{R_K}\colon M(G_f, R)^K \longrightarrow R, \qquad T_{R_K}(f) := \mu_{G_f}(K) \textstyle\sum_{x\in R_K} \frac{f(x)}{|\Gamma_K(x)|}. \tag{14}$$

Lemma 3.7. (1) *The quantity* $T_{R_K}(f)$ *depends only on* K *and not on* R_K, *justifying the notation* T_K.

(2) *If* $K_1 \subset K_2$, *so that* $M(G_f, R)^{K_1} \subset M(G_f, R)^{K_2}$, *then* $T_{R_{K_1}}(f) = T_{R_{K_2}}(f)$ *for all* $f \in M(G_f, R)^{K_1}$.

(3) *Let* ℓ_{g_f} *denote left multiplication by* $g \in G_f$. *Then the following diagram is commutative*:

$$\begin{array}{ccc} M(G_f, R)^K & \xrightarrow{\ \ell_g\ } & M(G_f, R)^{g^{-1}Kg} \\ & {\scriptstyle T_{R_K}} \searrow \quad \swarrow {\scriptstyle T_{R_{g^{-1}Kg}}} & \\ & \mathbb{C} & \end{array}$$

(4) *We have*

$$\mu_{G_0/\Gamma} = T_{G_0/\Gamma} \colon C(G_0/\Gamma)^{\mathcal{K}} = M(G_0, \mathbb{C}) \longrightarrow \mathbb{C}$$

and

$$\mu_{Z\backslash G_0/\Gamma} = T_{Z\backslash G_0/\Gamma} \colon C(Z\backslash G_0/\Gamma)^{\mathcal{K}} = M(Z\backslash G_0, \mathbb{C}) \longrightarrow \mathbb{C}.$$

Proof. One easily checks (1), (3) and (4). It's not hard to see that (2) is implied by the identity

$$\sum_{u \in K_1\backslash K_2/x\Gamma_{K_2}(x)x^{-1}} \frac{1}{|\Gamma_{K_1}(ux)|} = \frac{[K_2 : K_1]}{|\Gamma_{K_2}(x)|}. \tag{15}$$

One can verify this in case $R = \mathbb{C}$ as follows. First, the inclusion $K_1 y \subset K_1 y\Gamma$ induces a measure preserving homeomorphism $K_1 y/\Gamma_{K_1}(y) \xrightarrow{\sim} K_1 y\Gamma/\Gamma$ (by transport of the bijection). Putting these together and noticing that $K_1 ux\Gamma = K_1 u'x\Gamma$ if and only if $K_1 ux\Gamma_{K_2}(x)x^{-1} = K_1 u'x\Gamma_{K_2}(x)x^{-1}$ for every $u, u' \in K_2$, we obtain a measure preserving homeomorphism

$$\begin{aligned} &\bigsqcup_{u \in K_1\backslash K_2/x\Gamma_{K_2}(x)x^{-1}} K_1 ux/\Gamma_{K_1}(ux) \\ &\quad \xrightarrow{\sim} \bigsqcup_{u \in K_1\backslash K_2/x\Gamma_{K_2}(x)x^{-1}} K_1 ux\Gamma/\Gamma = K_2 x\Gamma/\Gamma. \end{aligned}$$

Therefore,

$$\sum_{u \in K_1\backslash K_2/x\Gamma_{K_2}(x)x^{-1}} \frac{\mu_{G_f}(K_1)}{|\Gamma_{K_1}(ux)|} = \mu_{G_f/\Gamma}(K_2 x\Gamma/\Gamma).$$

But the natural map $K_2 y \subset K_2 y\Gamma$ induces a measure preserving homeomorphism $K_2 y/\Gamma_{K_1}(y) \xrightarrow{\sim} K_2 y\Gamma/\Gamma$ (by transport of the bijection), implying $\mu_{G_f/\Gamma}(K_2 x\Gamma/\Gamma) = \frac{\mu_{G_f}(K_2)}{|\Gamma_{K_2}(x)|}$. □

It follows from parts (2) and (3) of Lemma 3.7 that the T_Ks fit together into R-linear functionals

(16) $$T_{G_f/\Gamma}\colon M(G_f, R) \longrightarrow R \quad \text{and} \quad T_{Z\backslash G_f/\Gamma}\colon M(Z\backslash G_f, R) \longrightarrow R$$

where $T_{G_f/\Gamma} = T_K$ on $M(G_f, R)^K$ and $T_{Z\backslash G_f/\Gamma} := T_{G_f/\Gamma|M(Z\backslash G_f,R)}$. Since we have assumed that $\mu_{Z\backslash G_f/\Gamma}$ is normalized so that it agrees with $\mu_{G_f/\Gamma}$ on $C(G_f/\Gamma)$, we see that

(17) $$T(f) = \textstyle\int_{Z\backslash G_f/\Gamma} f(g_f) d\mu_{Z\backslash G_f/\Gamma}(g_f)$$

for all $f \in M(G_f, \mathbb{C}, 1)$.

3.1.2 – Pairings and n-linear forms

We also have a natural map

(18) $$\otimes\colon M(G_f, \rho, \omega_0) \otimes_R M(G_f, \rho', \omega_0') \longrightarrow M(G_f, \rho \otimes_R \rho', \omega_0\omega_0')$$

defined by the rule $(\varphi \otimes \varphi')(x) := \varphi(x) \otimes \varphi'(x)$. In particular, writing $\rho^\vee$ for the R-dual representation $(v^\vee\gamma)(v) = v^\vee(v\gamma^{-1})$, we may define

(19) $$\begin{aligned}&\langle\cdot,\cdot\rangle\colon M(G_f, \rho, \omega_0) \otimes_R M(G_f, \rho^\vee, \omega_0^{-1}) \stackrel{\otimes}{\longrightarrow} M(G_f, \rho \otimes_R \rho^\vee)\\ &\qquad \longrightarrow M(Z\backslash G_f, R) \stackrel{T_{Z\backslash G_f/\Gamma}}{\longrightarrow} R.\end{aligned}$$

DEFINITION 3.8. We let $X(G_f, G_\infty, \omega_0) = X_\Gamma(G_f, G_\infty, Z_f, \omega_0)$ be the set of couples (χ_0, χ_∞) with the property that $\chi_0\colon G_f \to R^\times$ is a character such that $\chi_0(K) = 1$ for some $K \in \mathcal{K}$, $\chi_{0|Z_f} = \omega_0$ and $\chi_\infty\colon G_\infty \to R^\times$ is a character such that $\chi_{0|\Gamma} = \chi_{\infty|\Gamma}$.

Suppose that we are given a period morphism $\eta\colon (\Delta, H_f, H_\infty) \to (\Gamma, G_f, G_\infty)$, say $\eta = (\eta_f, \eta_\infty)$, that (V, ρ) is a representation of G_∞ with coefficients in some ring R and that $\omega_0\colon Z \to R^\times$ is a character. If $(\chi_0, \chi_\infty) \in X_\Delta(H_f, H_\infty, \eta_f^*(\omega_0))$ and we are given

$$\Lambda \in \mathrm{Hom}_{R[H_\infty]}(\eta_\infty^*(\rho), R(\chi_\infty)) = \rho^\vee(\chi_\infty)^{H_\infty}$$

then we get

$$M_\eta^{\chi_0,\chi_\infty}(\Lambda) \in \mathrm{Hom}_R(M(G_f, \rho, \omega_0), R)$$

by the rule

(20) $$\begin{aligned}&M_\eta^{\chi_0,\chi_\infty}(\Lambda)(\varphi)\\ &\quad := \mu_{H_f}(K) \textstyle\sum_{x \in K\backslash H_f/\Delta} \frac{\Lambda(\varphi(\eta_f(x)))\chi_0^{-1}(x)}{|\Delta_K(x)|} \quad \text{if } \varphi \in M(G_f, \rho, \omega_0)^K.\end{aligned}$$

Alternatively, we have

$$\begin{aligned}(21)\qquad M_\eta^{\chi_0,\chi_\infty}(\Lambda)\colon M_\Gamma(G_f,\rho,\omega_0) &\xrightarrow{\eta^*} M_\Delta(H_f,\eta_\infty^*(\rho),\eta_f^*(\omega_0))\\ &\xrightarrow{\Lambda_*} M(H_f,R(\chi_\infty),\eta_f^*(\omega_0)=\chi_{0|Z}) \xrightarrow{\langle\cdot,\chi_0^{-1}\rangle} R,\end{aligned}$$

where $\langle\cdot,\chi_0^{-1}\rangle$ is the pairing (19), which makes sense thanks to Remark 3.4 (2): it follows from this description that $M_\eta^{\chi_0,\chi_\infty}(\Lambda)$ is well defined. We write $M^{\chi_0,\chi_\infty} := M_\eta^{\chi_0,\chi_\infty}$.

In this case, we may define

$$(22)\qquad J^{\rho,\chi_0,\chi_\infty}\colon \rho^\vee(\chi_\infty)^{G_\infty}\otimes_R M(G_f,\rho,\omega_0)\longrightarrow R$$

by the rule

$$J^{\rho,\chi_0,\chi_\infty}(\Lambda\otimes_R\varphi) := M^{\chi_0,\chi_\infty}(\Lambda)(\varphi).$$

Finally, suppose that we are given a family $\{\rho_i\}_{i\in I}$ for representations, characters $\{\omega_{0,i}\}_{i\in I}$ and $\Lambda\in\mathrm{Hom}_{R[G_\infty]}(\rho,R(\chi_\infty))$, where $\rho := \otimes_{R,i\in I}\rho_i$. Then, assuming that $\prod_i\omega_{0,i}=\omega_0$ we generalize (19) as follows:

$$(23)\qquad \Lambda_M^{\chi_0,\chi_\infty}\colon \otimes_{R,i\in I}M(G_f,\rho_i,\omega_{0,i})\xrightarrow{\otimes} M(G_f,\rho,\omega_0)\xrightarrow{M^{\chi_0,\chi_\infty}(\Lambda)} R.$$

3.1.3 – The formal period integral

Suppose that we are given a period morphism $\eta\colon(\Delta,H_f,H_\infty)\to(\Gamma,G_f,G_\infty)$, say $\eta=(\eta_f,\eta_\infty)$, that (V,ρ) is a representation of G_∞ with coefficients in some ring R and that $\omega_0\colon Z\to R^\times$ is a character. Assume that we are given an H_∞-stable decomposition decomposition

$$(24)\qquad V^\vee = (\eta_\infty^*(\rho))^\vee(\chi_\infty)^{H_\infty}\oplus(\eta_\infty^*(\rho))^\vee(\chi_\infty)^{H_\infty,c}$$

where $(\chi_0,\chi_\infty)\in X_\Delta(H_f,H_\infty,\eta_f^*(\omega_0))$. It follows that we have a projection

$$p^{\rho,\chi_\infty}\colon V^\vee\longrightarrow(\eta_\infty^*(\rho))^\vee(\chi_\infty)^{H_\infty}.$$

Recall that we also have

$$\eta^*\colon M_\Gamma(G_f,\rho,\omega_0)\longrightarrow M_\Delta(H_f,\eta_\infty^*(\rho),\eta_f^*(\omega_0)).$$

It will be convenient to set

$$p_\eta^{\rho,\chi_\infty} := p^{\rho,\chi_\infty}\otimes_R\eta^*$$

and

$$M_\Gamma[G_f,\rho,\omega_0] := V^\vee\otimes_R M_\Gamma(G_f,\rho,\omega_0).$$

For every $(\chi_0,\chi_\infty) \in X_\Delta(H_f, H_\infty, \eta_f^*(\omega_0))$ and (24), we may define the formal period integral:

$$
\begin{aligned}
J_\eta^{\rho,\chi_0,\chi_\infty}\colon M_\Gamma[G_f,\rho,\omega_0] \\
&\xrightarrow{p^{\rho,\chi_\infty}_{\langle -,-\rangle_{V^\vee},\eta}} (\eta_\infty^*(\rho))^\vee(\chi_\infty)^{H_\infty} \otimes_R M_\Delta(H_f,\eta_\infty^*(\rho),\eta_f^*(\omega_0)) \\
&\xrightarrow{J^{\eta_\infty^*(\rho),\chi_0,\chi_\infty}} R.
\end{aligned}
\tag{25}
$$

4. Modular forms valued in algebraic representations and the algebraic period integral

Suppose that $F \subset E$ is a field extension such that $E/\mathbb{Q}$ is Galois and let $X_{E/F}$ be a set of embeddings $\sigma\colon E \hookrightarrow \mathbb{C}$ with the property that $\sigma \mapsto [\sigma_{|F}]$ defines a bijection between $X_{E/F}$ and the set (of equivalence classes) of archimedean places of F. We fix once and for all $\sigma_\infty \in X_{E/F}$, allowing us to regard $\mathbb{C}$ as an E-algebra, and an element $g_\sigma \in G_{F/\mathbb{Q}}$ with the property that $\sigma_\infty \circ g_\sigma = \sigma$ for every $\sigma \in X_{E/F}$, as granted by the fact that $E/\mathbb{Q}$ is Galois. Set

$$
\begin{aligned}
R^{X_{E/F}} &:= \textstyle\prod_{\sigma\in X_{E/F}} R,\\
\mathbf{G}^{X_{E/F}} &:= \textstyle\prod_{\sigma\in X_{E/F}} \mathbf{G},\\
\mathbf{H}^{X_{E/F}} &:= \textstyle\prod_{\sigma\in X_{E/F}} \mathbf{H}.
\end{aligned}
$$

We get a mapping

$$
E \longrightarrow R^{X_{E/F}}, \qquad x \longmapsto (g_\sigma(x))_{\sigma\in X_{E/F}},
$$

whose formation is functorial in R. We get an induced map

$$
\mathbf{G}(E) \longrightarrow \mathbf{G}(R^{X_{E/F}}) = \mathbf{G}^{X_{E/F}}(R) \tag{26}
$$

for every E-algebra. Note that this map is induced by $\prod_{\sigma\in X_{E/F}} \sigma\colon E \to \mathbb{C}^{X_{E/F}}$ when $R = \mathbb{C}$, thanks to $\sigma_\infty \circ g_\sigma = \sigma$.

Thanks to (A2) and (A3), the results of §3.1 apply to the triple

$$
(\Gamma, G_f, G_\infty) = (\mathbf{G}(F), \mathbf{G}(\mathbb{A}_f), \mathbf{G}^{X_{E/F}}(R))
$$

for every E-algebra R, where the required group homomorphism

$$
\Gamma = \mathbf{G}(F) \longrightarrow \mathbf{G}^{X_{E/F}}(R) = G_\infty
$$

is given by (26). The map $\eta\colon \mathbf{H} \to \mathbf{G}$ induces a morphism of triples

$$\eta_R := (\eta_{\mathbb{A}_f}, \eta_R^{X_{E/F}})\colon (\mathbf{H}(F), \mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}(R)) \longrightarrow (\mathbf{G}(F), \mathbf{G}(\mathbb{A}_f), \mathbf{G}^{X_{E/F}}(R)).$$

Let $\{(V_\sigma, \rho_\sigma)\}_{\sigma \in X_{E/F}}$ is a family of algebraic representations of $\mathbf{G}_{/E}$. Let

$$\omega_0\colon \mathbf{Z}^{\mathbf{G}}(\mathbb{A}_f) \longrightarrow E^\times$$

be a character. Let

$$(V, \rho) := \boxtimes_{\sigma \in X_{E/F}} (V_\sigma, \rho_\sigma),$$

be the external tensor product, a representation of $\mathbf{G}_{/E}^{X_{E/F}}$. For every E-algebra R, we have a representation (V_R, ρ_R) of $\mathbf{G}^{X_{E/F}}(R)$ and a character

$$\omega_{0,R}\colon \mathbf{Z}^{\mathbf{G}}(\mathbb{A}_f) \xrightarrow{\omega_0} E^\times \longrightarrow R^\times.$$

We may therefore form the spaces of *algebraic modular forms*

$$M(\mathbf{G}, \rho, \omega_0)(R) := M_{\mathbf{G}(F)}(\mathbf{G}(\mathbb{A}_f), \rho_R, \omega_{0,R}).$$

If $\psi\colon R \to R'$ is a homomorphism of E-algebras, then ψ induces a family $\psi_{\rho_\sigma}\colon \rho_{\sigma,R} \to \rho_{\sigma,R'}$ of morphisms of $\mathbf{G}(R)$-representations over R. Setting

$$\psi_\rho := \boxtimes_{\sigma \in X_{E/F}} \psi_{\rho_\sigma},$$

we get a morphism $\psi_\rho\colon \rho_R \to \rho_{R'}$ of $\mathbf{G}^{X_{E/F}}(R)$-representations over R and an induced $\mathbf{G}(\mathbb{A}_f)$-equivariant, R-linear map

$$\psi_{\rho,*}\colon M(\mathbf{G}, \rho, \omega_0)(R) \longrightarrow M(\mathbf{G}, \rho, \omega_0)(R').$$

If $(V^\vee, \rho^\vee)$ is the dual representation of (V, ρ), it will be convenient to define

$$M[\mathbf{G}, \rho, \omega_0](R) := V_R^\vee \otimes_R M_{\mathbf{G}(F)}(\mathbf{G}(\mathbb{A}_f), \rho_R, \omega_{0,R}). \tag{27}$$

Then

$$\psi_{\rho^\vee} \otimes_R \psi_{\rho,*}\colon M[\mathbf{G}, \rho, \omega_0](R) \longrightarrow M[\mathbf{G}, \rho, \omega_0](R').$$

Thus, we have defined two functors from E-algebras to $\mathbf{G}(\mathbb{A}_f)$-modules,

$$R \longmapsto M(\mathbf{G}, \rho, \omega_0) \quad \text{and} \quad R \longmapsto M[\mathbf{G}, \rho, \omega_0].$$

Let $X\big(\mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}, \eta^*_{\mathbb{A}_f}(\omega_0)\big)$ be the set of pairs

$$(\chi_0\colon \mathbf{H}(\mathbb{A}_f) \longrightarrow E^\times,\ \chi\colon \mathbf{H}^{X_{E/F}}_{/E} \longrightarrow \mathbf{G}_{m/E})$$

such that

$$(\chi_0, \chi_E) \in X_{\mathbf{H}(F)}\big(\mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}(E), \eta^*_{\mathbb{A}_f}(\omega_0)\big).$$

An element $(\chi_0, \chi) \in X\big(\mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}, \eta^*_{\mathbb{A}_f}(\omega_0)\big)$ naturally induces a family

$$\{(\chi_{0,R}, \chi_R) \in X_{\mathbf{H}(F)}(\mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}(R), \eta^*_{\mathbb{A}_f}(\omega_{0,R}))\},$$

where

$$\chi_{0,R}\colon \mathbf{H}(\mathbb{A}_f) \xrightarrow{\chi_0} E^\times \longrightarrow R^\times, \quad \chi_R\colon \mathbf{H}^{X_{E/F}}(R) \longrightarrow R^\times,$$

and

$$\omega_{0,R}\colon \mathbf{Z}^{\mathbf{G}}(\mathbb{A}_f) \xrightarrow{\omega_0} E^\times \longrightarrow R^\times.$$

Since $\mathbf{H}^{X_{E/F}}_{/E}$ is a reductive group over a characteristic zero field, the algebraic representation $\eta^*(\rho)^\vee$ admits a decomposition into isotypic components, one of which is $\eta^*(\rho)^\vee(\chi)^{\mathbf{H}}$. It follows that there is a canonical decomposition

$$V^\vee = \eta^*(\rho)^\vee(\chi)^{\mathbf{H}^{X_{E/F}}} \oplus \eta^*(\rho)^\vee(\chi)^{\mathbf{H}^{X_{E/F}},c}$$

which gives rise to a family of decompositions

$$V^\vee_R = \eta^*_R(\rho_R)^\vee(\chi_R)^{\mathbf{H}^{X_{E/F}}(R)} \oplus \eta^*_R(\rho_R)^\vee(\chi_R)^{\mathbf{H}^{X_{E/F}}(R),c}. \tag{28}$$

We write $J^{\rho_R,\chi_{0,R},\chi_R}_{\eta_R}$ for the period morphism (25) obtained from (28). We can now easily prove the following result.

Proposition 4.1. *Suppose that* (V, ρ) *is an algebraic representation of* $\mathbf{G}_{/E}$ *and that*

$$(\chi_0, \chi) \in X(\mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}(E), \eta^*_{\mathbb{A}_f}(\omega_0)).$$

(1) *The family* $J^{\rho,\chi_0,\chi}_\eta := \{J^{\rho_R,\chi_{0,R},\chi_R}_{\eta_R}\}$ *defines a morphism of functors*

$$J^{\rho,\chi_0,\chi}_\eta\colon M[\mathbf{G}, \rho, \omega_0] \longrightarrow \mathbf{A}^1_{/E}.$$

(2) *If* $\psi\colon R \to R'$ *is a morphism of* E*-algebras, there are canonical identifications*

$$M(\mathbf{G}, \rho, \omega_0)(R') = R' \otimes_R M(\mathbf{G}, \rho, \omega_0)(R)$$

and

$$M[\mathbf{G}, \rho, \omega_0](R') = R' \otimes_R M[\mathbf{G}, \rho, \omega_0](R)$$

such that

$$J^{\rho_{R'},\chi_{0,R'},\chi_{R'}}_{\eta_{R'}} = R' \otimes_R J^{\rho_R,\chi_{0,R},\chi_R}_{\eta_R}.$$

Proof. Claim (1) follows from the fact that (25) is functorial with respect to period morphisms and compatible decompositions (24) as those arising from (28) for different Rs. Claim (2) easily follows from Lemma 3.6 (2). □

5. Modular forms valued in complex representations and their rational models

If G is a real Lie group (resp. an algebraic group over some field), we let $\mathrm{Rep}(G)$ be the category of finite dimensional continuous complex representations (resp. finite dimensional algebraic representations defined over the field); we also let $\mathrm{Irr}(G,\omega)$ be the set of equivalence classes of irreducible representations in $\mathrm{Rep}(G)$ with central character ω and write $\mathrm{Irr}(G)$ for the union of them. For a reductive Lie group G which is compact modulo $Z \subset Z_G$ and an irreducible continuous complex representation (V_∞, π_∞), it can be proved that $(V_\infty, \pi_\infty) \in \mathrm{Rep}(G)$, i.e. it is finite dimensional, that there is unique up to non-zero scalar factor Hermitian product $\langle -,-\rangle_{V_\infty}$ which is G'-invariant (G' being the derived subgroup) and that there is a unique continuous (hence real Lie group) character $\delta_{\pi_\infty}\colon G \to \mathbb{C}^\times$ such that

$$\langle v_1 g, v_2 g\rangle_{V_\infty} = \delta_{\pi_\infty}(g)\langle v_1, v_2\rangle_{V_\infty} \quad \text{for every } v_1, v_2 \in V \text{ and } g \in G. \tag{29}$$

In particular, there is a natural inclusion $\mathrm{Irr}^u(G,\omega) \subset \mathrm{Irr}(G,\omega)$ with equality in case G is compact, ω needs to be unitary in this case, and every irreducible representation in $\mathrm{Rep}(G)$ is unitary up to twisting it by $\delta_{\pi_\infty}^{-1/2}$, which makes sense because δ_{π_∞} takes value in $\mathbb{R}_+^\times$ by (29) with $v_1 = v_2 \neq 0$. We write $\mathrm{Irr}^u(G)$ for the whole set of isomorphism classes of unitary Hilbert space representations (the isomorphism being only required to be G-equivariant).

Let $\omega_0\colon \mathbf{Z}^{\mathbf{G}}(\mathbb{A}_f) \to \mathbb{C}^\times$ be a *continuous* (not necessarily unitary) character and let (V_∞, π_∞) be a continuous complex right representation of $\mathbf{G}(F_\infty)$ with central character ω_{π_∞} (possibly not irreducible). We suppose that we are given a Hermitian scalar product

$$\langle -,-\rangle_{V_\infty}\colon V_\infty \times V_\infty \longrightarrow \mathbb{C}$$

satisfying (29). The scalar product on the dual $V_\infty^\vee$ of V_∞ is defined via the conjugate linear isomorphism $\Phi\colon V_\infty \to V_\infty^\vee$ defined by the rule

$$(\Phi(v), x) := \langle x, v\rangle_{V_\infty},$$

where $(-,-)$ denotes the evaluation pairing, and then setting

$$\langle v_1^\vee, v_2^\vee\rangle_{V_\infty^\vee} := \langle \Phi^{-1}(v_2^\vee), \Phi^{-1}(v_1^\vee)\rangle_{V_\infty}.$$

The dual representation $(V_\infty^\vee, \pi_\infty^\vee)$ is defined via

$$(\pi_\infty^\vee(g)v^\vee, v) := (v^\vee, v\pi_\infty(g));$$

regarding $V_\infty^\vee$ as a right $\mathbf{G}(F_\infty)$-module via

$$v^\vee \pi_\infty^\vee(g) := \pi_\infty^\vee(g^{-1})v^\vee,$$

it is easy to see that Φ is $\mathbf{G}(F_\infty)$-equivariant if and only if $\pi_\infty(g)^\vee = \pi_\infty(g^{-1})$, i.e. if and only if (V_∞, π_∞) is unitary.

If we are given a character $\chi_\infty\colon \mathbf{G}(F_\infty) \to \mathbb{C}^\times$, we may consider the representation $(V_\infty, \chi_\infty\pi_\infty) = (V_\infty, \pi_\infty(\chi_\infty))$, defined by the rule

$$(\chi_\infty\pi_\infty)(g_\infty) := \chi_\infty(g_\infty)\pi_\infty(g_\infty).$$

Writing $V_{\infty,\pi_\infty}^\vee$ be the underlying space of $\pi_\infty^\vee$, we can consider the orthogonal decomposition

$$V_\infty^\vee = V_{\infty,\pi_\infty}^\vee = (\eta_{F_\infty}^*(\pi_\infty))^\vee(\chi_\infty)^{\mathbf{H}(F_\infty)} \oplus (\eta_{F_\infty}^*(\pi_\infty))^\vee(\chi_\infty)^{\mathbf{H}(F_\infty),\perp}. \tag{30}$$

The following definition will be of crucial importance in order to connect automorphic forms and algebraic automorphic forms. Recall our fixed unitary and continuous character $\omega\colon \frac{\mathbf{Z}^{\mathbf{G}}(\mathbb{A})}{\mathbf{Z}^{\mathbf{G}}(F)} \to \mathbb{C}^\times$.

Definition 5.1. We say that a continuous character

$$\mathbf{N}\colon \mathbf{G}(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

with components

$$\mathbf{N}_f := \mathbf{N}_{|\mathbf{G}(\mathbb{A}_f)} \quad \text{and} \quad \mathbf{N}_\infty := \mathbf{N}_\infty := \mathbf{N}_{|\mathbf{G}(F_\infty)},$$

binds (V_∞, π_∞) *to* ω if

- $(\mathbf{N}_f, \mathbf{N}_\infty) \in X(\mathbf{G}(\mathbb{A}_f), \mathbf{G}(F_\infty), \mathbf{N}_{f|\mathbf{Z}^{\mathbf{G}}(\mathbb{A}_f)})$, i.e. $\mathbf{N}_f\mathbf{N}_\infty^{-1}\colon \mathbf{G}(\mathbb{A}) \to \mathbb{C}^\times$ is trivial on $\mathbf{G}(F)$;
- there is a continuous character $\omega_0\colon \mathbf{Z}^{\mathbf{G}}(\mathbb{A}_f) \to \mathbb{C}^\times$ with the property that

$$\omega_0\omega_{\pi_\infty}^{-1} = \omega\mathbf{N}_f\mathbf{N}_\infty^{-1} \quad \text{on } \mathbf{Z}_{\mathbf{G}}(\mathbb{A}) \text{ on } \mathbf{Z}^{\mathbf{G}}(\mathbb{A}) \tag{31}$$

  and $(V_\infty, \mathbf{N}_\infty^{-1}\pi_\infty)$, which has central character $\mathbf{N}_\infty^{-1}\omega_{\pi_\infty} = \omega_\infty^{-1}$, is a unitary representation of $\mathbf{G}(F_\infty)$.

If there exists $\mathbf{N}$ which binds (V_∞, π_∞) to ω_∞, we say that (V_∞, π_∞) belongs to ω.

Remarks 5.2. (1) If $\mathbf{N}$ binds (V_∞, π_∞) to ω then (31) determines $\omega_0 = \omega_f \mathbf{N}_f$. Conversely, if we only have $\omega_{\pi_\infty}^{-1} = \omega_\infty \mathbf{N}_\infty^{-1}$ and we define

$$\omega_0 \colon \mathbf{Z_G}(\mathbb{A}_f) \longrightarrow \mathbb{C}^\times, \quad \omega_0 := \omega_f \mathbf{N}_f,$$

then (31) is satisfied. For this reason, if $\mathbf{N}$ binds (V_∞, π_∞) to ω we will always write $\omega_0 := \omega_f \mathbf{N}_f$.

(2) If (V_∞, π_∞) belongs to ω, then (31) implies that $\omega_0 \omega_{\pi_\infty}^{-1}$ is trivial on $\mathbf{Z^G}(F)$ and this is compatible with Remark 3.2 (2) asserting that the space $M(\mathbf{G}(F_\infty), \pi_\infty, \omega_0)$ is non-zero only if $\omega_{\pi_\infty}(z_\infty) = \omega_0(z_f)$ for every $z \in \mathbf{Z^G}(F)$.

(3) If $\mathbf{N}$ binds (V_∞, π_∞) to ω_∞ and $\omega_0 := \omega_f \mathbf{N}_f$, then

$$\mathbf{N}_f^{-1} \colon M(\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0) \xrightarrow{\sim} M(\mathbf{G}(\mathbb{A}_f), \mathbf{N}_\infty^{-1}\pi_\infty, \omega_f).$$

is an isomorphism by Remark 3.4 (1).

(4) Suppose that $\mathbf{Z^G} = \mathbf{Z_G}$. If $\mathbf{N}$ binds (V_∞, π_∞) to ω_∞ and (V_∞, π_∞) is irreducible, then the equality $\mathbf{N}_\infty^{-1} \omega_{\pi_\infty} = \omega_\infty^{-1}$ already implies that $(V_\infty, \mathbf{N}_\infty^{-1} \pi_\infty)$ is unitary thanks to (29) because ω_∞^{-1} is and

$$\mathbf{Z^G}(F_\infty) = \mathbf{Z_G}(F_\infty) \longrightarrow \mathbf{G}(F_\infty) \longrightarrow \frac{\mathbf{G}(F_\infty)}{\mathbf{G}'(F_\infty)}$$

is an isogeny because $\mathbf{G}(F_\infty)$ is reductive.

(5) If $(V_\infty, \pi_\infty^u) \in \mathrm{Irr}^u(\mathbf{G}(F_\infty), \omega_\infty^{-1})$, we may always take $\mathbf{N} = 1$ and then (V_∞, π_∞^u) belongs to any Hecke character such that $\omega_{\pi_\infty^u}^{-1} = \omega_\infty$ and $\omega_0 = \omega_f$.

Suppose that $\mathbf{N}$ binds (V_∞, π_∞) to ω. Recall the scalar product $\langle -, - \rangle_{V_\infty}$ on V_∞: our assumption that $(V_\infty, \mathbf{N}_\infty^{-1} \pi_\infty)$ is unitary means that

$$\begin{aligned} &|\mathbf{N}_\infty(g_\infty)|^{-2} \langle v\pi_\infty(g_\infty), w\pi_\infty(g_\infty)\rangle_{V_\infty} \\ &\quad = \langle v(\mathbf{N}_\infty^{-1}\pi_\infty)(g_\infty), w(\mathbf{N}_\infty^{-1}\pi_\infty)(g_\infty)\rangle_{V_\infty} \\ &\quad = \langle v, w\rangle_{V_\infty}. \end{aligned} \tag{32}$$

It follows from Definition 5.1 and (32) that the following result is in force.

Lemma 5.3. *Suppose that (V_∞, π_∞) belongs to ω and that*

$$\varphi_1, \varphi_2 \in S(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0).$$

Then, for every $z \in \mathbf{Z^G}(\mathbb{A}_f)$ and $\gamma \in \mathbf{G}(F)$, we have the equality

$$|\mathbf{N}_f(zx\gamma_f)|^{-2} \langle \varphi_1(zx\gamma_f), \varphi_2(zx\gamma_f)\rangle_{V_\infty} = |\mathbf{N}_f(x)|^{-2} \langle \varphi_1(x), \varphi_2(x)\rangle_{V_\infty}.$$

It follows from (7) and 5.3 that, when (V_∞, π_∞) belongs to ω, the rule

$$\langle \varphi_1, \varphi_2 \rangle := \int_{[\mathbf{G}(\mathbb{A}_f)]} |\mathbf{N}_f(x_f)|^{-2} \langle \varphi_1(x_f), \varphi_2(x_f) \rangle_{V_\infty} d\mu_{[\mathbf{G}(\mathbb{A}_f)]}(x_f)$$

makes sense for the measurable functions $\varphi_1, \varphi_2 \in S(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0)$. We may therefore define the spaces $L^2(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0)$ in the usual way, by taking the finite normed vectors in the completion of the quotient by the kernel of $\langle -, - \rangle$ of the subset of measurable function on $S(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0)$. Since $\langle \varphi_1 g, \varphi_2 g \rangle = \langle \varphi_1, \varphi_2 \rangle$ for every $g \in \mathbf{G}(\mathbb{A}_f)$, we find a right Hilbert space representation of $\mathbf{G}(\mathbb{A}_f)$. Then

$$M(\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0) \subset L^2(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0)$$

is a dense right $\mathbf{G}(\mathbb{A}_f)$-submodule.

Since $\mathbf{G}(F_\infty)/\mathbf{Z}^{\mathbf{G}}(F_\infty)$ is compact and $\mathbf{G}(F_\infty)$ a reductive Lie group, every irreducible representation (V_∞, π_∞) of $\mathbf{G}(F_\infty)$ is finite dimensional and can be written as the product $(V_\infty, \pi_\infty) \simeq \boxtimes_{\sigma \in X_{E/F}, \mathbb{C}} (V_{\infty,\sigma}, \pi_{\infty,\sigma})$ where $(V_{\infty,\sigma}, \pi_{\infty,\sigma})$ is an irreducible representation of $\mathbf{G}(F_\sigma)$ (uniquely determined up to isomorphism). Here $F_\sigma \subset \mathbb{C}$ denotes the completion of F at $\sigma_{|F}$, so that $F_\infty = \prod_{\sigma \in X_{E/F}} F_\sigma$ canonically (once $X_{E/F}$ has been fixed) and

$$\mathbf{G}(F_\infty) = \prod_{\sigma \in X_{E/F}} \mathbf{G}(F_\sigma) \subset \mathbf{G}^{X_{E/F}}(\mathbb{C}). \tag{33}$$

This facts motivate the following definition, where (V_∞, π_∞) could be any continuous complex representation of $\mathbf{G}(F_\infty)$.

Definition 5.4. A model of (V_∞, π_∞) over E is a family $\{(V_\sigma, \rho_\sigma)\}_{\sigma \in X_{E/F}}$ of algebraic representations (V_σ, ρ_σ) of $\mathbf{G}_{/E}$ such that, setting

$$(V, \rho) := \boxtimes_{\sigma \in X_{E/F}, E} (V_\sigma, \rho_\sigma),$$

we have $(V_\infty, \pi_\infty) \simeq (V_{\mathbb{C}}, \rho_{\mathbb{C}})$ as representations of $\mathbf{G}(F_\infty) \subset \mathbf{G}^{X_{E/F}}(\mathbb{C})$ via (33).

It is not difficult to see that every irreducible (V_∞, π_∞) admits a model over some finite field extension E/F (and we may take E such that $\mathbf{G}_{/E}$ is split). This definition applies to characters: a model of a character $\chi_\infty \colon \mathbf{G}(F_\infty) \to \mathbb{C}^\times$ is a family of algebraic characters $\{\chi_\sigma \colon \mathbf{G}_{/E} \to \mathbf{G}_{m/E}\}$ with the property that, setting $\chi := \prod_{\sigma \in X_{E/F}} \chi_\sigma$, we have $\chi_{\mathbb{C}|\mathbf{G}(F_\infty)} = \chi_\infty$.

Suppose that $\{(V_\sigma, \rho_\sigma)\}_{\sigma \in X_{E/F}}$ is a model of (V_∞, π_∞) over E and that $\{\chi_\sigma\}_{\sigma \in X_{E/F}}$ is a model of $\chi_\infty \colon \mathbf{G}(F_\infty) \to \mathbb{C}^\times$. If (V, ρ) and χ are defined as above, then we can consider $p^{\pi_\infty, \chi_\infty}$ (resp. $p^{\rho_{\mathbb{C}}, \chi_{\mathbb{C}}}$) comes from (30) (resp. (28) with $R = \mathbb{C}$).

Lemma 5.5. *Up to the identification* $V_\infty^\vee \simeq V_{\mathbb{C}}^\vee$ *induced by* $(V_\infty, \pi_\infty) \simeq (V_{\mathbb{C}}, \rho_{\mathbb{C}})$*, we have* $p^{\pi_\infty,\chi_\infty} \simeq p^{\rho_{\mathbb{C}},\chi_{\mathbb{C}}}$.

Proof. Since $\mathbf{G}(F_\infty)/\mathbf{Z}^{\mathbf{G}}(F_\infty)$, the Schur orthogonality relations imply that $p^{\rho_{\mathbb{C}},\chi_{\mathbb{C}}}$ is the projection onto the isotypic χ_∞-component. Then, using the fact that we are working in characteristic zero and passing to the Lie algebras, the claim is easily deduced. □

Suppose now that $(\chi_0, \chi) \in X(\mathbf{H}(\mathbb{A}_f), \mathbf{H}^{X_{E/F}}, \eta^*_{\mathbb{A}_f}(\omega_0))$, implying that we may consider the formal period integrals $J^{\pi_\infty,\chi_0,\chi}_{\eta_{F_\infty}}$ (resp. $J^{\rho_{\mathbb{C}},\chi_{0,\mathbb{C}},\chi_{\mathbb{C}}}_{\eta_{\mathbb{C}}}$) (25) obtained from (30) (resp. (28)).

Proposition 5.6. *The identification* $(V_\infty, \pi_\infty) \simeq (V_{\mathbb{C}}, \rho_{\mathbb{C}})$ *induce isomorphisms of* $\mathbf{G}(\mathbb{A}_f)$*-modules*

$$M(\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0) \simeq M(\mathbf{G}, \rho, \omega_0)(\mathbb{C})$$

and

$$M[\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0] \simeq M[\mathbf{G}, \rho, \omega_0](\mathbb{C}).$$

The latter identifies $J^{\pi_\infty,\chi_0,\chi}_{\eta_{F_\infty}} \simeq J^{\rho_{\mathbb{C}},\chi_{0,\mathbb{C}},\chi_{\mathbb{C}}}_{\eta_{\mathbb{C}}}$.

Proof. Recall that the morphism $\mathbf{G}(E) \to \mathbf{G}^{X_{E/F}}(R)$ of (26) was defined so that it is induced by $\prod_{\sigma \in X_{E/F}} \sigma : E \to \mathbb{C}^{X_{E/F}}$ when $R = \mathbb{C}$. It follows that its restriction to $\mathbf{G}(F) \subset \mathbf{G}(E)$ equals the canonical morphism $\mathbf{G}(F) \to \mathbf{G}(F_\infty)$ followed by (33). Hence the identification $(V_\infty, \pi_\infty) \simeq (V_{\mathbb{C}}, \rho_{\mathbb{C}})$ induces a $(\mathbf{G}(\mathbb{A}_f), \mathbf{G}(F))$-equivariant identification

$$S(\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0) \simeq S(\mathbf{G}(\mathbb{A}_f), \rho_{\mathbb{C}}, \omega_0).$$

The two isomorphisms follows. Going back to (25), we see that it suffices to show that $p^{\pi_\infty,\chi_\infty} \simeq p^{\rho_{\mathbb{C}},\chi_{\mathbb{C}}}$ via $V_\infty^\vee \simeq V_{\mathbb{C}}^\vee$ in order to prove $J^{\pi_\infty,\chi_0,\chi}_{\eta_{F_\infty}} \simeq J^{\rho_{\mathbb{C}},\chi_{0,\mathbb{C}},\chi_{\mathbb{C}}}_{\eta_{\mathbb{C}}}$. Hence the claim follows from Lemma 5.5. □

6. The adelic Peter–Weyl theorem

Suppose that $\mathbf{N}$ binds (V_∞, π_∞) to ω, so that we write $\omega_0 = \omega_f \mathbf{N}_f$ (see Remark 5.2 (1)), and let $d_{\mathbf{N}_\infty^{-1}\pi_\infty,\mu_{\mathbf{G},\infty}}$ be the formal degree of the representation $\mathbf{N}_\infty^{-1}\pi_\infty$ with respect to $\mu_{\mathbf{G},\infty}$. Recall that we write $(V_\infty^\vee, \pi_\infty^\vee)$ for the dual left representation. The following result is an application of the Peter–Weyl theorem and Definition 5.1 (it generalizes [24, Theorem 1.3]).

Proposition 6.1 (adelic Peter–Weyl theorem). *Suppose that* $\mathbf{N}$ *binds* (V_∞, π_∞) *to* ω. *For every* $\Lambda \in V_\infty^\vee$ *there is an injective map*

$$f^{\mathbf{N}}_{\Lambda,\cdot} = f^{\mathbf{N},\pi_\infty}_{\Lambda,\cdot} \colon L^2(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0) \hookrightarrow L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega),$$

induced by the rule $f^{\mathbf{N}}_{\Lambda,\cdot}(x) := (\mathbf{N}_f^{-1}\mathbf{N}_\infty)(x)(\Lambda, \varphi(x_f)x_\infty^{-1})$, *which has the following properties.*

(1) *For every* $u \in \mathbf{G}(\mathbb{A}_f)$, *it satisfies the rule*

$$f^{\mathbf{N}}_{\Lambda,\varphi u} = \mathbf{N}_f(u) f^{\mathbf{N}}_{\Lambda,\varphi} u.$$

(2) *For every* $\varphi_1, \varphi_2 \in L^2(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0)$, *it holds the formula*

$$\langle f^{\mathbf{N}}_{\Lambda,\varphi_1}, f^{\mathbf{N}}_{\Lambda,\varphi_2} \rangle = \frac{\langle \Lambda, \Lambda \rangle_{V_\infty^\vee}}{d_{\mathbf{N}_\infty^{-1}\pi_\infty, \mu_{\mathbf{G},\infty}}} \langle \varphi_1, \varphi_2 \rangle.$$

(3) *It induces an embedding*

$$f^{\mathbf{N}}_{\Lambda,\cdot} \colon M(\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0) \hookrightarrow A(\mathbf{G}(\mathbb{A}), \omega)$$

and, setting $M[\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0] := V_\infty^\vee \otimes_{\mathbb{C}} M(\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0)$, *for varying* Λ*s they induce the* $\mathbf{G}(\mathbb{A}_f)$*-equivariant identification*:

$$f^{\mathbf{N}}_{\cdot,\cdot\cdot} = f^{\mathbf{N},\pi_\infty}_{\cdot,\cdot\cdot} \colon M[\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0](\mathbf{N}_f^{-1}) \xrightarrow{\sim} A(\mathbf{G}(\mathbb{A}), \omega)[\mathbf{N}_\infty^{-1}\pi_\infty].$$

(4) *The above rules* $f^{\pi^u_\infty}_{\cdot,\cdot\cdot} := f^{\mathbf{N},\pi^u_\infty}_{\cdot,\cdot\cdot}$ *with* $\mathbf{N} = 1$ *induce a* $\mathbf{G}(\mathbb{A}_f)$*-equivariant identifications*

$$\oplus_{\pi^u_\infty} f^{\pi^u_\infty}_{\cdot,\cdot\cdot} \colon \bigoplus_{\pi^u_\infty \in \mathrm{Irr}^u(\mathbf{G}(F_\infty), \omega_\infty^{-1})} M[\mathbf{G}(\mathbb{A}_f), \pi^u_\infty, \omega_0 = \omega_f] \xrightarrow{\sim} A(\mathbf{G}(\mathbb{A}), \omega)$$

and we have $\mathrm{Im}(f^{\mathbf{N},\pi_\infty}_{\cdot,\cdot\cdot}) = \mathrm{Im}(f^{\pi^u_\infty}_{\cdot,\cdot\cdot})$ *when* $\pi^u_\infty = \mathbf{N}_\infty^{-1}\pi_\infty$.

Proof. It follows from the Peter–Weyl theorem that, setting

$$\psi^{\pi^u_\infty}_{\Lambda,v}(x) := (\Lambda, v\pi^u_\infty(x_\infty)) \quad \text{for } \Lambda \in \pi^{u\vee}_\infty \text{ and } v \in \pi^u_\infty,$$

yields a $(\mathbf{G}(F_\infty), \mathbf{G}(F_\infty))$-equivariant identification of Hilbert spaces (up to a scalar factor on each component)

$$\begin{aligned} \oplus_{\pi^u_\infty} \psi^{\pi^u_\infty}_{\cdot\cdot,\cdot} \colon \bigoplus_{\pi^u_\infty \in \mathrm{Irr}^u(\mathbf{G}(F_\infty), \omega_\infty^{-1})} &(V_\infty^\vee, \pi^{u\vee}_\infty) \otimes_{\mathbb{C}} (V_\infty, \pi^u_\infty) \\ &\xrightarrow{\sim} S^{\infty\text{-fin}}(\mathbf{G}(F_\infty), \omega_\infty^{-1}), \end{aligned} \tag{34}$$

where the target denotes the subspace of right $\mathbf{G}(F_\infty)$-finite vectors in $S(\mathbf{G}(F_\infty), \omega_\infty^{-1})$. More explicitly, the fact that $\oplus_{\pi_\infty^u}\psi_{\cdot,\cdot}^{\pi_\infty^u}$ is an identification of Hilbert spaces (up to a scalar factor on each component) means that the spaces indexed by different π_∞^us are orthogonal, while $d_{\pi_\infty^u,\mu_{\mathbf{G},\infty}} \in \mathbb{R}_{>0}^\times$ is defined so that

(35)
$$\int_{\mathbf{G}(F_\infty)/\mathbf{Z}^{\mathbf{G}}(F_\infty)} \psi_{\Lambda_1,v_1}^{\pi_\infty^u}(x_\infty)\overline{\psi_{\Lambda_2,v_2}^{\pi_\infty^u}}(x_\infty)d\mu_{\mathbf{G},\infty}(x_\infty) = \frac{\langle\Lambda_1,\Lambda_2\rangle_{V_\infty^\vee}\langle v_1,v_2\rangle_{V_\infty}}{d_{\pi_\infty^u,\mu_{\mathbf{G},\infty}}}$$

for $v_1, v_2 \in \pi_\infty^u$ and $\Lambda_1, \Lambda_1 \in \pi_\infty^{u\vee}$. Since $\mathbf{G}(F_\infty)/\mathbf{Z}^{\mathbf{G}}(F_\infty)$ is compact, every irreducible and unitary representation is finite dimensional and it is a well known fact that an element of $S(\mathbf{G}(F_\infty), \omega_\infty)$ is right (or left) $\mathbf{G}(F_\infty)$-finite if and only if it is the matrix coefficient of a finite dimensional representation. Next one remarks that (see Remark 5.2 (1))

$$f_\varphi^{\Lambda,\mathbf{N}} = \psi_{\Lambda,\cdot}^{\mathbf{N}_\infty^{-1}\pi_\infty} \circ (\mathbf{N}_f^{-1}\varphi) \tag{36}$$

and check that

$$f_{g\varphi u}^{\Lambda,\mathbf{N}} = \mathbf{N}_f(u)(\mathbf{N}_f\mathbf{N}_\infty^{-1})(g)gf_\varphi^{\Lambda,\mathbf{N}}u \tag{37}$$

for every $(g,u) \in \mathbf{G}(\mathbb{A}) \times \mathbf{G}(\mathbb{A}_f)$ using Definition 5.1. Since $(\mathbf{N}_f\mathbf{N}_\infty^{-1})(g) = 1$ for $g \in \mathbf{G}(F)$ one finds

$$S(\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F), \pi_{\infty/\mathbf{G}(F)}, \omega_0) \hookrightarrow S^{\infty\text{-fin}}(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega)$$

The continuity of $\mathbf{N}_f$ and (37) give the inclusion in (3) after applying $(-)^{\mathcal{K}}$. Taking the completion gives the map $f_\cdot^{\Lambda,\mathbf{N}}$ between the L^2-spaces and (2), in view of (35) (which also implies the injectivity of $f_\cdot^{\Lambda,\mathbf{N}}$). In order to prove (4), from which (3) follows, we may assume that $\mathbf{N} = 1$ thanks to (36), so that $f_\varphi^{\Lambda,\mathbf{N}} = \psi_{\Lambda,\cdot}^{\pi_\infty^u}$ with $\pi_\infty^u := \mathbf{N}_\infty^{-1}\pi_\infty$. Applying $(S(\mathbf{G}(\mathbb{A}_f),-)^{(\mathbf{G}(F),1)})^{\mathcal{K}}$ to (34) and employing Lemma 3.6 (1) in order to express the left hand side, the claim is reduced to the obvious

$$M(\mathbf{G}(\mathbb{A}_f), S^{\infty\text{-fin}}(\mathbf{G}(F_\infty), \omega_\infty^{-1}), \omega_f) = A(\mathbf{G}(\mathbb{A}), \omega). \qquad \square$$

7. Period integrals and their algebraicity

As usual, suppose that $\mathbf{N}$ binds (V_∞, π_∞) to ω and recall our morphism of algebraic groups $\eta: \mathbf{H} \to \mathbf{G}$ and $\omega^\eta: \mathbf{H}(\mathbb{A}) \to \mathbb{C}^\times$ such that ω^η is trivial on $\mathbf{H}(F)$ and $\omega^\eta_{|\mathbf{Z}^{\mathbf{H}}(\mathbb{A})} = \omega \circ \eta_{\mathbb{A}|\mathbf{Z}^{\mathbf{H}}(\mathbb{A})}$. We set $\mathbf{N}^\eta := \mathbf{N} \circ \eta_{\mathbb{A}}$ and use the shorthands $\omega^{-\eta} := (\omega^\eta)^{-1}$ and $\mathbf{N}^{-\eta} := (\mathbf{N}^\eta)^{-1}$. In this section we prove "$m_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}^{-1} I_\eta \simeq J_{F_\infty}$": the first step consists of expressing the projection $p^{\pi_\infty,\chi_\infty}$

arising from (30) in case $\chi_\infty = \omega_\infty^{-\eta}\mathbf{N}_\infty^\eta$ in terms of integration. To this end we focus on the local period integral

$$I_{\eta,\infty}\colon L^2(\mathbf{Z}^{\mathbf{G}}(F_\infty)\backslash\mathbf{G}(F_\infty),\omega_\infty^{-1}) \longrightarrow \mathbb{C}$$

defined by the rule

$$I_{\eta,\infty}(f) := \int_{\mathbf{Z}^{\mathbf{H}}(F_\infty)\backslash\mathbf{H}(F_\infty)} f(\eta(x_\infty))\omega_\infty^\eta(x_\infty)d\mu_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}(x_\infty).$$

It is well defined because $x_\infty \mapsto f(\eta(x_\infty))\omega_\infty^\eta(x_\infty)$ is invariant under $\mathbf{Z}^{\mathbf{H}}(F_\infty)$. The above formula defines a linear functional which satisfies the $\mathbf{H}(F_\infty)$-equivariance property

(38) $$I_{\eta,\infty}(f\eta(h)) = \omega_\infty^{-\eta}(h)I_{\eta,\infty}(f) \quad \text{for every } h \in \mathbf{H}(F_\infty).$$

Recall the embedding

$$\psi_{\Lambda,\cdot}^{\mathbf{N}_\infty^{-1}\pi_\infty}\colon (V_\infty, \mathbf{N}_\infty^{-1}\pi_\infty) \hookrightarrow L^2(\mathbf{G}(F_\infty),\omega_\infty^{-1})$$

(see (34)) and define

$$r_\eta := I_{\eta,\infty}\circ\psi_{\Lambda,\cdot}^{\mathbf{N}_\infty^{-1}\pi_\infty}\colon V_{\infty,\pi_\infty}^\vee \longrightarrow V_{\infty,\pi_\infty}^\vee,$$

i.e.

$$r_\eta(\Lambda)(v) := I_{\eta,\infty}(\psi_{\Lambda,v}^{\mathbf{N}_\infty^{-1}\pi_\infty}).$$

LEMMA 7.1. *The map r_η induces*

$$m_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}p^{\pi_\infty,\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta}\colon V_{\infty,\pi_\infty}^\vee \longrightarrow \mathrm{Hom}_{\mathbf{H}(F_\infty)}(\omega_\infty^\eta\mathbf{N}_\infty^{-\eta}\pi_\infty,\mathbb{C}) = (\eta_{F_\infty}^*(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty)},$$

i.e. we have

$$r_\eta(\Lambda) = \begin{cases} m_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}\Lambda & \text{if } \Lambda \in (\eta_{F_\infty}^*(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty)},\\ 0 & \text{if } \Lambda \in (\eta_{F_\infty}^*(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty),\perp}.\end{cases}$$

PROOF. The embedding $\psi_{\Lambda,\cdot}^{\mathbf{N}_\infty^{-1}\pi_\infty}$ is $\eta(\mathbf{H}(F_\infty))$-equivariant and then (38) implies that

$$I_{\eta,\infty}\circ\psi_{\Lambda,\cdot}^{\mathbf{N}_\infty^{-1}\pi_\infty} \in \mathrm{Hom}_{\mathbf{H}(F_\infty)}(\omega_\infty^\eta\mathbf{N}_\infty^{-1}\pi_\infty,\mathbb{C}).$$

Suppose that $\Lambda_\eta \in \mathrm{Hom}_{\mathbf{H}(F_\infty)}(\omega_\infty^\eta\mathbf{N}_\infty^{-1}\pi_\infty,\mathbb{C})$, meaning that

(39) $$(\mathbf{N}_\infty^{-1}\pi_\infty)(\eta(x_\infty))\Lambda_\eta = \overline{\omega_\infty^\eta(x_\infty)}\Lambda_\eta.$$

Using (39) and exploring the definition of $\psi^{\mathbf{N}_\infty^{-1}\pi_\infty}_{\Lambda_\eta,v}$, we find

$$\begin{aligned}\psi^{\mathbf{N}_\infty^{-1}\pi_\infty}_{\Lambda_\eta,v}(\eta(x_\infty)) &= \psi^{\mathbf{N}_\infty^{-1}\pi_\infty}_{(\mathbf{N}_\infty^{-1}\pi_\infty)(\eta(x_\infty))\Lambda_\eta,v}(1)\\ &= \overline{\omega_\infty(\eta(x_\infty))}\psi^{\mathbf{N}_\infty^{-1}\pi_\infty}_{\Lambda_\eta,v}(1)\\ &= \overline{\omega_\infty(\eta(x_\infty))}(\Lambda_\eta, v).\end{aligned}$$

It easily follows that

$$r_\eta(\Lambda_\eta)(v) = I_{\eta,\infty}(\psi^{\mathbf{N}_\infty^{-1}\pi_\infty}_{\Lambda_\eta,v}) = \mu_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}(\mathbf{PH}_{\mathbf{Z}^{\mathbf{H}}}(F_\infty))\cdot(\Lambda_\eta, v),$$

proving that $r_\eta = \mu_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}(\mathbf{PH}_{\mathbf{Z}^{\mathbf{H}}}(F_\infty))\cdot 1$ on $\mathrm{Hom}_{\mathbf{H}(F_\infty)}(\omega_\infty^\eta\mathbf{N}_\infty^{-1}\pi_\infty, \mathbb{C})$.

Using the fact that $\mu_{\mathbf{H},\infty}$ is both right and left invariant, one checks that r_η is $\mathbf{H}(F_\infty)$-equivariant. Consider the orthogonal decomposition of the $\mathbf{H}(F_\infty)$-representation

$$V^\vee_{\infty,\pi_\infty} = (\eta^*_{F_\infty}(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty)} \oplus (\eta^*_{F_\infty}(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty),\perp}.$$

The irreducible representations appearing in the orthogonal complement are not isomorphic to the representation $(\eta^*_{F_\infty}(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty)}$, because this latter is an $\mathbf{H}(F_\infty)$-isotypic component. The $\mathbf{H}(F_\infty)$-equivariance of r_η, which maps to $(\eta^*_{F_\infty}(\pi_\infty))^\vee(\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)^{\mathbf{H}(F_\infty)}$, implies that $r_\eta = 0$ on $\mathrm{Hom}_{\mathbf{H}(F_\infty)}(\omega_\infty^\eta\mathbf{N}_\infty^{-1}\pi_\infty, \mathbb{C})^\perp$. □

Recall the $\mathbf{G}(\mathbb{A}_f)$-equivariant identification

$$f^{\mathbf{N},\pi_\infty}_{\cdot,\cdot\cdot}\colon M[\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0](\mathbf{N}_f^{-1}) \xrightarrow{\sim} A(\mathbf{G}(\mathbb{A}), \omega)[\mathbf{N}_\infty^{-1}\pi_\infty]$$

from Proposition 6.1 (3). Since $\mathbf{N}$ binds (V_∞, π_∞) to ω, one easily checks that

$$(\omega_f^\eta\mathbf{N}_f^\eta, \omega_\infty^{-\eta}\mathbf{N}_\infty^\eta) \in X(\mathbf{H}(\mathbb{A}_f), \mathbf{H}(F_\infty), \eta^*_{\mathbb{A}_f}(\omega_0)).$$

Let us write $J^{\pi_\infty,\omega_f^\eta\mathbf{N}_f^\eta,\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta}_{\eta F_\infty}$ for the period morphism (25) obtained from (30) with $(\chi_0, \chi_\infty) = (\omega_f^\eta\mathbf{N}_f^\eta, \omega_\infty^{-\eta}\mathbf{N}_\infty^\eta)$. The following result expresses $I_\eta \circ f^{\mathbf{N},\pi_\infty}_{\cdot,\cdot\cdot}$ in terms of $J^{\pi_\infty,\omega_f^\eta\mathbf{N}_f^\eta,\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta}_{\eta F_\infty}$, thus characterizing the restriction of I_η to the $\mathbf{N}_\infty^{-1}\pi_\infty$-isotypic component of $A(\mathbf{G}(\mathbb{A}), \omega)$.

THEOREM 7.2. *We have* $(\omega_f^\eta\mathbf{N}_f^\eta, \omega_\infty^{-\eta}\mathbf{N}_\infty^\eta) \in X(\mathbf{H}(\mathbb{A}_f), \mathbf{H}(F_\infty), \eta^*_{\mathbb{A}_f}(\omega_0))$ *and*

$$I_\eta \circ f^{\mathbf{N},\pi_\infty}_{\cdot,\cdot\cdot} = m_{\mathbf{Z}^{\mathbf{H}}\backslash\mathbf{H},\infty}\cdot J^{\pi_\infty,\omega_f^\eta\mathbf{N}_f^\eta,\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta}_{\eta F_\infty}.$$

PROOF. We may applying (7) to $\mathbf{H}$ and we find, also using (36),

$$I_\eta(f_{\Lambda,\varphi}^{\mathbf{N},\pi_\infty}) = \int_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}^\mathbf{H}}} I_\infty \omega_f^{-1}(\eta(x_f))d\mu_{[\mathbf{H}(\mathbb{A}_f)]}(x_f),$$

where

$$\begin{aligned} I_\infty &= \int_{\mathbf{Z}^\mathbf{H}(F_\infty)\backslash\mathbf{H}(F_\infty)} \psi_{\Lambda,(\mathbf{N}_f^{-1}\varphi)(\eta(x_f))}^{\mathbf{N}_\infty^{-1}\pi_\infty}(\eta(x_\infty^{-1}))\omega_\infty(\eta(x_\infty^{-1}))d\mu_{\mathbf{H},\infty}(x_\infty) \\ &= \int_{\mathbf{Z}^\mathbf{H}(F_\infty)\backslash\mathbf{H}(F_\infty)} \psi_{\Lambda,(\mathbf{N}_f^{-1}\varphi)(\eta(x_f))}^{\mathbf{N}_\infty^{-1}\pi_\infty}(\eta(x_\infty))\omega_\infty(\eta(x_\infty))d\mu_{\mathbf{H},\infty}(x_\infty). \end{aligned}$$

Here we used $\mu_{\mathbf{H},\infty}(x_\infty) = \mu_{\mathbf{H},\infty}(x_\infty^{-1})$ by unimodularity of $\mathbf{H}(F_\infty)/\mathbf{Z}^\mathbf{H}(F_\infty)$. By definition this is $I_{\eta,\infty}(\psi_{\Lambda,(\mathbf{N}_f^{-1}\varphi)(\eta(x_f))}^{\mathbf{N}_\infty^{-1}\pi_\infty})$, so that we find

$$\begin{aligned} I_\infty &= r_\eta(\Lambda)((\mathbf{N}_f^{-1}\varphi)(\eta(x_f))) \\ &= \mathbf{N}_f^{-1}(\eta(x_f))r_\eta(\Lambda)(\varphi(\eta(x_f))). \end{aligned}$$

Hence we find

$$\begin{aligned} &I_\eta(f_{\Lambda,\varphi}^{\mathbf{N},\pi_\infty}) \\ &\quad= \int_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}^\mathbf{H}}} r_\eta(\Lambda)(\varphi(\eta(x_f)))\omega_f^{-1}(\eta(x_f))\mathbf{N}_f^{-1}(\eta(x_f))d\mu_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}^\mathbf{H}}}(x_f). \end{aligned}$$

Applying Lemma 7.1 gives the claim, thanks to Lemma 3.7 (4). □

If $\{(V_\sigma,\rho_\sigma)\}_{\sigma\in X_{E_0/F}}$ is a model over E of (V_∞,π_∞), we set

$$(V,\rho) := \boxtimes_{\sigma\in X_{E/F},E}(V_\sigma,\rho_\sigma).$$

We can now prove "$m_{\mathbf{Z}^\mathbf{H}\backslash\mathbf{H},\infty}^{-1} I_\eta \simeq J_{F_\infty} \simeq J_\mathbb{C}$."

COROLLARY 7.3. *Let $E/\mathbb{Q}$ be a Galois extension such that $\omega_0\colon \mathbf{Z}^\mathbf{G}(\mathbb{A}_f) \to E^\times$, that $\{(V_\sigma,\rho_\sigma)\}_{\sigma\in X_{E/F}}$ is a model of (V_∞,π_∞) over E and that $\{(\omega^{-\eta}\mathbf{N}^\eta)_\sigma\}_{\sigma\in X_{E/F}}$ is a model of $\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta$ over E. Then $(V_\infty,\pi_\infty) \simeq (V_\mathbb{C},\rho_\mathbb{C})$ provided by Definition 5.4 induces an isomorphism of $\mathbf{G}(\mathbb{A}_f)$-modules*

$$M(\mathbf{G}(\mathbb{A}_f),\pi_\infty,\omega_0) \simeq M(\mathbf{G},\rho,\omega_0)(\mathbb{C})$$

and

$$M[\mathbf{G}(\mathbb{A}_f),\pi_\infty,\omega_0] \simeq M[\mathbf{G},\rho,\omega_0](\mathbb{C}).$$

Using this identification, the morphism $m_{\mathbf{Z}\backslash\mathbf{H},\infty}^{-1} I_\eta \circ f_{\cdot,\pi_\infty}^{\cdot\cdot,\mathbf{N}}$ extends to the morphism of functors

$$J_\eta^{\rho,\omega_f^\eta\mathbf{N}_f^\eta,(\omega^{-\eta}\mathbf{N}^\eta)^{X_{E/F}}} : M[\mathbf{G},\rho,\omega_0] \longrightarrow \mathbf{A}^1_{/E}$$

of Proposition 4.1.

Proof. Set $\chi_\infty = \omega_\infty^{-\eta}\mathbf{N}_\infty^\eta$ and $\chi := \prod_{\sigma\in X_{E/F}}(\omega^{-\eta}\mathbf{N}^\eta)_\sigma$, so that $\chi_{\mathbb{C}|\mathbf{H}(F_\infty)} = \chi_\infty$. Then Proposition 4.1 gives the morphism of functors and the claim follows from Proposition 5.6 and Theorem 7.2. $\square$

7.1 – *The rationality of the period integrals*

Recall that $\mathbf{S_G} \subset \mathbf{Z_G}$ denotes the maximal split torus in the center of $\mathbf{G}$. Let $\pi: \mathbf{G} \to \mathbf{S}'_\mathbf{G}$ be the maximal quotient of $\mathbf{G}$ which is a split torus. Then $\varphi_\mathbf{G}: \mathbf{S_G} \to \mathbf{G} \to \mathbf{S}'_\mathbf{G}$ is an isogeny of tori and we define $\mathbf{G}_1 := \ker(\pi)$. Recall our given η and define $\mathbf{Z}_\eta := \eta^{-1}(\mathbf{Z_G}) \cap \mathbf{Z_H} \subset \mathbf{Z_H}$. We specialize the setting pictured just after (8) to the case where

$$(\mathbf{H}, \mathbf{Z^H}) \quad \text{with } \mathbf{S_H} \subset \mathbf{Z^H} \subset \mathbf{Z}_\eta \text{ and } (\mathbf{G}, \mathbf{Z^G}) = (\mathbf{G}, \mathbf{Z_G}).$$

Furthermore, we suppose that F is totally real (see Remark 2.1).

Also, we fix an extension E/F such that $\mathbf{G}_{/E}$ is a split reductive group and $E/\mathbb{Q}$ is Galois: we also fix a set $X_{E/F}$ of embeddings $\sigma: E \hookrightarrow \mathbb{C}$ extending the (classes of) archimedean places of F. Recall that we view $\mathbb{C}$ as an E-algebra via $\sigma_\infty: E \hookrightarrow \mathbb{C}$ (see (26) and the discussion around there for the notations). If $(V_\infty, \pi_\infty) \in \mathrm{Rep}(\mathbf{G}(F_\infty))$ (resp. $(V, \rho) \in \mathrm{Rep}(\mathbf{G}_{/E}^{X_{E/F}})$) has central character ω_{π_∞} (resp. ω_ρ), we call $\omega_{\pi_\infty}^{\mathrm{s}} := \omega_{\pi_\infty|\mathbf{S_G}(F_\infty)}$ (resp. $\omega_\rho^{\mathrm{s}} := \omega_{\rho|\mathbf{S_G}}$) the split central character of the representation.

Definition 7.4. Suppose that $(V_\infty, \pi_\infty) \in \mathrm{Rep}(\mathbf{G}(F_\infty))$. We say that it is pseudo-algebraic if $\omega_{\pi_\infty}^{\mathrm{s}}$ has a model over F, i.e. if there is a family of algebraic characters $\{\omega_\sigma: \mathbf{S_G} \to \mathbf{G}_m\}_{\sigma\in X_{E/F}}$ with the property that, setting

$$\omega := \boxtimes_{\sigma\in X_{E/F}}\omega_\sigma: \mathbf{S}_\mathbf{G}^{X_{E/F}} \longrightarrow \mathbf{G}_m,$$

we have $\omega_{\mathbb{C}|\mathbf{S_G}(F_\infty)} = \omega_{\pi_\infty}^{\mathrm{s}}$, where (33): $\mathbf{S_G}(F_\infty) \subset \mathbf{S}_\mathbf{G}^{X_{E/F}}(\mathbb{C})$. We say that it is parallel (resp. even) if it is pseudo-algebraic and $\omega_\sigma = \omega_{\sigma_\infty}$ for every $\sigma \in X_{E/F}$, i.e. ω has all the components which are equal in $X^*(\mathbf{S}_\mathbf{G}^{X_{E/F}}) = X^*(\mathbf{S_G})^{X_{E/F}}$ (resp. ω is a square in $X^*(\mathbf{S}_\mathbf{G}^{X_{E/F}}) = X^*(\mathbf{S_G})^{X_{E/F}}$).

Consider the (normalized) absolute value functions

$$|-|_v : F_v^\times \longrightarrow \mathbb{R}_+^\times,$$
$$|-|_{\mathbb{A}_f} : \mathbb{A}_f^\times \longrightarrow \mathbb{Q}_+^\times,$$
$$|-|_{\mathbb{A}} : \mathbb{A}^\times \longrightarrow \mathbb{R}_+^\times.$$

Setting

$$\mathrm{N} := |-|_{\mathbb{A}_f}^{-1}\, |-|_\infty : \mathbf{G}_m(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

gives a function such that $\mathrm{N}_f\, \mathrm{N}_\infty^{-1} = |-|_{\mathbb{A}}^{-1}$ is trivial on $\mathbf{G}_m(F)$ by the product formula. Suppose that $\chi\colon \mathbf{G} \to \mathbf{G}_m$ is an algebraic character and that $\tau\colon R^\times \to G$ is a character. Then we define $\tau_\chi\colon \mathbf{G}(R) \stackrel{\chi_R}{\to} R^\times \stackrel{\tau}{\to} G$. In particular, we have the continuous character

$$\mathrm{N}_\chi\colon \mathbf{G}(\mathbb{A}) \stackrel{\chi_{\mathbb{A}}}{\longrightarrow} \mathbb{A}^\times \stackrel{\mathrm{N}}{\longrightarrow} \mathbb{R}_+^\times$$

and, recalling that $\mathrm{N}_f = |-|_{\mathbb{A}_f}^{-1}$ and $\mathrm{N}_\infty = |-|_\infty$,

$$\mathrm{N}_{\chi,f}\colon \mathbf{G}(\mathbb{A}_f) \stackrel{\chi_{\mathbb{A}_f}}{\longrightarrow} \mathbb{A}_f^\times \stackrel{|-|_{\mathbb{A}_f}^{-1}}{\longrightarrow} \mathbb{Q}_+^\times$$

and

$$\mathrm{N}_{\chi,\infty}\colon \mathbf{G}(F_\infty) \stackrel{\chi_\infty}{\longrightarrow} F_\infty^\times \stackrel{|-|_\infty}{\longrightarrow} \mathbb{R}_+^\times.$$

Of course $\mathrm{N}_{\chi,f}$ (resp. $\mathrm{N}_{\chi,\infty}$) is the finite adele (resp. ∞) component of N_χ, as suggested by the notation. If $\kappa\colon \mathbb{Q}_+^\times \to R^\times$ is a character (that we usually write exponentially $r \mapsto r^\kappa$), we can also define

$$\mathrm{N}_{\chi,f}^\kappa\colon \mathbf{G}(\mathbb{A}_f) \stackrel{\mathrm{N}_{\chi,f}}{\longrightarrow} \mathbb{Q}_+^\times \stackrel{\kappa}{\longrightarrow} R^\times$$

Note that $\chi_\infty(\mathbf{G}(F_\infty)) = \chi_\infty(\mathbf{G}(F_\infty)^\circ) \subset \mathbb{R}_+^\times$, implying that $\chi_F(\mathbf{G}(F)) \subset F_+^\times$ and we may consider $\kappa_\chi := \kappa \circ N_{F/\mathbb{Q}} \circ \chi_F$. If $V = (V, \rho)$ is a representation of $\mathbf{G}(F_\infty)$ with coefficients in R, we write $V(\kappa_\chi) = (V, \rho(\kappa_\chi))$ for the representation $\rho(\kappa_\chi)(g)(v) := \kappa_\chi(g)\rho(g)v$.

Remark 7.5. The continuous character N_χ is such that $\mathrm{N}_{\chi,f}\, \mathrm{N}_{\chi,\infty}^{-1}$ is trivial on $\mathbf{G}(F)$ and we have

$$\mathrm{N}_{\chi,f}^\kappa \in M(\mathbf{G}(\mathbb{A}_f), R(\kappa_\chi), \mathrm{N}^\kappa_{\chi,f|\mathbf{Z_G}(\mathbb{A}_f)})^K$$

for every open and compact $K \in \mathcal{K}$.

Proof. This is an application of the product formula and the fact that we have $\chi_F(\mathbf{G}(F)) \subset F_+^\times$. □

The main result that we want to prove in this §7.1 is the following.

THEOREM 7.6. *Suppose that* $\mathbf{H}(F_\infty)$ *and* $\mathbf{G}(F_\infty)$ *are connected and that* F *is totally real.*

(1) *The association*

$$\boxtimes_{\sigma\in X_{E/F},E}(V_\sigma,\rho_\sigma)=(V,\rho)\longmapsto(V_{\mathbb{C}},\rho_{\mathbb{C}|\mathbf{G}(F_\infty)})$$

obtained from (33): $\mathbf{G}(F_\infty)\subset\mathbf{G}^{X_{E/F}}(\mathbb{C})$ *induces an injection*

$$\prod\nolimits_{\sigma\in X_{E/F}}\operatorname{Irr}(\mathbf{G}_{/E})=\operatorname{Irr}(\mathbf{G}_{/E}^{X_{E/F}})\hookrightarrow\operatorname{Irr}(\mathbf{G}(F_\infty))$$

and this is a bijection when $\mathbf{G}(F_\infty)$ *is compact, which happens if and only if* $\mathbf{S}_{\mathbf{G}}=\{1\}$.

(2) *If* $[(V_\infty,\pi_\infty)]\in\operatorname{Irr}(\mathbf{G}(F_\infty))$ *we have that* $[(V_\infty,\pi_\infty)]$ *belongs to the image of the map in* (1) *if and only if it is pseudo-algebraic. In this case,* $\delta_{\pi_\infty}\colon\mathbf{G}(F_\infty)\to\mathbb{C}^\times$ *has a model over* F*, i.e. there is a family*

$$\{\nu_\sigma=\nu_{\pi_\infty,\sigma}\colon\mathbf{G}\longrightarrow\mathbf{G}_m\}_{\sigma\in X_{E/F}}$$

with the property that, setting

$$\nu:=\boxtimes_{\sigma\in X_{E/F}}\nu_\sigma\colon\mathbf{G}^{X_{E/F}}\longrightarrow\mathbf{G}_m,$$

we have

$$\nu_{\mathbb{C}|\mathbf{G}(F_\infty)}=\delta_{\pi_\infty}.$$

(3) *If* $[(V_\infty,\pi_\infty)]\in\operatorname{Irr}(\mathbf{G}(F_\infty))$ *is parallel, taking* $\mathbf{N}=\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty}}\colon\mathbf{G}(\mathbb{A})\to\mathbb{R}_+^\times$ *we have* $\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty},\infty}=\delta^{1/2}_{\pi_\infty}$ *and* $\mathbf{N}$ *binds* (V_∞,π_∞) *to* ω (*see Definition* 5.1) *for every unitary Hecke character* $\omega\colon\frac{\mathbf{Z_G}(\mathbb{A})}{\mathbf{Z_G}(F)}\to\mathbb{C}^\times$ *such that* $\omega_\infty=\omega^{-1}_{\pi_\infty}\delta^{1/2}_{\pi_\infty}$.

Suppose that $[(V_\infty,\pi_\infty)]\in\operatorname{Irr}(\mathbf{G}(F_\infty))$ *is parallel, that* $(V_\infty,\pi_\infty\circ\eta_\infty)\in\operatorname{Rep}(\mathbf{H}(F_\infty))$ *is even and that the extension* $\omega^\eta\colon\mathbf{H}(\mathbb{A})\to\mathbb{C}^\times$ *of* $\omega\circ\eta_{\mathbb{A}|\mathbf{Z^H}(\mathbb{A})}$ *is such that* $\omega^\eta_\infty\in\operatorname{Irr}(\mathbf{H}(F_\infty))$ *is pseudo-algebraic (for example because* $\omega\circ\eta_{\mathbb{A}|\mathbf{Z^H}(\mathbb{A})}=1$ *and we take* $\omega^\eta=1$*). Then* $m_{\mathbf{Z^H}\backslash\mathbf{H},\infty}(\mathbf{PH}_{\mathbf{Z^H}}(F_\infty))^{-1}I_\eta\circ f^{\cdot\cdot,\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty}}}_{\cdot,\pi_\infty}$ *extends to the morphism of functors*

$$J_\eta^{\rho,\omega_f^\eta\mathrm{N}^{1/2,\eta}_{\nu_{\pi_\infty,\sigma_\infty}},(\omega^{-\eta}\nu^{\eta,1/2}_{\pi_\infty})^{X_{E/F}}}\colon M[\mathbf{G},\rho,\omega_0]_{/E(\omega_0)}\longrightarrow\mathbf{A}^1_{/E(\omega_0)}$$

of Proposition 4.1 *with* $\omega_0=\omega_f\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty},f}$ *and* $E(\omega_0)=E(\omega_f)$ *obtained from* E *adding the values of either* ω_0 *or* ω_f *as in* (3)*. If* (V_∞,π_∞) *and* ω^η_∞ *have models over* $E'/\mathbb{Q}$ *Galois with* $F\subset E'\subset E$*, then we can descend to* $E'(\omega_0)$.

Before proving the result we make the following remark.

REMARK 7.7. Let F be totally real in the following observations.

(1) If $\eta\colon \mathbf{H} \to \mathbf{G}$ is an algebraic subgroup then $\eta^{-1}(\mathbf{S_G}) = \mathbf{S_H}$ and $\eta^{-1}(\mathbf{G}(F)) = \mathbf{H}(F)$. In particular, when $(\mathbf{G}, \mathbf{S_G})$ satisfies the assumptions (A1)–(A3), it follows that $(\mathbf{H}, \mathbf{S_H})$ satisfies the assumptions (A1)–(A3) and $\mathbf{S_H} = \mathbf{Z^H} \subset \mathbf{Z}_\eta$. In other words, Theorem 7.6 applies in this case with $\mathbf{Z^H} = \mathbf{S_H}$ simply assuming that $(\mathbf{G}, \mathbf{S_G})$ satisfies (A1)–(A3) and that $\mathbf{H}(F_\infty)$ and $\mathbf{G}(F_\infty)$ are connected.

(2) When $(\mathbf{G}, \mathbf{S_G})$ satisfies (A1)–(A3), and $\mathbf{G}(F_\infty)$ is compact, i.e. $\mathbf{S_G} = \{1\}$ (for example because $\mathbf{Z_G}$ is finite, under (A1) for $(\mathbf{G}, \mathbf{S_G})$), every $[(V_\infty, \pi_\infty)] \in \mathrm{Irr}(\mathbf{G}(F_\infty))$ is even and parallel. Furthermore, $\mathbf{G}(F_\infty) \subset \mathbf{G}^{X_{E/F}}(\mathbb{C})$ is a maximal compact subgroup which is therefore connected because its complexification $\mathbf{G}^{X_{E/F}}(\mathbb{C})$ is connected (see the proof of the following Lemma 7.8 (2)). Hence assuming that $\eta\colon \mathbf{H} \to \mathbf{G}$ is an algebraic subgroup, Theorem 7.6 applies removing all references to being even or parallel and the connectedness assumptions.

(3) When $\eta\colon \mathbf{H} \to \mathbf{G}$ is the diagonal immersion in the product of totally definite quaternion algebra over a totally real field F, Theorem 7.6 applies.

Fix $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}_{/E}$, where $\mathbf{T}$ (resp. $\mathbf{B}$) is a split maximal torus over E (resp. a Borel subgroup defined over E). We write $\mathbf{N} \subset \mathbf{B}$ for the maximal unipotent subgroup. Let $K \subset \mathbf{G}^{X_{E/F}}(\mathbb{C})$ be a maximal compact (Lie) subgroup. The Borel–Weil theorem implies that the representation theory of $\mathbf{G}^{X_{E/F}}_{/E}$ and K are obtained as follows (see for example [25, Chapter VII, §7] and [22, Part II, §5] for an algebraic point of view). For every dominant weight λ of $\mathbf{G}^{X_{E/F}}_{/E}$, we may naturally extend it to a morphism $\lambda\colon \mathbf{B}^{X_{E/F}} \to \mathbf{G}_m$ by setting $\lambda(n) = 1$ for every $n \in \mathbf{N}^{X_{E/F}}$. Then we can form the $\mathbf{B}^{X_{E/F}}$-equivariant sheaf $\mathcal{O}_{\mathbf{G}^{X_{E/F}}_{/E}}(\lambda)$ on $\mathbf{G}^{X_{E/F}}_{/E}$, which is simply $\mathcal{O}_{\mathbf{G}^{X_{E/F}}_{/E}}$ endowed with the $\mathbf{B}^{X_{E/F}}$-action defined by the rule

$$(fb)(x) := b^{-1} \cdot_\lambda f(bx) = \lambda(b)^{-1} f(bx).$$

Consider the quotient $\pi\colon \mathbf{G}^{X_{E/F}}_{/E} \to \mathbf{B}^{X_{E/F}} \backslash \mathbf{G}^{X_{E/F}}_{/E}$ and let $\mathcal{O}_{/E}(\lambda)$ be the sheaf on $\mathbf{B}^{X_{E/F}} \backslash \mathbf{G}^{X_{E/F}}_{/E}$ which corresponds to $\mathcal{O}_{\mathbf{G}^{X_{E/F}}_{/E}}(\lambda)$, i.e. the sheaf defined by the rule

$$\mathcal{O}_{/E}(\lambda)(U) := \Gamma(\pi^{-1}(U), \mathcal{O}_{\mathbf{G}^{X_{E/F}}_{/E}}(\lambda))^{\mathbf{B}^{X_{E/F}}}.$$

Setting

$$P_{\lambda,R} := \Gamma(\mathbf{B}^{X_{E/F}}_{/R}\backslash\mathbf{G}^{X_{E/F}}_{/R}, \mathcal{O}_{/R}(\lambda))$$

for every E-algebra R, yields a left irreducible algebraic representation P_λ of $\mathbf{G}^{X_{E/F}}_{/E}$ by right translations $(gf)(x) := f(xg)$: it has highest weight λ and central character $\omega_\lambda = \lambda_{|\mathbf{Z}_{\mathbf{G}^{X_{E/F}}_{/E}}}$. Since we work with right representations, we let L_λ be the dual representation with right action $(\Lambda g)(f) := \Lambda(gf)$: it has highest weight λ and central character $\omega_\lambda = \lambda_{|\mathbf{Z}_{\mathbf{G}^{X_{E/F}}_{/E}}}$. Note that $L_{\lambda,\mathbb{C}}$ is the $\mathbb{C}$-dual of

$$P_{\lambda,\mathbb{C}} = \{f \in \mathcal{O}_{\mathbf{G}^{X_{E/F}}_{/\mathbb{C}}}(\mathbf{G}^{X_{E/F}}_{/\mathbb{C}}) \colon \text{for all } b \in \mathbf{B}^{X_{E/F}}(\mathbb{C}),\ f(bx) = \lambda(b)f(x)\}.$$

Furthermore, $T := K \cap \mathbf{B}^{X_{E/F}}(\mathbb{C})$ is a maximal connected commutative Lie subgroup and the inclusion $K \subset \mathbf{G}^{X_{E/F}}(\mathbb{C})$ induces $T\backslash K \xrightarrow{\sim} \mathbf{B}^{X_{E/F}}(\mathbb{C})\backslash\mathbf{G}^{X_{E/F}}(\mathbb{C})$. The choice of a Haar measure μ_K on K fixes a K-invariant Hermitian scalar product on $P_{\lambda,\mathbb{C}}$ by the rule

$$\langle f_1, f_2\rangle_\lambda := \textstyle\int_K f_1(x)\overline{f_2(x)}d\mu_K(x).$$

Letting $\Phi\colon P_{\lambda,\mathbb{C}} \to V_{\lambda,\mathbb{C}}$ be the conjugate linear morphism $(\Phi(v), x) := \langle x, v\rangle_\lambda$, we transport $\langle -, -\rangle_\lambda$ to a pairing Hermitian scalar product on $V_{\lambda,\mathbb{C}}$ by the rule $\langle v_1^\vee, v_2^\vee\rangle_\lambda := \langle \Phi^{-1}(v_2^\vee), \Phi^{-1}(v_1^\vee)\rangle_\lambda$ (as we did before Definition 5.1). Then $\langle vk, wk\rangle_\lambda = \langle v, w\rangle_\lambda$ for every $k \in K$. The Borel–Weil theorem asserts that the association $\lambda \mapsto L_\lambda$ (resp. $\lambda \mapsto (L_{\lambda,\mathbb{C}|K}, \langle -, -\rangle_\lambda)$) realizes a bijection of the set of dominant weight with (an explicit) set of representatives for $\mathrm{Irr}(\mathbf{G}^{X_{E/F}}_{/E})$ (resp. $\mathrm{Irr}(K)$). In particular, $(V, \rho) \mapsto (V_\mathbb{C}, \rho_{\mathbb{C}|K})$ is a bijection

(40) $$\mathrm{Irr}(\mathbf{G}^{X_{E/F}}_{/E}) \xrightarrow{\sim} \mathrm{Irr}(K).$$

We now need the following result.

Lemma 7.8. *The following facts hold.*

(1) *The morphism of algebraic groups $\mathbf{S_G} \times \mathbf{G}_1 \to \mathbf{G}$ defined on points by the rule $(s, g_1) \mapsto sg_1$ is an epimorphism of fppf sheaves whose kernel is a finite group. Furthermore, it induces an isomorphism*

$$\mathbf{S_G}(F_\infty)^\circ \times \mathbf{G}_1(F_\infty) \longrightarrow \mathbf{G}(F_\infty).$$

(2) *The inclusion (33): $\mathbf{G}_1(F_\infty) \subset \mathbf{G}_1^{X_{E/F}}(\mathbb{C})$ makes $\mathbf{G}_1(F_\infty)$ a maximal compact subgroup of $\mathbf{G}_1^{X_{E/F}}(\mathbb{C})$ and $\mathbf{G}_1^{X_{E/F}}$ is connected and reductive.*

Proof. The first statement is a formal consequence of the fact that $\varphi_{\mathbf{G}}$ is an isogeny and $\mathbf{G}(F_\infty) = \mathbf{G}(F_\infty)^\circ$. We remark that, if $\mathbf{K}$ is an algebraic group over $\mathbb{R}$ such that $\mathbf{K}(\mathbb{R})$ is compact, then $\mathbf{K}(\mathbb{R}) \subset \mathbf{K}(\mathbb{C})$ is the complexification of the real Lie group $\mathbf{K}(\mathbb{R})$ and then it is known that $\mathbf{K}(\mathbb{R})$ is connected if and only if $\mathbf{K}(\mathbb{C})$ is connected, meaning that $\mathbf{K}$ is connected (by [27, Aside 2.45 (a)]). It follows from (1) that $\frac{\mathbf{G}(F_\infty)}{\mathbf{S}_{\mathbf{G}}(F_\infty)} \to \mathbf{G}_1(F_\infty)$ is a continuous surjection (indeed an isomorphism): hence $\mathbf{G}_1(F_\infty)$ is connected and, thanks to (A1) for $(\mathbf{G}, \mathbf{S}_{\mathbf{G}})$, it is also compact. But we have $\mathbf{G}_1(F_\infty) = \mathbf{G}_1^{X_{E/F}}(\mathbb{R})$ (because F is totally real); the above remark implies that $\mathbf{G}_1(F_\infty) \subset \mathbf{G}_1^{X_{E/F}}(\mathbb{C})$ is the complexification of $\mathbf{G}_1(F_\infty)$ and that $\mathbf{G}_1^{X_{E/F}}$ is connected. It is obviously reductive and we are done. □

Corollary 7.9. *Suppose that the representation* $(V, \rho) \in \mathrm{Rep}(\mathbf{G}_{/E}^{X_{E/F}})$ *(resp.* $(V_\infty, \pi_\infty) \in \mathrm{Rep}(\mathbf{G}(F_\infty))$*) is irreducible.*

(1) *If* $(V, \rho) \in \mathrm{Rep}(\mathbf{G}_{/E}^{X_{E/F}})$ *(resp.* $(V_\infty, \pi_\infty) \in \mathrm{Rep}(\mathbf{G}(F_\infty))$*) is an irreducible representation, the representation is still irreducible when restricted to* $\mathbf{G}_{1/E}^{X_{E/F}}$ *(resp.* $\mathbf{G}_1(F_\infty)$*).*

(2) *If* $(V_\infty, \pi_\infty) \in \mathrm{Rep}(\mathbf{G}(F_\infty))$*, then there is* $\rho \in \mathrm{Rep}(\mathbf{G}_{/\mathbb{C}}^{X_{E/F}})$ *such that* $\rho_{\mathbb{C}|\mathbf{G}(F_\infty)} = \pi_\infty$*, in short* π_∞ *"*$\in \mathrm{Rep}(\mathbf{G}_{/\mathbb{C}}^{X_{E/F}})$*," if and only if* $\pi_{\infty|\mathbf{G}_1(F_\infty)}$ *"*$\in \mathrm{Rep}(\mathbf{G}_{/\mathbb{C}}^{X_{E/F}})$*" and* $\omega_{\pi_\infty}^{\mathrm{s}} := \pi_{\infty|\mathbf{S}_{\mathbf{G}}(F_\infty)}$ *"*$\in \mathrm{Rep}(\mathbf{S}_{\mathbf{G}/\mathbb{C}}^{X_{E/F}})$*."*

(3) *If* $(V, \rho), (V', \rho') \in \mathrm{Rep}(\mathbf{G}_{/\mathbb{C}}^{X_{E/F}})$ *and* $\rho_{\mathbb{C}|\mathbf{G}(F_\infty)} = \rho'_{\mathbb{C}|\mathbf{G}(F_\infty)}$ *then* $\rho = \rho'$*.*

Proof. Claim (1) follows from Lemma 7.8 (1) and the fact that $\mathbf{S}_{\mathbf{G}}^{X_{E/F}}$ (resp. $\mathbf{S}_{\mathbf{G}}(F_\infty)$) acts by means of the central character on an irreducible representation. In order to prove (2), suppose that we are given a morphism $f : \mathbf{H} \to \mathbf{G}$ of algebraic groups over $\mathbb{R}$ which is an epimorphism of fppf sheaves whose kernel is a finite group which is still surjective when taking the real points. Let $\pi_\infty : \mathbf{G}(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{C})$ be a morphism of real Lie groups which pull-back to $\pi_\infty \circ f : \mathbf{H}(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{C})$ which is algebraic: we claim that f is algebraic. We recall that, for an algebraic group $\mathbf{K}$ over $\mathbb{R}$, we have that $\mathbf{K}(\mathbb{R}) \subset \mathbf{K}(\mathbb{C})$ is the algebraic complexification, meaning that we have

$$\mathrm{Hom}_{\mathbb{C}\text{-alg-gr}}(\mathbf{K}(\mathbb{C}), \mathbf{GL}_n(\mathbb{C})) = \mathrm{Hom}_{\mathbb{R}\text{-alg-gr}}(\mathbf{K}(\mathbb{R}), \mathbf{GL}_n(\mathbb{C})). \tag{41}$$

The identification is a consequence of the universal property of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{GL}_{n,\mathbb{R}})$ after identifying a morphism of schemes with the morphism induced on points (by smoothness of $\mathbf{K}$ in characteristic zero). Taking $\mathbf{K} = \mathbf{H}$, it follows that there is a unique algebraic $\rho_f\colon \mathbf{H}(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$ such that $\rho_{f|\mathbf{H}(\mathbb{R})} = \pi_\infty \circ f_\mathbb{R}$. We claim that $\rho_{f|\ker(f)(\mathbb{C})} = 1$. Once this result has been proved, we will deduce that there is a unique algebraic morphism $\rho\colon \mathbf{G}(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$ such that $\rho \circ f = \rho_f$. Since $f_\mathbb{R}$ is surjective, $\rho_{\mathbb{C}|\mathbf{G}(\mathbb{R})} \circ f_\mathbb{R} = \rho_{f|\mathbf{H}(\mathbb{R})} = \pi_\infty \circ f_\mathbb{R}$ will imply that $\pi_\infty = \rho_{\mathbb{C}|\mathbf{G}(\mathbb{R})}$ is algebraic. But $\rho_{f|\ker(f)(\mathbb{C})} \in \mathrm{Hom}_{\mathbb{C}\text{-alg-gr}}(\mathbf{H}(\mathbb{C}), \mathbf{GL}_n(\mathbb{C}))$ maps to $\rho_{f|\ker(f)(\mathbb{R})} = \pi_\infty \circ f_{\mathbb{R}|\ker(f)(\mathbb{R})} = 1$ and (41) implies $\rho_{f|\ker(f)(\mathbb{C})} = 1$ as wanted. It follows from Lemma 7.8 (1) that we can apply this result to $\mathbf{S}_\mathbf{G}^{X_{E/F}} \times \mathbf{G}_1^{X_{E/F}} \to \mathbf{G}^{X_{E/F}}$; since F is totally real, the real points of these groups $\mathbf{H}$ are (33): $\mathbf{H}(F_\infty) = \mathbf{H}^{X_{E/F}}(\mathbb{R}) \subset \mathbf{H}^{X_{E/F}}(\mathbb{C})$. Claim (3) is clear from (41). □

We can now prove (1) and the first statement in (2) of Theorem 7.6. First of all, if $(V, \rho) \in \mathrm{Rep}(\mathbf{G}_{/E}^{X_{E/F}})$ then we know from Corollary 7.9 (1) that $\rho_{|\mathbf{G}_{1/E}^{X_{E/F}}}$ is still irreducible. It follows from Lemma 7.8 (2) that we may apply (40) with $K = \mathbf{G}_1(F_\infty) \subset \mathbf{G}_1^{X_{E/F}}(\mathbb{C})$ and we deduce that $\rho_{\mathbb{C}|\mathbf{G}_1(F_\infty)}$ is irreducible. Then $\rho_{\mathbb{C}|\mathbf{G}_1(F_\infty)}$ is irreducible a fortiori. Hence the map $\rho \mapsto \rho_{\mathbb{C}|\mathbf{G}(F_\infty)}$ induces a map between the irreducible classes. The fact that it is injective follows from Corollary 7.9 (3). The characterization of its image follows from Corollary 7.9 (2), since the condition $\pi_{\infty|\mathbf{G}_1(F_\infty)}$"$\in \mathrm{Rep}(\mathbf{G}_{1/\mathbb{C}}^{X_{E/F}})$" is free; indeed, we may apply (40) with $K = \mathbf{G}_1(F_\infty) \subset \mathbf{G}_1^{X_{E/F}}(\mathbb{C})$, thanks to Lemma 7.8 (2). Finally, the equivalence between $\mathbf{G}(F_\infty)$ being compact and $\mathbf{S}_\mathbf{G} = \{1\}$ follows from Lemma 7.8 (1).

As remarked $\mathbf{G}_1(F_\infty)$ is compact. Hence (we may assume) $\mathbf{G}_1(F_\infty) \subset K$: writing every element $g \in \mathbf{G}(F_\infty)$ in the form $g = s_g g_1$ with $(s_g, g_1) \in \mathbf{S}_\mathbf{G}(F_\infty)^\circ \times \mathbf{G}_1(F_\infty)$, as granted by Lemma 7.8 (1), we see that $\langle f_1 g, f_2 g\rangle_\lambda = \left|\omega_\lambda(s_g)\right|^2 \langle f_1, f_2\rangle_\lambda$. Since F is totally real it is easy to see that $\omega_\lambda(s_g)^2 \in \mathbb{R}_+^\times$, so that $\left|\omega_\lambda(s_g)\right|^2 = \omega_\lambda(s_g)^2$. But for an arbitrary element $z \in \mathbf{S}_\mathbf{G}(F_\infty)$ we have $z^2 \in \mathbf{S}_\mathbf{G}(F_\infty)^\circ$, so that one finds $\omega_\lambda(s_z)^2 = \omega_\lambda^2(z)$. It follows that, setting $\pi_\infty := \rho_{\mathbb{C}|\mathbf{G}(F_\infty)}$, we have $\delta_{\pi_{\infty|\mathbf{G}_1(F_\infty)}} = 1$"$\in \mathrm{Rep}(\mathbf{G}_{1/\mathbb{C}}^{X_{E/F}})$" and $\delta_{\pi_\infty|\mathbf{S}_\mathbf{G}(F_\infty)} = \omega^2_{\lambda|\mathbf{S}_\mathbf{G}^{X_{E/F}}}$"$\in \mathrm{Rep}(\mathbf{S}_{\mathbf{G}/\mathbb{C}}^{X_{E/F}})$." Corollary 7.9 (2) yields δ_{π_∞}"$\in \mathrm{Rep}(\mathbf{G}_{/\mathbb{C}}^{X_{E/F}})$." This is the second second statement in (2), once we remark that δ_{π_∞}"$\in \mathrm{Rep}(\mathbf{G}^{X_{E/F}})$," because its pull-back $(1, \omega^2_{\lambda|\mathbf{S}_\mathbf{G}^{X_{E/F}}})$ is defined over F and $\mathbf{S}_\mathbf{G}^{X_{E/F}} \times \mathbf{G}_1^{X_{E/F}} \to \mathbf{G}^{X_{E/F}}$ is an fppf quotient over F. When (V_∞, π_∞) is parallel, it is easy to deduce that ν is parallel. This fact implies that $\nu_{\sigma_\infty,\infty} = \nu_{|\mathbf{G}(F_\infty)}$. Since $\nu_{\sigma_\infty,\infty}(\mathbf{G}(F_\infty)) = \nu_{\sigma_\infty,\infty}(\mathbf{G}(F_\infty)^\circ) \subset \mathbb{R}_+^\times$, we have $\mathrm{N}_{\nu_{\sigma_\infty},\infty} = \nu_{\sigma_\infty,\infty}$. It follows that $\mathrm{N}_{\nu_{\sigma_\infty},\infty} = \nu_{\mathbb{C}|\mathbf{G}(F_\infty)} = \delta_{\pi_\infty}$. Since $\delta_{\pi_\infty}^{-1/2}\pi_\infty$ is unitary, the statement (3) follows from Remark 7.5.

Finally, the last statement of Theorem 7.6 follows from Corollary 7.3, as far as we know that $\omega_\infty^{-\eta}\,\mathrm{N}^{1/2,\eta}_{\nu_{\pi\infty,\sigma\infty},\infty}$ has a model over E, since then all assumptions required for its application are satisfied, thanks to (3). If ξ is a representation of $\mathbf{G}(F_\infty)$ or $\mathbf{Z}_{\mathbf{G}}^{X_{E/F}}$, let us abusively write $\xi_{|\mathbf{S}_{\mathbf{G}}(F_\infty)}$ or $\xi_{|\mathbf{S}_{\mathbf{H}}^{X_{E/F}}}$ to mean the restriction of $\xi\circ\eta$: it makes sense because $\mathbf{S}_{\mathbf{H}}\subset\mathbf{Z}_\eta$. Since $\pi_\infty\circ\eta_\infty$ is even, we have $(\delta^{1/2}_{\pi\infty|\mathbf{S}_{\mathbf{H}}(F_\infty)})^2=\omega^2_{\lambda|\mathbf{S}_{\mathbf{H}}^{X_{E/F}}}$ with $\omega_{\lambda|\mathbf{S}_{\mathbf{H}}^{X_{E/F}}}=(\omega^{1/2})^2$ for some $\omega^{1/2}\in X^*(\mathbf{S}_{\mathbf{H}}^{X_{E/F}})$. We deduce $\delta^{1/2}_{\pi\infty|\mathbf{S}_{\mathbf{H}}(F_\infty)}=\omega_{\lambda|\mathbf{S}_{\mathbf{H}}^{X_{E/F}}}$ "$\in\mathrm{Rep}(\mathbf{S}_{\mathbf{H}/\mathbb{C}}^{X_{E/F}})$" because $\delta^{1/2}_{\pi\infty}(\mathbf{S}_{\mathbf{H}}(F_\infty))\subset\mathbb{R}_+^\times$ and $\omega_\lambda(\mathbf{S}_{\mathbf{H}}(F_\infty))\subset\mathbb{R}_+^\times$ in light of $\omega_{\lambda|\mathbf{S}_{\mathbf{H}}^{X_{E/F}}}=(\omega^{1/2})^2$. Another application of Corollary 7.9 (2) as above yields $\delta^{1/2}_{\pi\infty}\circ\eta_\infty$"$\in\mathrm{Rep}(\mathbf{H}^{X_{E/F}})$." This means that there is some $\nu^{\eta,1/2}\colon\mathbf{H}^{X_{E/F}}\to\mathbf{G}_m$ such that $\nu\circ\eta=(\nu^{\eta,1/2})^2$. But then we see that $(\nu^{\eta,1/2}_{\mathbb{C}|\mathbf{H}(F_\infty)})^2=\mathrm{N}^\eta_{\nu_{\pi\infty,\sigma\infty},\infty}$; since both $\nu^{\eta,1/2}_{\mathbb{C}|\mathbf{H}(F_\infty)}$ and $\mathrm{N}^\eta_{\nu_{\pi\infty,\sigma\infty},\infty}$ takes values in $\mathbb{R}_+^\times$, we deduce that $\nu^{\eta,1/2}_{\mathbb{C}|\mathbf{H}(F_\infty)}=\mathrm{N}^{\eta,1/2}_{\nu_{\pi\infty,\sigma\infty},\infty}=\mathrm{N}^{1/2,\eta}_{\nu_{\pi\infty,\sigma\infty},\infty}$. This means that $\mathrm{N}^{1/2,\eta}_{\nu_{\pi\infty,\sigma\infty},\infty}$ has a model over F and we are done.

7.2 – *Proof of Theorem C of the introduction*

Here we prove Theorem C of the introduction assuming that $F=\mathbb{Q}$ to simplify a bit the notations. Recall that we have fixed $E(\pi,\mathrm{Ad})\subset\mathbb{C}$ and that we write $x\mapsto\bar{x}$ for the induced complex conjugation on x. We also set $E^*(\pi):=E(\pi,\mathrm{Ad})\overline{E(\pi,\mathrm{Ad})}$ and, for an $E^*(\pi)$-vector space, we write $\overline{V}$ for the conjugate vector space. We sketch the proof, leaving to the reader the proofs of the following (42), (43), and Remark 7.10.

In the paper we have defined a global sub $E^*(\pi)[\mathbf{G}(\mathbb{A}_f)\times\mathbf{G}(\mathbb{Q})]$-module $\pi_{E^*(\pi)}$ of $\pi=\pi_f\otimes\pi_\infty$, namely $\pi_{E^*(\pi)}:=M[\mathbf{G},\rho,\omega_0](E(\omega_f))[\pi]$ (see Proposition 4.1), such that $\mathbb{C}\otimes_{E^*(\pi)}\pi_{E^*(\pi)}\simeq\pi$ and $\mathbb{C}\otimes_{E^*(\pi)}\rho\simeq\pi_\infty$. An algebraic argument shows that there are $E^*(\pi)[\mathbf{G}(\mathbb{Q}_v)]$-submodules $\pi_{E^*(\pi),v}\subset\pi_v$ such that $\pi_v\simeq\mathbb{C}\otimes_{E^*(\pi)}\pi_{E^*(\pi),v}$ for every finite v with the property that, as $E^*(\pi)[\mathbf{G}(\mathbb{A}_f)\times\mathbf{G}(\mathbb{Q})]$-modules,

$$\pi_{E^*(\pi)}\simeq\Big(\bigotimes\nolimits'_{E^*(\pi),v<\infty}\pi_{E^*(\pi),v}\Big)\otimes_{E^*(\pi)}\rho. \tag{42}$$

Furthermore, it follows from Theorem 7.2 applied to the diagonal $\mathbf{G}\subset\mathbf{G}\times\mathbf{G}$ (and $\mathbf{Z}^{\mathbf{G}}=\mathbf{Z}_{\mathbf{G}}$) that the Petersson inner product $\langle -,-\rangle_\pi$ on π is the base change of $\langle -,-\rangle_{\pi_{E^*(\pi)}}\colon\pi_{E^*(\pi)}\otimes_{E^*(\pi)}\overline{\pi_{E^*(\pi)}}\to E^*(\pi)$. Since $\pi_{E^*(\pi)}$ is irreducible (because π is), we can write

$$\langle -,-\rangle_{\pi_{E^*(\pi)}}=\bigotimes\nolimits'_{E^*(\pi),v}\langle -,-\rangle_{\pi_{E^*(\pi),v}} \tag{43}$$

where $\langle -,-\rangle_{\pi_{E^*(\pi),v}}\colon \pi_{E^*(\pi),v} \otimes_{E^*(\pi)} \overline{\pi_{E^*(\pi),v}} \to E^*(\pi)$ is $\mathbf{G}(\mathbb{Q}_v)$-invariant (resp. $\mathbf{G}(\mathbb{Q})$-invariant) for finite v (resp. $v = \infty$). Recall that, since $\mathbf{H}(\mathbb{R})$ is compact, $\mathbf{Z}^{\mathbf{H}} = \mathbf{S}_{\mathbf{H}} = \{1\}$.

Remark 7.10. Let $\mu_{\mathbf{H}(\mathbb{A})}$, $\mu_{\mathbf{H}(\mathbb{A}_f)}$ and $\mu_{\mathbf{H},\infty}$be any measures on $\mathbf{H}(\mathbb{A})$, $\mathbf{H}(\mathbb{A}_f)$ and $\mathbf{H}(\mathbb{R})$ such that $\mu_{\mathbf{H}(\mathbb{A})} = \mu_{\mathbf{H}(\mathbb{A}_f)} \times \mu_{\mathbf{H},\infty}$. Then (7) is satisfied by the couple $(\mathbf{H},\{1\})$ and, when $\mu_{\mathbf{H}(\mathbb{A}_f)}(K) \in \mathbb{Q}$ for some (and hence every) $K \in \mathcal{K}(\mathbf{H}(\mathbb{A}_f))$, then $\mu_{\mathbf{H}(\mathbb{R})}(\mathbf{H}(\mathbb{R})) \sim \mu_{\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})}(\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q}))$.

Let $\mu_{\mathbf{H}(\mathbb{A})}$ be the Tamagawa measure, choose local measures $\mu_{\mathbf{H},v}$ at the finite primes so that $\mu_{\mathbf{H}(\mathbb{A}_f)} = \prod_{v<\infty} \mu_{\mathbf{H},v}$ and $\mu_{\mathbf{H}(\mathbb{A}_f)}(K) \in \mathbb{Q}$ for some (and hence every) $K \in \mathcal{K}(\mathbf{H}(\mathbb{A}_f))$ by imposing a similar local conditions $\mu_{\mathbf{H}(\mathbb{Q}_v)}(K_v) \in \mathbb{Q}$ for $K_v \in \mathcal{K}(\mathbf{H}(\mathbb{Q}_v))$ and fix $\mu_{\mathbf{H},\infty}$ so that $\mu_{\mathbf{H}(\mathbb{A})} = \mu_{\mathbf{H}(\mathbb{A}_f)} \times \mu_{\mathbf{H},\infty}$. Then it follows from Remark 7.10 that the the normalizations imposed on the couple $(\mathbf{H},\{1\})$ after (7) are satisfied. Since $\mathbf{H} := \mathbf{N} \rtimes \mathbf{G}_W$, we may further suppose that $\mu_{\mathbf{G}_W(\mathbb{A})} = \prod_v \mu_{\mathbf{G}_W,v}$ with $\mu_{\mathbf{G}_W(\mathbb{A})}$ the Tamagawa measure. It follows from $\mu_{\mathbf{G}_W(\mathbb{A})} = \prod_v \mu_{\mathbf{G}_W,v}$ with $\mu_{\mathbf{G}_W(\mathbb{A})}$ the Tamagawa measure and (43) that (5) is in force (see [26, Remark 2.6]).

We embed $\pi_{E^*(\pi)}$ in the space of automorphic forms via $f^{\mathbf{N}}_{\Lambda,\cdot}$ with $\Lambda \in \overline{\pi_{E^*(\pi)}} \simeq \pi_{E^*(\pi)}$ which is non-zero and $\mathbf{H}(\mathbb{Q})$-invariant (the $\simeq$ because we are in a self-dual situation, since $\mathbf{G}(\mathbb{R})$ is compact). Note that the non-zero Λ exists, unique up to a non-zero constant, thanks to (3) at $v = \infty$, because we have the equality there thanks to $\mathbf{G} = \mathbf{G}'_\pi$. If follows from (42) that a $E^*(\pi)$-rational global test vector can be chosen so that $f = (\otimes'_{v<\infty} f_v) \otimes \Lambda$ is a pure tensor of $E^*(\pi)$-rational local test vectors f_v. Thus, the matrix coefficients of f_v with respect to the $E^*(\pi)$-rational $\mathbf{H}(\mathbb{Q}_v)$-invariant bilinear pairing $\langle -,-\rangle_{\pi_{E^*(\pi),v}}$ are $E^*(\pi)$-valued. We have

$$\alpha_v(f_v) = \frac{L_v(1,\pi_V,\mathrm{Ad})L_v(1,\pi_W,\mathrm{Ad})}{L_v(1/2,\pi_V \boxtimes \pi_W)\Delta_{\mathbf{G}_V,v}} I_v(f_v),$$

where $I_v(f_v)$ is the (stable) matrix coefficient [26, (2.2)] and $I_v(f_v) \in \mathbb{C}^\times$ because f_v is a local test vector (note that our α_v is denoted $\alpha_v^\natural$ in loc.cit.). Let S_π be the set of bad primes (i.e. not good according to [26, After Theorem 2.1]). Since $\pi_v^{K_v} \simeq \mathbb{C} \otimes_{E^*(\pi)} \pi^{K_v}_{E^*(\pi),v}$, according to [26, Conjecture 2.3] (that we assume) and [26, Theorem 2.2] with $K_v = \mathcal{K}_{0,v} \times \mathcal{K}_{2,v}$ in loc. cit., we may assume that $\alpha_v(f_v) = 1$ for every $v \notin S_\pi$. Let us suppose for the moment that, at finite $v \in S_\pi$, we have $I_v(f_v) \in E(\pi)$ and, hence $I_v(f_v), \alpha_v(f_v) \in E^*(\pi)^\times$. At $v = \infty$, we have that $f_\infty = \Lambda$ is $\mathbf{H}(\mathbb{R})$-invariant (by density of $\mathbf{H}(\mathbb{Q}) \subset \mathbf{H}(\mathbb{R})$) and we notice that (cf. [26, Proposition 3.15] to see that α_∞ can be defined as above in our case)

$$\begin{aligned}\alpha_\infty(f_\infty) &= \frac{L_\infty(1,\pi_V,\mathrm{Ad})L_\infty(1,\pi_W,\mathrm{Ad})}{L_\infty(1/2,\pi_V\boxtimes\pi_W)\Delta_{\mathbf{G}_V,\infty}}\langle f_\infty, f_\infty\rangle^2_{\pi_{E^*(\pi),v}}\mu_{\mathbf{H},\infty}(\mathbf{H}(\mathbb{R}))\\ &\sim \frac{L_\infty(1,\pi_V,\mathrm{Ad})L_\infty(1,\pi_W,\mathrm{Ad})}{L_\infty(1/2,\pi_V\boxtimes\pi_W)\Delta_{\mathbf{G}_V,\infty}}\mu_{\mathbf{H},\infty}(\mathbf{H}(\mathbb{R})).\end{aligned}$$

The result now follows from Theorem A of the introduction, in view of (5) and $\mu_{\mathbf{H}(\mathbb{R})}(\mathbf{H}(\mathbb{R})) \sim \mu_{\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})}(\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})) \in \mathbb{Q}^\times$ (Tamagawa number conjecture).

It remains to explain our assumption $I_v(f_v) \in E(\pi)$. We remark that the global model of $\pi = \pi_V \boxtimes \pi_W$ is indeed the tensor product of models of π_V and π_W. We can repeat the above consideration componentwisely and decompose both f, the f_vs, $\langle -,-\rangle_{\pi_{E^*(\pi)}}$ and the $\langle -,-\rangle_{\pi_{E^*(\pi),v}}$s as a tensor product of their analogues for π_V and π_W. Then (see [26, (2.2)]), we have to integrate the function c_{f_v} defined by the formula

$$\begin{aligned}c_{f_v}(h_v) &:= \langle \pi_{E^*(\pi),v}(h_v)f_v, f_v\rangle_{\pi_{E^*(\pi),v}}\\ &= \langle \pi_{V,E^*(\pi),v}(h_v)f_{V,v}, f_{V,v}\rangle_{\pi_{E^*(\pi),v}}\\ &\qquad \langle \pi_{W,E^*(\pi),v}(h_v)f_{W,v}, f_{W,v}\rangle_{\pi_{E^*(\pi),v}}\end{aligned}$$

on $\mathbf{H}(\mathbb{Q}_v) = \mathbf{N}(\mathbb{Q}_v) \rtimes \mathbf{G}_W(\mathbb{Q}_v)$, where $\mathbf{N}(\mathbb{Q}_v)$ acts via the projection to $\mathbf{G}_W(\mathbb{Q}_v)$ on $\pi_{W,v}$. On $\mathbf{N}(\mathbb{Q}_v)$ the integral is stable and, if we assume that $\pi_{V,v|\mathbf{G}_W(\mathbb{Q}_v)}$ or $\pi_{W,v}$ are compactly supported, then [26, (2.2)] is reduced to a $\mathbb{Q}$-linear combination of integrals of matrix coefficients over $\mathbf{G}_W(\mathbb{Q}_v)$ (see [26, pag. 16]). Then, because $\mu_{\mathbf{H}(\mathbb{Q}_v)}(K_v) \in \mathbb{Q}$ for $K_v \in \mathcal{K}(\mathbf{H}(\mathbb{Q}_v))$, the integral is a sum of integrals of compactly supported locally constant functions and it is therefore a $\mathbb{Q}$-linear combination of values $c_{f_v}(h_v) \in E^*(\pi)$. In general, without making any assumption on the support of the matrix coefficients, we can conclude as follows. Because $\mathbf{G}(\mathbb{R})$ is compact, we are in case $r = 0$ (see Remark 1.2) and, hence, the proof of [28, §1.7 Lemme] (which also works in the unitary case, as remarked in [15, §4.1.5]) shows how to reduce the integral $I_v(f_v)$ to a $\mathbb{Q}$-linear combination of values in $E^*(\pi)$, using the fact that $c_{f_v}(h_v) \in E^*(\pi)$.

8. Examples

Let B be a definite quaternion division $\mathbb{Q}$-algebra and let $\mathbf{B}$ (resp. $\mathbf{B}^\times$) be the associated ring scheme (resp. algebraic group). We set $B_f := \mathbf{B}(\mathbb{A}_f)$ (resp. $B_f^\times := \mathbf{B}^\times(\mathbb{A}_f)$) and $B_v = \mathbf{B}(\mathbb{Q}_v)$ (resp. $B_v^\times := \mathbf{B}^\times(\mathbb{Q}_v)$) if v is either a finite place or $v = \infty$. We write $b \mapsto b^\iota$ for the main involution and $\mathrm{nrd}\colon \mathbf{B}^\times \to \mathbf{G}_m$ for the reduced norm. Suppose that $\underline{k} := (k_1, \ldots, k_r) \in \mathbb{Z}^r$, naturally regarded as a character of $\mathbf{G}_m^r$. Then we can consider the algebraic character

$$\mathrm{nrd}^{\underline{k}}\colon \times^r \mathbf{B}^\times \xrightarrow{\times^r \mathrm{nrd}} \mathbf{G}_m^r \xrightarrow{\underline{k}} \mathbf{G}_m.$$

Explicitly, $\mathrm{nrd}^{\underline{k}}(b_1, \ldots, b_r) = \mathrm{nrd}(b_1)^{k_1} \ldots \mathrm{nrd}(b_r)^{k_r}$; when $r = 1$, we write $\mathrm{nrd}^k := \mathrm{nrd}^{(k)}$. We note that $\times^r \mathrm{nrd}$ realizes the maximal quotient which is a split torus, so that we get a description of $X^*(\mathbf{S}'_{\mathbf{G}})$, the characters of $\times^r \mathbf{B}^\times$ defined over $\mathbb{Q}$. More generally, if $\underline{\kappa} = (\kappa_1, \ldots, \kappa_r)$ is a family of characters $\kappa_i\colon \mathbb{R}^\times \to R^\times$ regarded as a character of $R^{\times r}$ via $\underline{x}^{\underline{\kappa}} := x_1^{\kappa_1} \ldots x_r^{\kappa_r}$, we define

$$\mathrm{nrd}^{\underline{\kappa}}\colon \times^r \mathbf{B}^\times(R) \xrightarrow{\times^r \mathrm{nrd}} R^{\times r} \xrightarrow{\underline{\kappa}} R^\times.$$

If $V = (V, \rho)$ is a representation of either $\times^r \mathbf{B}^\times$ or $\times^r B_\infty^\times$ with coefficients in R and $\underline{\kappa}$ is as above, we $V(\underline{\kappa}) = (V, \rho(\underline{\kappa}))$ for the representation $\rho(\underline{\kappa})(b)(v) := \mathrm{nrd}^{\underline{\kappa}}(b)\rho(b)v$. Taking $\chi = \mathrm{nrd}^{\underline{k}}$ in the discussion before Remark 7.5 with $\underline{k} \in \mathbb{Q}^r$ yields the functions $\mathrm{Nrd}_f^{\underline{k}} := \mathrm{N} \circ \mathrm{nrd}_f^{\underline{k}}$, $\mathrm{Nrd}_f^{\underline{k}} := |{-}|_{\mathbb{A}_f}^{-1} \circ \mathrm{nrd}_f^{\underline{k}}$ and $\mathrm{Nrd}_\infty^{\underline{k}} := |{-}|_\infty \circ \mathrm{nrd}_f^{\underline{k}}$. Remark 7.5 gives

$$\mathrm{Nrd}_f^{\underline{k}} := |\,\mathrm{nrd}_f^{\underline{k}}\,|_{\mathbb{A}_f}^{-1} \in M(\times^r \mathbf{B}^\times, R(\underline{k}), \mathrm{Nrd}_f^{\underline{k}})^K \subset M(\times^r \mathbf{B}^\times, R(\underline{k}), \mathrm{Nrd}_f^{\underline{k}})$$

for every open and compact $K \in \mathcal{K}$. Take $\eta = \Delta$

$$\Delta\colon \mathbf{B}^\times \longrightarrow \mathbf{B}^\times \times \mathbf{B}^\times \times \mathbf{B}^\times,$$

the diagonal inclusion.

Let $E/\mathbb{Q}$ be a Galois splitting field for B and fix $\mathbf{B}_{/E} \simeq \mathbf{M}_{2/E}$ inducing $\mathbf{B}_{/E}^\times \simeq \mathbf{GL}_{2/E}$. If $k \in \mathbb{N}$ we let $\mathbf{P}_{k/E}$ be the left $\mathbf{GL}_{2/E}$-representation on two variables polynomials of degree k, the action being defined by the rule $(gP)(X, Y) = P((X, Y)g)$. We write $\mathbf{V}_k$ for the dual right representation. If $\underline{k} := (k_1, \ldots, k_r) \in \mathbb{N}^r$, we may identify $\mathbf{P}_{k_1/E} \otimes \cdots \otimes \mathbf{P}_{k_r/E}$ with the space of $2r$-variable polynomials $\mathbf{P}_{\underline{k}/E}$ which are homogeneous of degree k_i in the i-th couple of variables $W_i := (X_i, Y_i)$. Then $\mathbf{V}_{k_1/E} \otimes \cdots \otimes \mathbf{V}_{k_r/E}$ is identified with the dual $\mathbf{V}_{\underline{k}/E}$ of $\mathbf{P}_{\underline{k}/E}$ and any $P \in \mathbf{P}_{\underline{k}/E}(-r)^{\mathbf{GL}_{2/E}}$, i.e. such that $gP = \det(g)^r P$, induces

$$\Lambda_P \in \mathrm{Hom}_{\mathbf{GL}_{2/E}}(\mathbf{V}_{\underline{k}/E}, \mathbf{1}_{/E}(r))$$

by the rule $\Lambda_P(l) := l(P)$. Note also that, if $P \neq 0$ then there is l such that $l(P) = 1$ and we see that $\Lambda_P \neq 0$. Setting $0 \neq \delta^k(X_1, Y_1, X_2, Y_2) := \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}^k$, we have $\delta^1(W_1 g, W_2 g) = \det(g)\delta^1(W_1, W_2)$, from which it follows that $\delta_k \in \mathbf{P}_{k,k/E}$ and $g\delta^k = \det(g)^k \delta^k$. We deduce that $\langle -, - \rangle_k := \Lambda_{\delta^k} \neq 0$ satisfies the above requirement: then the irreducibility of $\mathbf{V}_{k/E}$ implies that it is perfect and symmetric.

If $\underline{k} := (k_1, k_2, k_3) \in \mathbb{N}^3$, we define the quantities $\underline{k}^* := \frac{k_1+k_2+k_3}{2}$, $\underline{k}_1^* := \frac{-k_1+k_2+k_3}{2}$, $\underline{k}_2^* := \frac{k_1-k_2+k_3}{2}$ and $\underline{k}_3^* := \frac{k_1+k_2-k_3}{2}$. With a slight abuse of notation, we write $\mathbf{P}_{\underline{k}/E}$ and $\mathbf{V}_{\underline{k}/E}$ to denote the external tensor product, which is a representation of $\mathbf{GL}_{2/E}^3$. When $\underline{k}^* \in \mathbb{N}$ and $\underline{k}$ is balanced, we can also define

$$\Lambda_{\underline{k}/E} \in \mathrm{Hom}_{\mathbf{GL}_{2/E}}(\mathbf{V}_{\underline{k}/E}, \mathbf{1}_{/E}(\underline{k}^*))$$

as follows. The balanced condition precisely means that $\underline{k}_i^* \geq 0$ for $i = 1, 2, 3$, so that we can consider

$$0 \neq \Delta_{\underline{k}/E} := \delta^{\underline{k}_1^*}(W_2, W_3)\delta^{\underline{k}_2^*}(W_1, W_3)\delta^{\underline{k}_3^*}(W_1, W_2) \in \mathbf{P}_{\underline{k}/E}.$$

We have $g\Delta_{\underline{k}/E} = \det(g)^{\underline{k}^*}\Delta_{\underline{k}/E}$. Hence $\Delta_{\underline{k}/E} \in \mathbf{P}_{\underline{k}/E}(-\underline{k}^*)^{\mathbf{GL}_{2/E}}$ and we may set $\Lambda_{\underline{k}/E} := \Lambda_{\Delta_{\underline{k}/E}} \neq 0$. The following result is an application of the Clebsch-Gordan decomposition that we leave to the reader.

LEMMA 8.1. *Suppose that* $2\underline{k}^* = k_1 + k_2 + k_3 \in 2\mathbb{N}$ *and* $\underline{k}$ *is balanced.*

(1) *There is a representation* $\mathbf{V}_{\underline{k}}$ *of* $\mathbf{B}^\times$ *such that*

$$E \otimes \mathbf{V}_{\underline{k}} \simeq \mathbf{V}_{\underline{k}/E}$$

via $\mathbf{B}^\times_{/E} \simeq \mathbf{GL}_{2/E}$ *and* $\langle -, - \rangle_{\underline{k}} \in \mathrm{Hom}_{\mathbf{B}^\times}(\mathbf{V}_{\underline{k}} \otimes \mathbf{V}_{\underline{k}}, \mathbf{1}(\underline{k}))$ *such that*

$$E \otimes \langle -, - \rangle_{\underline{k}} \simeq \langle -, - \rangle_{\underline{k}/E}.$$

(2) *We have, setting* $\mathbf{B}_1^\times := \ker(\mathrm{nrd})$,

$$\dim(\mathrm{Hom}_{\mathbf{B}_1^\times}(\mathbf{V}_{\underline{k}}, \mathbf{1})) = \dim(\mathrm{Hom}_{\mathbf{SL}_{2/E}}(\mathbf{V}_{\underline{k}/E}, \mathbf{1}_{/E})) = 1.$$

8.1 – *An explicit Harris–Kudla–Ichino's formula*

All the representations of $\mathbf{B}^{\times 3}(\mathbb{R})$ are pseudo-algebraic, arising from twists of the representations $\mathbf{V}_{\underline{k}/E}$, whose diagonal restrictions are even precisely when $2\underline{k}^*$ is even (according to Definition 7.4). Taking $(V_\infty, \pi_\infty) = \mathbf{V}_{\underline{k}}(\mathbb{R})$ in Theorem 7.6 (3), we find $\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty}} = \mathrm{Nrd}^{\underline{k}/2}(\underline{t})$, $\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty},f} = \mathrm{Nrd}_f^{\underline{k}/2}(\underline{t})$ and $\mathrm{N}^{1/2}_{\nu_{\pi_\infty,\sigma_\infty},\infty} = \mathrm{Nrd}_\infty^{\underline{k}/2}(\underline{t})$, where $\underline{k}/2 = (k_1/2, k_2/2, k_3/2)$ and $\underline{t} = (t_1, t_2, t_3)$. Similarly we find that $\mathbf{V}_{k_i,\mathbb{C}}$ belongs to ω_i for every $\omega_i = \omega_{f,i} \otimes \mathrm{sgn}(-)^{k_i}$ and, in this case, we have $\omega_{0,i} = \omega_{f,i}\,\mathrm{Nrd}_f^{k_i/2}$. We note that we have $\mathrm{Nrd}_f^{\underline{k}^*} = \Delta^*_{\mathbb{A}_f}(\omega_{f,1}\,\mathrm{Nrd}_f^{k_1/2}, \omega_{f,2}\,\mathrm{Nrd}_f^{k_2/2}, \omega_{f,3}\,\mathrm{Nrd}_f^{k_3/2})$ when $\omega_1\omega_2\omega_3 = 1$. It follows that we can consider the quantity $t_{\underline{k}} := M_{\Delta_{\mathbb{A}_f}}^{\mathrm{Nrd}_f^{\underline{k}^*},\mathrm{nrd}_\infty^{\underline{k}^*}}(\Lambda_{\underline{k}/E})$ defined by (21). The following result is now a consequence of Ichino's formula (see [20]), rephrased by means of Theorem 7.2, and Theorem 7.6.

THEOREM 8.2. *Suppose that $\underline{k}$ is balanced and that $\omega_i = \omega_{i,f} \otimes \mathrm{sgn}(-)^{k_i}$ are unitary Hecke characters such that $\omega_1\omega_2\omega_3 = 1$, implying $\underline{k}^* \in \mathbb{N}$. Consider the quantity*

$$t_{\underline{k}}(\varphi) = \mu_{\mathbf{B}(\mathbb{A}_f)}(K_\varphi) \sum_{x \in K_\varphi \backslash \mathbf{B}(\mathbb{A}_f)/\mathbf{B}(F)} \frac{\Lambda_{\underline{k}}(\varphi(x,x,x))}{|\Gamma_{K_\varphi}(x)| \, \mathrm{Nrd}_f^{\underline{k}^*}(x)},$$

where $K_\varphi \in \mathcal{K}(\mathbf{G}(\mathbb{A}_f))$ is such that $\Delta\colon K_\varphi \subset K_1 \times K_2 \times K_3$ and

$$\varphi \in M(\mathbf{B}^{\times 3}, \mathbf{V}_{\underline{k},\mathbb{C}}, \mathrm{Nrd}_f^{\underline{k}/2})^{K_\varphi} = \bigotimes_{i=1}^{3} M(\mathbf{B}^{\times}, \mathbf{V}_{k_i,\mathbb{C}}, \omega_{i,f}\, \mathrm{Nrd}_f^{k_i/2})^{K_i}.$$

(1) *We have*

$$t_{\underline{k}}^2 = \frac{1}{2^3 m^2_{\mathbf{Z_B}\backslash\mathbf{B},\infty}} \frac{\zeta_{\mathbb{Q}}^2(2) L(1/2, \pi_-)}{L(1,-,\mathrm{Ad})} \prod_v \alpha_v(-)$$

as functionals on $f_{\Lambda_{\underline{k}},\cdot}^{\cdot \mathrm{Nrd}^{\underline{k}/2}, \mathbf{V}^u_{\underline{k},\mathbb{C}}}\colon M(\mathbf{B}^{\times 3}, \mathbf{V}^u_{\underline{k},\mathbb{C}}, \omega_f\, \mathrm{Nrd}_f^{\underline{k}/2}) \subset A(\mathbf{B}^{\times 3}(\mathbb{A}), \omega)$ with $\omega = (\omega_1, \omega_2, \omega_3)$. Here the quantities appearing in right hand side have a similar nature as those in (5) (*see* [20]).

(2) *If $\Pi = \Pi_f \otimes \mathbf{V}^u_{\underline{k},\mathbb{C}}$ is an automorphic representation of $\mathbf{B}^{\times 3}$, we have*

$$f_{\cdot,\cdot\cdot}^{\cdot \mathrm{Nrd}^{\underline{k}/2}, \mathbf{V}^u_{\underline{k},\mathbb{C}}}\colon M[\mathbf{B}^{\times 3}, \mathbf{V}_{\underline{k},\mathbb{C}}, \omega_f\, \mathrm{Nrd}_f^{\underline{k}/2}][\mathrm{Nrd}_f^{-\underline{k}/2}\, \Pi_f] \simeq A(\mathbf{G}(\mathbb{A}), \omega)[\Pi].$$

Setting $J(\Lambda \otimes_{\mathbb{C}} \varphi) := \lambda t_{\underline{k}}(\varphi)$ when $\Lambda = \lambda \Lambda_{\underline{k}}$ and $J(\Lambda \otimes_{\mathbb{C}} \varphi) := 0$ for Λ orthogonal to $\Lambda_{\underline{k}}$ in $\mathbf{V}^{\vee}_{\underline{k},\mathbb{C}}$, we have

$$I_\Delta \circ f_{\cdot,\cdot\cdot}^{\cdot \mathrm{Nrd}^{\underline{k}/2}, \mathbf{V}^u_{\underline{k},\mathbb{C}}} = m_{\mathbf{Z_B}\backslash\mathbf{B},\infty} J$$

on $M[\mathbf{B}^{\times 3}, \mathbf{V}^u_{\underline{k},\mathbb{C}}, \omega_f\, \mathrm{Nrd}_f^{\underline{k}/2}]$ and this rule extends to a morphism of functors from modular forms with coefficients in $\mathbb{Q}(\omega_f)$-algebras to $\mathbf{A}^1$.

(3) *Suppose that Π' is an automorphic representation of $\mathbf{GL}_2^3$ and that the discriminant predicted by* [29] *is that of the quaternion algebra B. Then*

$$L(\Pi', 1/2) \neq 0 \iff M(\Lambda_{\underline{k}}) \neq 0 \quad \text{on } M(\mathbf{B}^{\times 3}, \mathbf{V}^u_{\underline{k},\mathbb{C}}, \omega_f\, \mathrm{Nrd}_f^{\underline{k}/2})[\mathrm{Nrd}_f^{-\underline{k}/2}\, \Pi_f],$$

with $\Pi = \Pi_f \otimes \mathbf{V}^u_{\underline{k},\mathbb{C}}$ corresponding to Π' by the Jacquet–Langlands correspondence.

8.2 – *An explicit Waldspurger's formula*

Let $j: K \hookrightarrow B$ be an embedding of a quadratic imaginary field K in a definite quaternion $\mathbb{Q}$-algebra B (so that $\mathbf{B}^\times(\mathbb{R})/\mathbf{S}_{\mathbf{B}^\times}(\mathbb{R})$ is compact). This embedding induces $\mathbf{j}^\times: \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times) \subset \mathbf{B}^\times$, where $\mathbf{B}^\times$ (resp. $\mathbf{K}^\times$) is the algebraic group attached to B (resp. K). We consider

$$\eta := \mathbf{j}^\times \times 1: \mathbf{H} := \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times) \subset \mathbf{B}^\times \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times) =: \mathbf{G}$$

(so that $\mathbf{S_H} = \mathbf{G}_m$). We fix $\mathbf{B}_{/K} \simeq \mathbf{M}_{2/K}$ inducing $\mathbf{B}^\times_{/K} \simeq \mathbf{GL}_{2/K}$ and can take $E/\mathbb{Q}$ any Galois extension such that $K \subset E$. We may also view $\mathbf{V}_{k/E}$ as a representation of $\mathbf{G}_{/E}$ letting $\mathbf{H}_{/E}$ acts trivially.Let π_g be the automorphic representation of $A(\mathbf{B}^\times(\mathbb{A}),\varepsilon)[\mathbf{V}^u_{k,\mathbb{C}}]$ obtained as the Jacquet–Langlands lift of the representation π'_g of $\mathbf{GL}_2$ attached to a modular form g of weight $k+2$ and let $\chi: \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times)(\mathbb{A}) \to \mathbb{C}^\times$ be a Hecke character of K. Let the assumptions be as in [31, III, §3]: π_g is unitary, $\chi_{|\mathbf{G}_m(\mathbb{A})} = \varepsilon = 1$ (i.e. g has trivial nebentype) and χ is a finite order character. Then $\pi_g \times \chi^{-1} \in A(\mathbf{G}(\mathbb{A}),1)[\pi^u_\infty]$ where $\pi_\infty = \mathbf{V}^u_{k,\mathbb{C}} \times \chi^{-1}_\infty = \mathbf{V}^u_{k,\mathbb{C}}$.

The maximal split toric quotient of $\mathbf{G}$ (resp. $\mathbf{H}$) is

$$\mathrm{nr}_{\mathbf{G}} := (\mathrm{nrd}, \mathrm{nr}_{K/\mathbb{Q}}): \mathbf{G} = \mathbf{B}^\times \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times) \longrightarrow \mathbf{G}_m \times \mathbf{G}_m \text{ (resp. } \mathrm{nr}_{K/\mathbb{Q}}\text{)}$$

Hence the algebraic characters of $\mathbf{G}$ (resp. $\mathbf{H}$) can be describes as follows: if $(k,l) \in \mathbb{Z}^2 = \mathrm{Hom}(\mathbf{G}_m^2, \mathbf{G}_m)$, we set $\mathrm{nr}^{k,l}_{\mathbf{G}}(g) := \mathrm{nr}_{\mathbf{G}}(g)^{(k,l)}$ (resp. $\mathrm{nr}^l_{K/\mathbb{Q}}(h) := \mathrm{nr}_{K/\mathbb{Q}}(h)^l$). We define $\mathrm{Nr}^{k,l}_{\mathbf{G}} := \mathrm{N} \circ \mathrm{nr}^{k,l}_{\mathbf{G}}$ (resp. $\mathrm{Nr}^l_{K/\mathbb{Q}} := \mathrm{N} \circ \mathrm{nr}^l_{K/\mathbb{Q}}$), so that $\mathrm{Nr}^{k,l}_{\mathbf{G}} \circ \eta = \mathrm{Nr}^{k+l}_{K/\mathbb{Q}}$. Then Theorem 7.6 applied to $\pi_\infty = \mathbf{V}_k(\mathbb{C})$ implies that Theorem 7.2 in force with $\mathbf{N} = \mathrm{Nr}^{k/2,0}_{\mathbf{G}}$, so that $(\omega^\eta_f \mathbf{N}^\eta_f, \omega^{-\eta}_\infty \mathbf{N}^\eta_\infty) = (\mathrm{Nr}^{k/2}_{K/\mathbb{Q},f}, \mathrm{nr}^{k/2}_{K/\mathbb{Q}})$.

If $Q_{j/E} \in \mathbf{P}_{2/E}$ be defined as in [12, §2.3.2] (which applies with no changes when K is imaginary), then the evaluation at $Q_j^{k/2} \in \mathbf{P}_{k/E}$ gives (see [12, (3.5)])

$$\Lambda_{j,k/E} \in \mathrm{Hom}_{\mathbf{H}_{/E}}(\mathbf{V}_k, \mathbf{1}(k/2)).$$

It follows from [12, §2.3.2] that there are models $\mathbf{V}_k$ and $\Lambda_{j,k}$ over $\mathbb{Q}$ for the representation $\mathbf{V}_{k/E}$ and $\Lambda_{j,k/E}$. In this case, Proposition 6.1 gives the identification

$$f^{\cdot \mathrm{Nrd}^{k/2}, \mathbf{V}^u_{k,\mathbb{C}}}_{\cdot,\cdot\cdot}: M[\mathbf{B}^{\times 3}, \mathbf{V}_{k,\mathbb{C}}, \mathrm{Nrd}^{k/2}_f][\mathrm{Nrd}^{-k/2}_f \pi_{g,f}] \simeq A(\mathbf{B}^\times(\mathbb{A}),1)[\pi_g].$$

Hence, if $K_{\varphi,\chi} \in \mathcal{K}(\mathbf{H}(\mathbb{A}_f))$ is such that $\eta(K_{\varphi,\chi}) \subset K_\varphi \times K_\chi$ and

$$\begin{aligned}\varphi \times \chi^{-1} \in\ & M(\mathbf{G}(\mathbb{A}_f), \mathbf{V}_{k,\mathbb{C}}, \mathrm{Nr}^{k/2,0}_{\mathbf{G},f|\mathbf{G}_m(\mathbb{A}_f)})^{K_{\varphi,\chi}} \\ &= M(\mathbf{B}^\times, \mathbf{V}_{k,\mathbb{C}}, \mathbf{N}^k_f)^{K_\varphi} \otimes M(\mathbf{B}^\times, \mathbf{1}, 1)^{K_\chi},\end{aligned}$$

we have

$$J_{\chi^{-1}}(\varphi) := J_\eta^{\mathbf{V}_{k,\mathbb{C}},\mathrm{Nr}^{k/2}_{K/\mathbb{Q},f},\mathrm{nr}^{k/2}_{K/\mathbb{Q}}}(\Lambda_{j,k} \otimes (\varphi \times \chi^{-1}))$$
$$= \mu_{\mathbf{H}(\mathbb{A}_f)}(K_{\varphi,\chi}) \sum\nolimits_{x \in K_{\varphi,\chi}\backslash \mathbf{H}(\mathbb{A}_f)/\mathbf{H}(\mathbb{Q})} \frac{\chi^{-1}(x)\Lambda_{j,k}(\varphi(j(x)))}{|\Gamma_{K_{\varphi,\chi}}(x)|\,\mathrm{Nr}^{k/2}_{K/\mathbb{Q},f}(x)}.$$

Let $\pi_{\chi^{-1}}$ be the representation attached to the theta lift $\theta_{\chi^{-1}}$ of χ^{-1}. Then Theorem 7.2, together with (5) (see [31, Proposition 7]), gives

$$J_\chi(\varphi)J_{\chi^{-1}}(\varphi) = J_\chi(\varphi)\overline{J_\chi(\varphi)}$$
$$= \frac{1}{4m_{\mathbf{S_H}\backslash\mathbf{H},\infty}} \frac{\zeta_{\mathbb{Q}}(2)L(1/2, \pi'_g \times \pi_{\chi^{-1}})}{L(1,\pi'_g,\mathrm{Ad})L(1,\pi_{\chi^{-1}},\mathrm{Ad})} \prod\nolimits_v \alpha_v(\varphi_v). \tag{44}$$

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