Pseudo-dualizing complexes and pseudo-derived categories

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Abstract – The definition of a pseudo-dualizing complex is obtained from that of a dualizing complex by dropping the injective dimension condition, while retaining the finite generatedness and homothety isomorphism conditions. In the specific setting of a pair of associative rings, we show that the datum of a pseudo-dualizing complex induces a triangulated equivalence between a pseudo-coderived category and a pseudo-contraderived category. The latter terms mean triangulated categories standing "in between" the conventional derived category and the coderived or the contraderived category. The constructions of these triangulated categories use appropriate versions of the Auslander and Bass classes of modules. The constructions of derived functors providing the triangulated equivalence are based on a generalization of a technique developed in our previous paper [45].

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0. Introduction

0.1. – According to the philosophy elaborated in the introduction to [45], the choice of a dualizing complex induces a triangulated equivalence between the coderived category of (co)modules and the contraderived category of (contra)modules, while in order to construct an equivalence between the conventional derived categories of (co)modules and (contra)modules one needs a dedualizing complex. In particular, an associative ring A is a dedualizing complex of bimodules over itself, while a coassociative coalgebra $\mathcal{C}$ over a field k is a dualizing complex of bicomodules over itself. The former assertion refers to the identity equivalence

$$(1) \qquad\qquad \mathsf{D}(A\text{–mod}) = \mathsf{D}(A\text{–mod}),$$

while the latter one points to the natural triangulated equivalence between the coderived category of comodules and the contraderived category of contramodules

$$(2) \qquad\qquad \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–contra}),$$

which is known as the *derived comodule-contramodule correspondence* [42, Sections 0.2.6–7 and 5.4], [43, Sections 4.4 and 5.2].

Given a left coherent ring A and a right coherent ring B, the choice of a dualizing complex of A-B-bimodules $D^\bullet$ induces a triangulated equivalence between the coderived and the contraderived category [28, Theorem 4.8], [46, Theorem 4.5]

$$(3) \qquad\qquad \mathsf{D}^{\mathsf{co}}(A\text{--mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{--mod}).$$

Given a left cocoherent coalgebra $\mathcal{C}$ and a right cocoherent coalgebra $\mathcal{D}$ over a field k, the choice of a dedualizing complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules $\mathcal{B}^\bullet$ induces a triangulated equivalence between the conventional derived categories of comodules and contramodules [49, Theorem 3.6]

$$(4) \qquad\qquad \mathsf{D}(\mathcal{C}\text{--comod}) \simeq \mathsf{D}(\mathcal{D}\text{--contra}).$$

0.2. – The equivalences (1)–(4) of Section 0.1 are the "pure types." The more complicated and interesting triangulated equivalences of the "broadly understood co-contra correspondence" kind are obtained by mixing these pure types, or maybe rather building these elementary blocks on top of one another.

In particular, let R be a commutative ring and $I \subset R$ be an ideal. An R-module M is said to be I-*torsion* if

$$R[s^{-1}] \otimes_R M = 0 \quad \text{for all } s \in I.$$

Clearly, it suffices to check this condition for a set of generators $\{s_j\}$ of the ideal I. An R-module P is said to be an I-*contramodule* if

$$\mathrm{Hom}_R(R[s^{-1}], P) = 0 = \mathrm{Ext}^1_R(R[s^{-1}], P) \quad \text{for all } s \in I.$$

Once again, it suffices to check these conditions for a set of generators $\{s_j\}$ of the ideal I [64, Theorem 5 and Lemma 7(1)], [47, Theorem 5.1]. Both the full subcategory of I-torsion R-modules R–$\mathrm{mod}_{I\text{-tors}} \subset R$–mod and the full subcategory of I-contramodule R-modules R–$\mathrm{mod}_{I\text{-ctra}} \subset R$–mod are abelian categories.

Assume for simplicity that R is a Noetherian ring. Then, using what the paper [45] calls a *dedualizing complex* $B^\bullet$ for the ring R with the ideal $I \subset R$, one can construct a triangulated equivalence between the conventional derived categories of the abelian categories of I-torsion and I-contramodule R-modules

$$(5) \qquad\qquad \mathsf{D}(R\text{--mod}_{I\text{-tors}}) \simeq \mathsf{D}(R\text{--mod}_{I\text{-ctra}}).$$

This result can be generalized to the so-called weakly proregular finitely generated ideals I in the sense of [57, 40] in not necessarily Noetherian commutative rings R [45, Corollary 3.5 or Theorem 5.10].

Using what is generally known as a *t-dualizing* (torsion-dualizing) *complex* [1, Definition 2.5.1], [41, Section 3] (called a "dualizing complex for a ring R with an ideal $I \subset R$" in [44]), one can construct a triangulated equvalence between the coderived category of I-torsion R-modules and the contraderived category of I-contramodule R-modules [44, Theorem C.1.4] (see also [44, Theorem C.5.10])

$$(6) \qquad \mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}_{I\text{-tors}}) \simeq \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod}_{I\text{-ctra}}).$$

This result can be generalized from affine Noetherian formal schemes to ind-affine ind-Noetherian or ind-coherent ind-schemes with a dualizing complex [44, Theorem D.2.7] (see also [45, Remark 4.10]).

Informally, one can view the I-adic completion of a ring R as "a ring in the direction of R/I and a coalgebra in the transversal direction of R relative to R/I." In this sense, one can say that (the formulation of) the triangulated equivalence (5) is obtained by building (4) on top of (1), while (the idea of) the triangulated equivalence (6) is the result of bulding (2) on top of (3).

0.3. – A number of other triangulated equivalences appearing in the present author's work can be described as mixtures of some of the equivalences (1)–(4). In particular, the equivalence between the coderived category of comodules and the contraderived category of contramodules over a pair of corings over associative rings in [44, Corollaries B.4.6 and B.4.10] is another way of building (2) on top of (3).

The equivalence between the conventional derived categories of semimodules and semicontramodules in [49, Theorem 4.3] is obtained by building (1) on top of (4). The equivalence between the semicoderived and the semicontraderived categories of modules in [46, Theorem 5.6] is the result of building (1) on top of (3).

The most deep and difficult in this series of triangulated equivalences is the *derived semimodule-semicontramodule correspondence* of [42, Section 0.3.7] (see the proof in a greater generality in [42, Section 6.3]). The application of this triangulated equivalence to the categories O and $\mathsf{O}^{\mathsf{ctr}}$ over Tate Lie algebras in [42, Corollary D.3.1] is of particular importance. This is the main result of the book [42]. It can be understood as obtainable by building (1) on top of (2).

Note that all the expressions like "can be obtained by" or "is the result of" above refer, at best, to the *formulations* of the mentioned theorems, rather than to their *proofs*. For example, the derived semimodule-semicontramodule correspondence, even in the generality of [42, Section 0.3.7], is a difficult theorem. There is no way to deduce it from the easy (2) and the trivial (1). The formulations of (2) and (1)

serve as an inspiration and the guiding heuristics for arriving to the formulation of the derived semimodule-semicontramodule correspondence. Subsequently, one has to develop appropriate techniques leading to a proof.

0.4. – More generally, beyond building things on top of one another, one may wish to develop notions providing a kind of "smooth interpolation" between various concepts. In particular, the notion of a discrete module over a topological ring can be viewed as interpolating between those of a module over a ring and a comodule over a coalgebra over a field, while the notion of a contramodule over a topological ring (see [42, Remark A.3] or [50]) interpolates between those of a module over a ring and a contramodule over a coalgebra over a field.

The notion of a *pseudo-dualizing complex* (known as a "semi-dualizing complex" in the literature) interpolates between those of a dualizing and a dedualizing complex. Similarly, the notions of a *pseudo-coderived* and a *pseudo-contraderived* category interpolate between those of the conventional derived category and the co- or contraderived category. The aim of this paper is to construct the related interpolation between the triangulated equivalences (1) and (3).

Let us mention that a family of "intermediate" model structures between conventional derived ones ("of the first kind") and the coderived ones ("of the second kind") was constructed, in the case of DG-coalgebras and DG-comodules, in the paper [17]. There is some vague similarity between our construction and the one in [17]. The differences are that we start from a pseudo-dualizing complex and obtain a triangulated equivalence for our intermediate triangulated categories in the context of the comodule-contramodule correspondence, while the authors of [17] start from a twisting cochain and obtain a Quillen adjunction in the context of Koszul duality.

0.5. – Let A and B be associative rings. A *pseudo-dualizing complex* $L^\bullet$ for the rings A and B is a finite complex of A-B-bimodules satisfying the following two conditions:

 (ii) as a complex of left A-modules, $L^\bullet$ is quasi-isomorphic to a bounded above complex of finitely generated projective A-modules, and similarly, as a complex of right B-modules, $L^\bullet$ is quasi-isomorphic to a bounded above complex of finitely generated projective B-modules;

(iii) the homothety maps

$$A \longrightarrow \mathrm{Hom}_{\mathsf{D}^\mathrm{b}(\mathsf{mod}-B)}(L^\bullet, L^\bullet[*]) \quad \text{and} \quad B^\mathrm{op} \longrightarrow \mathrm{Hom}_{\mathsf{D}^\mathrm{b}(A-\mathsf{mod})}(L^\bullet, L^\bullet[*])$$

are isomorphisms of graded rings.

This definition is obtained by dropping the injectivity (or finite injective dimension, or fp-injectivity, etc.) condition (i) from the definition of a *dualizing* or ("cotilting") complex of A-B-bimodules $D^\bullet$ in the papers [34, 65, 12, 46], removing the Noetherianness/coherence conditions on the rings A and B, and rewriting the finite generatedness/presentability condition (ii) accordingly.

For example, when the rings A and B coincide, the one-term complex $L^\bullet = A = B$ becomes the simplest example of a pseudo-dualizing complex. This is what can be called a *dedualizing complex* in this context. More generally, a "dedualizing complex of A-B-bimodules" is the same thing as a "(two-sided) tilting complex" $T^\bullet$ in the sense of Rickard's derived Morita theory [53, 54, 18].

What in our terminology would be called "pseudo-dualizing complexes of modules over commutative Noetherian rings" were studied in the paper [11] and the references therein under some other names, such as "semi-dualizing complexes". What the authors call "semidualizing bimodules" for pairs of associative rings were considered in the paper [26]. We use this other terminology of our own in this paper, because in the context of the present author's work the prefix "semi" means something related but different and more narrow (as in [42] and [46, Sections 5–6]).

The main result of this paper provides the following commutative diagram of triangulated functors associated with a pseudo-dualizing complex of A-B-bimodules $L^\bullet$:

(7)
$$
\begin{array}{ccc}
\mathsf{D}^{\mathsf{co}}(A\text{--mod}) & & \mathsf{D}^{\mathsf{ctr}}(B\text{--mod}) \\
\downarrow & & \downarrow \\
\mathsf{D}^{L^\bullet}_r(A\text{--mod}) & =\!=\!= & \mathsf{D}^{L^\bullet}_l(B\text{--mod}) \\
\downarrow & & \downarrow \\
\mathsf{D}'_{L^\bullet}(A\text{--mod}) & =\!=\!= & \mathsf{D}''_{L^\bullet}(B\text{--mod}) \\
\downarrow & & \downarrow \\
\mathsf{D}(A\text{--mod}) & & \mathsf{D}(B\text{--mod})
\end{array}
$$

Here the vertical arrows are Verdier quotient functors, while the horizontal double lines are triangulated equivalences.

Thus $\mathsf{D}^{L^\bullet}_r(A\text{--mod})$ and $\mathsf{D}'_{L^\bullet}(A\text{--mod})$ are certain intermediate triangulated categories between the coderived category of left A-modules $\mathsf{D}^{\mathsf{co}}(A\text{--mod})$ and their conventional unbounded derived category $\mathsf{D}(A\text{--mod})$. Similarly, $\mathsf{D}^{L^\bullet}_l(B\text{--mod})$ and $\mathsf{D}''_{L^\bullet}(B\text{--mod})$ are certain intermediate triangulated categories between the

contraderived category of left B-modules $\mathsf{D}^{\mathrm{ctr}}(B\text{–mod})$ and their conventional un-bounded derived category $\mathsf{D}(B\text{–mod})$. These intermediate triangulated quotient categories depend on, and are determined by, the choice of a pseudo-dualizing complex $L^\bullet$ for a pair of associative rings A and B.

The triangulated category $\mathsf{D}'_{L^\bullet}(A\text{–mod})$ is called the *lower pseudo-coderived category* of left A-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}''_{L^\bullet}(B\text{–mod})$ is called the *lower pseudo-contraderived category* of left B-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}^{L^\bullet}_r(A\text{–mod})$ is called the *upper pseudo-coderived category* of left A-modules corresponding to $L^\bullet$. The triangulated category $\mathsf{D}^{L^\bullet}_{r'}(B\text{–mod})$ is called the *upper pseudo-contraderived category* of left B-modules corresponding to $L^\bullet$. The choice of a pseudo-dualizing complex $L^\bullet$ also induces triangulated equivalences $\mathsf{D}'_{L^\bullet}(A\text{–mod}) \simeq \mathsf{D}''_{L^\bullet}(B\text{–mod})$ and $\mathsf{D}^{L^\bullet}_r(A\text{–mod}) \simeq \mathsf{D}^{L^\bullet}_{r'}(A\text{–mod})$ forming the commutative diagram (7).

In particular, when $L^\bullet = D^\bullet$ is a dualizing complex, i. e., the condition (i) of [46, Section 4] is satisfied, assuming additionally that all fp-injective left A-modules have finite injective dimensions, one has $\mathsf{D}^{L^\bullet}_r(A\text{–mod}) = \mathsf{D}^{\mathrm{co}}(A\text{–mod})$ and $\mathsf{D}^{L^\bullet}_{r'}(B\text{–mod}) = \mathsf{D}^{\mathrm{ctr}}(B\text{–mod})$, that is the upper two vertical arrows in the diagram (7) are isomorphisms of triangulated categories. The upper triangulated equivalence in the diagram (7) coincides with the one provided by [46, Theorem 4.5] in this case.

When $L^\bullet = A = B$, one has $\mathsf{D}'_{L^\bullet}(A\text{–mod}) = \mathsf{D}(A\text{–mod})$ and $\mathsf{D}''_{L^\bullet}(B\text{–mod}) = \mathsf{D}(B\text{–mod})$, that is the lower two vertical arrows in the diagram (7) are isomorphisms of triangulated categories. The lower triangulated equivalence in the diagram (7) is just the identity isomorphism $\mathsf{D}(A\text{–mod}) = \mathsf{D}(B\text{–mod})$ is this case. More generally, the lower triangulated equivalence in the diagram (7) corresponding to a tilting complex $L^\bullet = T^\bullet$ recovers Rickard's derived Morita equivalence [53, Theorem 6.4], [54, Theorem 3.3 and Proposition 5.1] (see also [18, Theorem 4.2]).

0.6. – A delicate point is that when $A = B = R$ is, e. g., a Gorenstein Noetherian commutative ring of finite Krull dimension, the ring R itself can be chosen as a dualizing complex of R-R-bimodules. So we are in both of the above-described situations at the same time. Still, the derived category of R-modules $\mathsf{D}(R\text{–mod})$, the coderived category $\mathsf{D}^{\mathrm{co}}(R\text{–mod})$, and the contraderived category $\mathsf{D}^{\mathrm{ctr}}(R\text{–mod})$ are three quite different quotient categories of the homotopy category of (complexes of) R-modules $\mathsf{Hot}(R\text{–mod})$. In this case, the commutative diagram (7)

takes the form

$$D^{co}(R\text{–mod}) \;=\!=\!=\; D^{ctr}(R\text{–mod})$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$D(R\text{–mod}) \;=\!=\!=\; D(R\text{–mod})$$

More precisely, the two Verdier quotient functors $\mathsf{Hot}(R\text{–mod}) \to \mathsf{D}^{co}(R\text{–mod})$ and $\mathsf{Hot}(R\text{–mod}) \to \mathsf{D}^{ctr}(R\text{–mod})$ both factorize naturally through the Verdier quotient functor $\mathsf{Hot}(R\text{–mod}) \to \mathsf{D}^{abs}(R\text{–mod})$ from the homotopy category onto the absolute derived category of R-modules $\mathsf{D}^{abs}(R\text{–mod})$. But the two resulting Verdier quotient functors $\mathsf{D}^{abs}(R\text{–mod}) \to \mathsf{D}^{co}(R\text{–mod})$ and $\mathsf{D}^{abs}(R\text{–mod}) \to \mathsf{D}^{ctr}(R\text{–mod})$ do *not* form a commutative triangle with the equivalence

$$\mathsf{D}^{co}(R\text{–mod}) \simeq \mathsf{D}^{ctr}(R\text{–mod}).$$

Rather, they are the two adjoint functors on the two sides to the fully faithful embedding of a certain (one and the same) triangulated subcategory in $\mathsf{D}^{abs}(R\text{–mod})$ [43, proof of Theorem 3.9].

This example shows that one cannot hope to have a procedure recovering the conventional derived category $\mathsf{D}(A\text{–mod}) = \mathsf{D}(B\text{–mod})$ from the dedualizing complex $L^\bullet = A = B$, and at the same time recovering the coderived category $\mathsf{D}^{co}(A\text{–mod})$ and the contraderived category $\mathsf{D}^{ctr}(B\text{–mod})$ from a dualizing complex $L^\bullet = D^\bullet$. Thus the distinction between the lower and and the upper pseudo-co/contraderived category constructions is in some sense inevitable.

0.7. – Before we finish this introduction, let us say a few words about where the pseudo-coderived and pseudo-contraderived categories come from in Section 0.5. We use "pseudo-derived categories" as a generic term for the pseudo-coderived and pseudo-contraderived categories. The constructions of such triangulated categories that we use in this paper were originally introduced for the purposes of the infinitely generated Wakamatsu tilting theory [52, Section 4] (an even more general approach to exotic derived categories is suggested in [52, Section 5]).

The pseudo-derived categories are constructed as the conventional unbounded derived categories of certain exact subcategories $\mathsf{E}_{/} \subset \mathsf{E}' \subset \mathsf{A}$ and $\mathsf{F}_{/\!/} \subset \mathsf{F}'' \subset \mathsf{B}$ in the abelian categories $\mathsf{A} = A\text{–mod}$ and $\mathsf{B} = B\text{–mod}$. The idea is that shrinking an abelian (or exact) category to its exact subcategory leads, under certain assumptions, to a bigger derived category, as complexes in the exact subcategory are considered up to a finer equivalence relation in the derived category construction.

In the situation at hand, the larger subcategories E' and F'' are our versions of what are called the *Auslander and Bass classes* in the literature [11, 24, 21, 12, 26]. Specifically, F'' is the Auslander class and E' is the Bass class. The two full subcategories $E_{,}$ and $F_{,,}$ are certain natural smaller classes. One can say, in some approximate sense, that E' and F'' are the *maximal corresponding classes*, while $E_{,}$ and $F_{,,}$ are the *minimal corresponding classes* in the categories A and B.

More precisely, there is a natural single way to define the full subcategories $E' \subset A$ and $F'' \subset B$ when the pseudo-dualizing complex $L^\bullet$ is a one-term complex. In the general case, we have two sequences of embedded subcategories $E_{d_1} \subset E_{d_1+1} \subset E_{d_1+2} \subset \cdots \subset A$ and $F_{d_1} \subset F_{d_1+1} \subset F_{d_1+2} \subset \cdots \subset B$ indexed by large enough integers. All the subcategories E_{l_1} with varying index $l_1 = d_1$, $d_1 + 1$, $d_1 + 2$, ... are "the same up to finite homological dimension", and so are all the subcategories F_{l_1}. Hence the triangulated functors $D(E_{l_1}) \to D(E_{l_1+1})$ and $D(F_{l_1}) \to D(F_{l_1+1})$ induced by the exact embeddings $E_{l_1} \to E_{l_1+1}$ and $F_{l_1} \to F_{l_1+1}$ are triangulated equivalences, so the pseudo-derived categories $D'_{L^\bullet}(A\text{–mod}) = D(E_{l_1})$ and $D''_{L^\bullet}(B\text{–mod}) = D(F_{l_1})$ do not depend on the choice of a number l_1.

The idea of the construction of the triangulated equivalence between the two lower pseudo-derived categories is that the functor $D'_{L^\bullet}(A) \to D''_{L^\bullet}(B)$ should be a version of $\mathbb{R}\,\mathrm{Hom}_A(L^\bullet, -)$, while the inverse functor $D''_{L^\bullet}(B) \to D'_{L^\bullet}(A)$ is a version of derived tensor product $L^\bullet \otimes_B^{\mathbb{L}} -$. The full subcategories $E_{l_1} \subset A$ and $F_{l_1} \subset B$ are defined by the conditions of uniform boundedness of cohomology of such Hom and tensor product complexes (hence dependence on a fixed bound l_1) and the composition of the two operations leading back to the original object.

The point is that the two functors $\mathbb{R}\,\mathrm{Hom}_A(L^\bullet, -)$ and $L^\bullet \otimes_B^{\mathbb{L}} -$ are mutually inverse when viewed as acting between the pseudo-derived categories $D(E)$ and $D(F)$, but objects of the pseudo-derived categories are complexes viewed up to a more delicate equivalence relation than in the conventional derived categories $D(A)$ and $D(B)$. When this subtlety is ignored, the two functors cease to be mutually inverse, generally speaking, and such mutual inverseness needs to be enforced as an additional adjustness restriction on the objects one is working with.

Similarly, there is a natural single way to define the full subcategories $E_{,} \subset A$ and $F_{,,} \subset F$ when the pseudo-dualizing complex $L^\bullet$ is a one-term complex. In the general case, we have two sequences of embedded subcategories $E^{d_2} \supset E^{d_2+1} \supset E^{d_2+2} \supset \cdots$ in A and $F^{d_2} \supset F^{d_2+1} \supset F^{d_2+2} \supset \cdots$ in B, indexed by large enough integers. As above, all the subcategories E^{l_2} with varying $l_2 = d_2$, $d_2 + 1$, $d_2 + 2$, ... are "the same up to finite homological dimension", and so are all the subcategories F^{l_2}. Hence the triangulated functors $D(E^{l_2+1}) \to D(E^{l_2})$
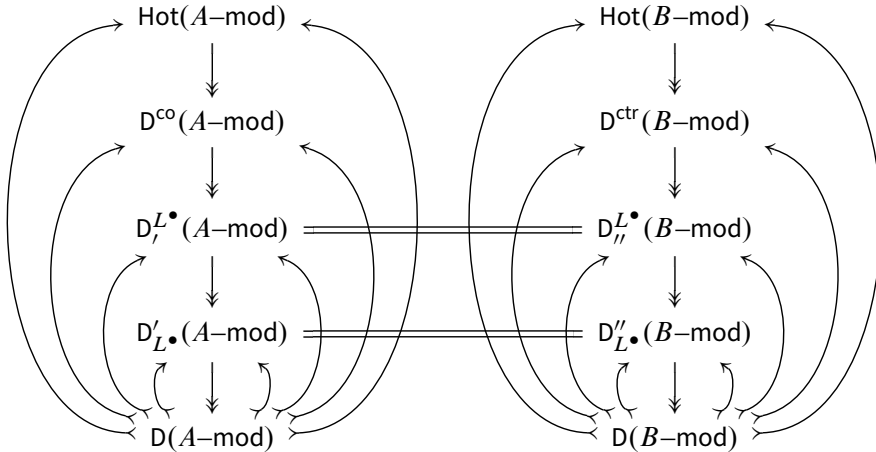
and $\mathsf{D}(\mathsf{F}^{l_2+1}) \to \mathsf{D}(\mathsf{F}^{l_2})$ induced by the exact embeddings $\mathsf{E}^{l_2+1} \to \mathsf{E}^{l_2}$ and $\mathsf{F}^{l_2+1} \to \mathsf{F}^{l_2}$ are triangulated equivalences, so the pseudo-derived categories $\mathsf{D}_r^{L^\bullet}(A\text{--mod}) = \mathsf{D}(\mathsf{E}^{l_2})$ and $\mathsf{D}_{ll}^{L^\bullet}(B\text{--mod}) = \mathsf{D}(\mathsf{F}^{l_2})$ do not depend on the choice of a number l_2.

The triangulated equivalence between the two upper pseudo-derived categories is also provided by some versions of derived functors $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ and $L^\bullet \otimes_B^{\mathbb{L}} -$. The full subcategories $\mathsf{E}^{l_2} \subset \mathsf{A}$ and $\mathsf{F}^{l_2} \subset \mathsf{B}$ are produced by a kind of generation process. One starts from declaring that all the injectives in A belong to E^{l_2} and all the projectives in B belong to F^{l_2}. Then one proceeds with generating further objects of F^{l_2} by applying $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ to objects of E^{l_2}, and further objects of E^{l_2} by applying $L^\bullet \otimes_B^{\mathbb{L}} -$ to objects of F^{l_2}. One needs to resolve the complexes so obtained to produce objects of the abelian module categories, and the number l_2 indicates the length of the resolutions used. More objects are added to E^{l_2} and F^{l_2} to make these full subcategories closed under certain operations.

We refer to the main body of the paper for further details.

0.8. – Inspired by the recent paper [3], in the last three sections of the present paper we address the question of existence of left and right adjoint functors to the Verdier quotient functors in the diagram (7) from Section 0.5. More precisely, for any pseudo-dualizing complex of A-B-bimodules $L^\bullet$ the adjoint functors shown by curvilinear arrows on the following diagram exist:

(8)

The functors shown by curvilinear arrows, being adjoints to Verdier quotient functors, are fully faithful. Their existence follows straightforwardly from the existence of left and right adjoint functors to the natural Verdier quotient functor $\mathsf{Hot}(R\text{--mod}) \to \mathsf{D}(R\text{--mod})$ for any associative ring R.

Moreover, assuming that the ring A is left coherent, fp-injective left A-modules have finite injective dimensions, the ring B is right coherent, and flat left B-modules have finite projective dimensions, the adjoint functors shown by the curvilinear arrows on the following diagram also exist:

(9)

$$
\begin{array}{ccc}
\mathsf{D}^{\mathsf{co}}(A\text{–mod}) & & \mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \\
\mathsf{D}'_{L^\bullet}(A\text{–mod}) & =\!\!=\!\!= & \mathsf{D}''_{L^\bullet}(B\text{–mod}) \\
\mathsf{D}(A\text{–mod}) & & \mathsf{D}(B\text{–mod})
\end{array}
$$

As compared to the previous diagram, the nontrivial additional part here is the existence of the left and right adjoints to the natural Verdier quotient functors $\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \to \mathsf{D}'_{L^\bullet}(A\text{–mod})$ and $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \to \mathsf{D}''_{L^\bullet}(B\text{–mod})$ between the co/contraderived categories and the lower pseudo-derived categories of modules over A and B. In order to prove that these functors exist, we show that the Auslander and Bass classes of modules $\mathsf{F}_{l_1}(L^\bullet) \subset B\text{–mod}$ and $\mathsf{E}_{l_1}(L^\bullet) \subset A\text{–mod}$ are *deconstructible*, deduce from this the *existence of Hom sets* in the lower pseudo-derived categories, recall from the literature that the co/contraderived categories are *compactly generated* in the above assumptions, and observe that the Verdier quotient functors in question preserve infinite direct sums and products.

0.9. – To end, let us say a few words about how the results of this paper compare to the theory developed in the papers [51, 52]. Both [51, 52] and this paper can be viewed as generalizations of the classical finitely generated n-tilting theory, as developed, e. g., in [35] and [13], albeit in different subsets of directions.

Basically, using the notion of a finitely generated n-tilting module as the starting point, one can

(a) drop the condition that the tilting module is finitely generated; and/or

(b) drop the condition that the tilting module has finite projective dimension n; and/or

(c) replace a tilting module by a tilting complex; and/or

(d) replace the category of modules over an associative ring by an abelian category of more general nature (like, e. g., a Grothendieck category).

All these four lines of thought have been explored by many authors over the years; we refer to the introduction to [52] for an (admittedly very possibly incomplete) collection of references. In particular, it seems that (a) was first done in [15], (b) was initiated in [61, 62], (c) was introduced in [53, 54], and (d) was first attempted in [14].

Our paper [51] does (a) and (d). In particular, the generalization to infinitely generated tilting modules/objects is the reason why contramodule categories appear (as the tilting hearts) in [51]. The paper [52] does (a), (b), and (d). In particular, the generalization to tilting modules/objects of infinite projective dimension is the reason why pseudo-derived categories were needed in [52, Section 4] (and an even more general approach using t-structures was developed in [52, Section 5]).

The present paper does (b) and (c). The generalization from ∞-tilting *modules* to pseudo-dualizing *complexes* in this paper (cf. [26] and [52, Example 6.6]) is what makes many aspects of our exposition here so much more complicated technically than in [52]. In fact, the triangulated equivalences $D^\star(E) \simeq D^\star(F)$ come for free in [51, 52], being immediate consequences of the equivalences of exact categories $E \simeq F$. In this paper, a whole (rather long and complicated) appendix is needed just to construct the pair of adjoint derived functors providing this triangulated equivalence. On the other hand, our pseudo-dualizing complexes are presumed to be strongly finitely generated (on each side and up to quasi-isomorphism) in this paper. That is why the underlying abelian categories on both sides of our equivalences are the conventional categories of modules over associative rings, and no contramodules appear.

0.10. – One terminological disclaimer is in order. With the exception of the final Sections 10–12, throughout this paper we adopt the policy of benign neglect of the issue of *existence* of Verdier quotients (cf. [63, Set-Theoretic Remark 10.3.3]).

Generally speaking, given a triangulated subcategory in a triangulated category, their Verdier quotient (such as, e. g., the derived category of an abelian or exact category) may not exist as a category, in the sense that morphisms between a fixed pair of objects in the Verdier quotient may not form a set. So we tacitly presume that a Grothendieck universe has been chosen, and all our sets (rings, modules, etc.) and categories are sets and categories in this universe. If the need arises, the problem of existence of a Verdier quotient category can be then resolved by considering such Verdier quotient as a category in a larger universe.

When (in the final two sections) we will need to emphasize that a particular Verdier quotient category *does* exist (in the fixed, unchanged universe), we will say that such a Verdier quotient *has Hom sets*.

1. Pseudo-coderived and pseudo-contraderived categories

For any additive category A, we denote by $\mathrm{Hot}^\star(A)$, where $\star = b, +, -$, or $\varnothing$, the categories of (respectively bounded or unbounded) cochain complexes in A with the morphisms considered up to the cochain homotopy.

Let A be an exact category (in Quillen's sense). A complex in A is said to be *exact* if it is obtained by splicing short exact sequences in A. A complex in A is *acyclic* if it is homotopy equivalent to an exact complex, or equivalently, if it is a direct summand of an exact complex. The (bounded or unbounded) conventional derived category $D^\star(A)$ with the symbol $\star = b, +, -$, or $\varnothing$ is defined as the quotient category of $\mathrm{Hot}^\star(A)$ by the thick subcategory of (respectively bounded or unbounded) acyclic complexes (see the paper [36] and the overviews [30, 10]).

A short exact sequence of complexes in A can be viewed as a bicomplex with three rows. Taking the total complex of such a bicomplex, one obtains an exact complex in A. A $\star$-bounded complex in A is said to be *absolutely acyclic* if it belongs to the thick subcategory of $\mathrm{Hot}^\star(A)$ generated by the totalizations of short exact sequences of $\star$-bounded complexes in A. In fact, a $\star$-bounded complex in A is absolutely acyclic if and only if it is absolutely acyclic as an unbounded complex [44, Lemma A.1.1]. Any bounded acyclic complex is absolutely acyclic (as a bounded complex, i.e., for $\star = b$). The *absolute derived categories* $D^{abs+}(A)$, $D^{abs-}(A)$, and $D^{abs}(A)$ are defined as the quotient categories of the respective homotopy categories $\mathrm{Hot}^+(A)$, $\mathrm{Hot}^-(A)$, and $\mathrm{Hot}(A)$ by their thick subcategories of absolutely acyclic complexes (see [44, Section A.1] or [45, Appendix A] for a further discussion).

We will say that a full subcategory $E \subset A$ is *coresolving* if E is closed under extensions and the passages to the cokernels of admissible monomorphisms in A, and every object of A is the source of an admissible monomorphism into an object of E. This definition slightly differs from that in [59, Section 2] in that we do not require E to be closed under direct summands (cf. [44, Section A.3]). Obviously, any coresolving subcategory E inherits an exact category structure from the ambient exact category A.

Let A be an exact category in which the functors of infinite direct sum are everywhere defined and exact. A complex in A is said to be *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category Hot(A) containing the totalizations of short exact sequences of complexes in A and closed under infinite direct sums. The *coderived category* $D^{co}(A)$ is defined as the Verdier quotient category of Hot(A) by the thick subcategory of coacyclic complexes (see [42, Section 2.1], [44, Section A.1], or [45, Appendix A]).

The conventional unbounded derived category D(A) is naturally a quotient category of $D^{co}(A)$. A triangulated category D' is called a *pseudo-coderived* category of A if triangulated Verdier quotient functors $D^{co}(A) \to D' \to D(A)$ are given forming a commutative triangle with the canonical Verdier quotient functor $D^{co}(A) \to D(A)$ between the coderived and the conventional derived category of the exact category A.

Let $E \subset A$ be a coresolving subcategory closed under infinite direct sums. According to the dual version of [44, Proposition A.3.1(b)] (formulated explicitly in [46, Proposition 2.1]), the triangulated functor between the coderived categories $D^{co}(E) \to D^{co}(A)$ induced by the direct sum-preserving embedding of exact categories $E \to A$ is an equivalence of triangulated categories. From the commutative diagram of triangulated functors

$$
\begin{array}{ccc}
D^{co}(E) & =\!=\!=\!=\!= & D^{co}(A) \\
\downarrow & & \downarrow \\
D(E) & \longrightarrow & D(A)
\end{array}
$$

one can see that the lower horizontal arrow is a Verdier quotient functor. Thus $D' = D(E)$ is a pseudo-coderived category of A [52, Proposition 4.2(a)].

Furthermore, let $E_, \subset E' \subset A$ be two embedded coresolving subcategories, both closed under infinite direct sums in A. Then the canonical Verdier quotient functor $D^{co}(A) \to D(A)$ decomposes into a sequence of Verdier quotient functors

$$
D^{co}(A) \longrightarrow D(E_,) \longrightarrow D(E') \longrightarrow D(A).
$$

In other words, when the full subcategory $E \subset A$ is expanded, the related pseudo-coderived category D(E) gets deflated [52, Remark 4.3].

Notice that, as a coresolving subcategory closed under infinite direct sums $E \subset A$ varies, its conventional derived category behaves in quite different ways depending on the boundedness conditions. The functor $D^b(E_,) \to D^b(E')$ induced by the embedding $E_, \to E'$ is fully faithful, the functor $D^+(E_,) \to D^+(E')$ is a

triangulated equivalence (by the assertion dual to [44, Proposition A.3.1(a)]), and the functor $D(E_\prime) \to D(E')$ is a Verdier quotient functor.

Let B be another exact category. We will say that a full subcategory $F \subset B$ is *resolving* if F is closed under extensions and the passages to the kernels of admissible epimorphisms, and every object of B is the target of an admissible epimorphism from an object of F. Obviously, a resolving subcategory F inherits an exact category structure from the ambient exact category B.

Let B be an exact category in which the functors of infinite product are everywhere defined and exact. A complex in B is said to be *contraacyclic* if it belongs to the minimal triangulated subcategory of Hot(B) containing the totalizations of short exact sequences of complexes in B and closed under infinite products. The *contraderived category* $D^{ctr}(B)$ is defined as the Verdier quotient category of Hot(B) by the thick subcategory of contraacyclic complexes (see [42, Section 4.1], [44, Section A.1], or [45, Appendix A]).

The conventional unbounded derived category D(B) is naturally a quotient category of $D^{ctr}(B)$. A triangulated category D'' is called a *pseudo-contraderived* category of B if Verdier quotient functors $D^{ctr}(B) \to D'' \to D(B)$ are given forming a commutative triangle with the canonical Verdier quotient functor $D^{ctr}(B) \to D(B)$ between the contraderived and the conventional derived categories of the exact category B.

Let $F \subset B$ be a resolving subcategory closed under infinite products. According to [44, Proposition A.3.1(b)], the triangulated functor between the contraderived categories $D^{ctr}(F) \to D^{ctr}(B)$ induced by the product-preserving embedding of exact categories $F \to B$ is an equivalence of triangulated categories. From the commutative diagram of triangulated functors

$$
\begin{array}{ccc}
D^{ctr}(F) & =\!=\!=\!=\!=\!= & D^{ctr}(B) \\
\downarrow & & \downarrow \\
D(F) & \longrightarrow & D(B)
\end{array}
$$

one can see that the lower horizontal arrow is a Verdier quotient functor. Thus $D'' = D(F)$ is a pseudo-contraderived category of B [52, Proposition 4.2(b)].

Let $F_{\prime\prime} \subset F'' \subset B$ be two embedded resolving subcategories, both closed under infinite products in F. Then the canonical Verdier quotient functor $D^{ctr}(B) \to D(B)$ decomposes into a sequence of Verdier quotient functors

$$
D^{ctr}(B) \longrightarrow D(F_{\prime\prime}) \longrightarrow D(F'') \longrightarrow D(B).
$$

In other words, when the full subcategory F $\subset$ B is expanded, the related pseudo-contraderived category D(F) gets deflated [52, Remark 4.3].

Once again, we notice that, as a resolving subcategory closed under infinite products F $\subset$ B varies, the behavior of its conventional derived category depends on the boundedness conditions. The functor $D^b(F_{\prime\prime}) \to D^b(F'')$ is fully faithful, the functor $D^-(F_{\prime\prime}) \to D^-(F'')$ is a triangulated equivalence [44, Proposition A.3.1(a)], and the functor $D(F_{\prime\prime}) \to D(F'')$ is a Verdier quotient functor.

Some of the simplest examples of coresolving subcategories E closed under infinite direct sums and resolving subcategories F closed under infinite products in the abelian categories of modules over associative rings will be given in Examples 2.5–2.6; and more complicated examples will be discussed in Sections 3, 5, and 8.

2. Strongly finitely presented modules

Let A be an associative ring. We denote by A–mod the abelian category of left A-modules and by mod–A the abelian category of right A-modules.

We say that an A-module is *strongly finitely presented* if it has a projective resolution consisting of finitely generated projective A-modules.

REMARK 2.1. In the traditional terminology, such modules are said to be *pseudo-coherent* [27, 29] or *of type FP_∞* [5, 8, 9]. More generally, an A-module M is said to be *n-pseudo-coherent* or *of type FP_n* if it has a projective resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ such that the A-modules P_i are finitely generated for all $0 \leqslant i \leqslant n$. We prefer our (admittedly less conventional) terminology for several reasons, one of them being that we would like to speak also about strongly finitely presented *complexes* of modules (see below in this section), and the term "chain complex of type FP_∞" already means something else [7, Section 2.1]. On the other hand, the prefix "pseudo-" is used for quite different purposes in this paper. Besides, "strongly finitely presented" abbreviates to a convenient prefix "sfp-", upon which we build our terminological system.

LEMMA 2.2. *Let* $0 \to K \to L \to M \to 0$ *be a short exact sequence of A-modules. Then whenever two of the three modules K, L, M are strongly finitely presented, so is the third one.*

PROOF. This result, in a more general version for modules of type FP_{n-1} and FP_n, goes back to [6, Exercise I.2.6]. For a different proof, see [5, Proposition 1.4],

and for a discussion with further references, [9, Section 1]. Here is yet another proof.

If $P_\bullet \to K$ and $R_\bullet \to M$ are projective resolutions of the A-modules K and M, then there is a projective resolution $Q_\bullet \to L$ of the A-module L with the terms $Q_i \simeq P_i \oplus R_i$. If $P_\bullet \to K$ and $Q_\bullet \to L$ are projective resolutions of the A-modules K and L, then there exists a morphism of complexes of A-modules $P_\bullet \to Q_\bullet$ inducing the given morphism $K \to L$ on the homology modules. The cone $R_\bullet$ of the morphism of complexes $P_\bullet \to Q_\bullet$ is a projective resolution of the A-module M with the terms $R_i \simeq Q_i \oplus P_{i-1}$.

If $Q_\bullet \to L$ and $R_\bullet \to M$ are projective resolutions of the A-modules L and M, then there exists a morphism of complexes of A-modules $Q_\bullet \to R_\bullet$ inducing the given morphism $L \to M$ on the homology modules. The cocone $P'_\bullet$ of the morphism of complexes $Q_\bullet \to R_\bullet$ is a bounded above complex of R-modules with the terms $P'_i = Q_i \oplus R_{i+1}$ and the only nonzero cohomology module $H_0(P'_\bullet) \simeq K$. Still, the complex $P'_\bullet$ is not yet literally a projective resolution of K, as its term $P'_{-1} \simeq R_0$ does not vanish. Setting $P_{-1} = 0$, $P_0 = \ker(P'_0 \to P'_{-1})$, and $P'_i = P_i$ for $i \geqslant 1$, one obtains a subcomplex $P_\bullet \subset P'_\bullet$ with a termwise split embedding $P_\bullet \to P'_\bullet$ such that $P_\bullet$ is a projective resolution of the R-module K. □

Abusing terminology, we will say that a bounded above complex of A-modules $M^\bullet$ is *strongly finitely presented* if it is quasi-isomorphic to a bounded above complex of finitely generated projective A-modules. (Such complexes are called "pseudo-coherent" in [27, 29].) Clearly, the class of all strongly finitely presented complexes is closed under shifts and cones in $\mathsf{D}^-(A\text{–mod})$.

LEMMA 2.3. (a) *Any bounded above complex of strongly finitely presented A-modules is strongly finitely presented.*

(b) *Let $M^\bullet$ be a complex of A-modules concentrated in the cohomological degrees $\leqslant n$, where n is a fixed integer. Then $M^\bullet$ is strongly finitely presented if and only if it is quasi-isomorphic to a complex of finitely generated projective A-modules concentrated in the cohomological degrees $\leqslant n$.*

(c) *Let $M^\bullet$ be a finite complex of A-modules concentrated in the cohomological degrees $n_1 \leqslant m \leqslant n_2$. Then $M^\bullet$ is strongly finitely presented if and only if it is quasi-isomorphic to a complex of A-modules $R^\bullet$ concentrated in the cohomological degrees $n_1 \leqslant m \leqslant n_2$ such that the A-modules R^m are finitely generated and projective for all $n_1 + 1 \leqslant m \leqslant n_2$, while the A-module R^{n_1} is strongly finitely presented.*

PROOF. Part (a) is provable using Lemma 2.2. Part (b) holds, because the kernel of a surjective morphism of finitely generated projective A-modules is a finitely generated projective A-module; and part (c) is also easy.   □

Let A and B be associative rings. A left A-module J is said to be *sfp-injective* if $\operatorname{Ext}_A^1(M, J) = 0$ for all strongly finitely presented left A-modules M, or equivalently, $\operatorname{Ext}_A^n(M, J) = 0$ for all strongly finitely presented left A-modules M and all $n > 0$. A left B-module P is said to be *sfp-flat* if $\operatorname{Tor}_1^B(N, P) = 0$ for all strongly finitely presented right B-modules N, or equivalently, $\operatorname{Tor}_n^B(N, P) = 0$ for all strongly finitely presented right B-modules N and all $n > 0$.

What we call "sfp-injective modules" are otherwise known as "FP_∞-injective", and what we call "sfp-flat" modules would be usually called "FP_∞-flat" [9, Section 3]. In the terminology of the papers [8, Section 2] and [7], the former modules are also called "absolutely clean", and the latter ones are called "level".

LEMMA 2.4. (a) *The class of all sfp-injective left A-modules is closed under extensions, the cokernels of injective morphisms, filtered inductive limits, infinite direct sums, and infinite products.*

(b) *The class of all sfp-flat left B-modules is closed under extensions, the kernels of surjective morphisms, filtered inductive limits, infinite direct sums, and infinite products.*

PROOF. This is [8, Propositions 2.7 and 2.10]. For a generalization to FP_n-injective and FP_n-flat modules, $n \geqslant 2$, see [9, Propositions 3.10 and 3.11].   □

EXAMPLES 2.5. (1) The following construction using strongly finitely presented modules provides some examples of pseudo-coderived categories of modules over an associative ring in the sense of Section 1. Let A be an associative ring and S be a set of strongly finitely presented left A-modules. Denote by E $\subset$ A $= A$–mod the full subcategory formed by all the left A-modules E such that $\operatorname{Ext}_A^i(S, E) = 0$ for all $S \in$ S and all $i > 0$. Then the full subcategory E $\subset A$–mod is a coresolving subcategory closed under infinite direct sums (and products). So the induced triangulated functor between the two coderived categories $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \to \mathsf{D}^{\mathsf{co}}(A$–mod$)$ is a triangulated equivalence by the dual version of [44, Proposition A.3.1(b)] (cf. [46, Proposition 2.1]). Thus the derived category $\mathsf{D}(\mathsf{E})$ of the exact category E is a pseudo-coderived category of the abelian category A–mod, that is an intermediate quotient category between the coderived category $\mathsf{D}^{\mathsf{co}}(A$–mod$)$ and the derived category $\mathsf{D}(A$–mod$)$.

(2) In particular, if $S = \varnothing$, then one has $E = A-\text{mod}$. On the other hand, if S is the set of all strongly finitely presented left A-modules, then the full subcategory $E \subset A-\text{mod}$ consists of all the sfp-injective modules. When the ring A is left coherent, all the finitely presented left A-modules are strongly finitely presented, and objects of the class E are called *fp-injective* left A-modules. In this case, the derived category $D(E)$ of the exact category E is equivalent to the homotopy $\text{Hot}(A-\text{mod}_{\text{inj}})$ of the additive category of injective left A-modules [60, Theorem 6.12].

(3) More generally, for any associative ring A, the category $\text{Hot}(A-\text{mod}_{\text{inj}})$ can be called the *coderived category in the sense of Becker* [4] of the category of left A-modules. A complex of left A-modules $X^\bullet$ is called *coacyclic in the sense of Becker* if the complex of abelian groups $\text{Hom}_A(X^\bullet, J^\bullet)$ is acyclic for any complex of injective left A-modules $J^\bullet$. According to [4, Proposition 1.3.6(2)], the full subcategories of complexes of injective modules and coacyclic complexes in the sense of Becker form a semiorthogonal decomposition of the homotopy category of left A-modules $\text{Hot}(A-\text{mod})$. According to [43, Theorem 3.5(a)], any coacyclic complex of left A-modules in the sense of [42, Section 2.1], [45, Appendix A] is also coacyclic in the sense of Becker. Thus $\text{Hot}(A-\text{mod}_{\text{inj}})$ occurs as an intermediate triangulated quotient category between $D^{\text{co}}(A-\text{mod})$ and $D(A-\text{mod})$. So the coderived category in the sense of Becker is a pseudo-coderived category in our sense.

We do *not* know whether the Verdier quotient functor $D^{\text{co}}(A-\text{mod}) \rightarrow \text{Hot}(A-\text{mod}_{\text{inj}})$ is a triangulated equivalence (or, which is the same, the natural fully faithful triangulated functor $\text{Hot}(A-\text{mod}_{\text{inj}}) \rightarrow D^{\text{co}}(A-\text{mod})$ is a triangulated equivalence) for an arbitrary associative ring A. Partial results in this direction are provided by [43, Theorem 3.7] and [46, Theorem 2.4] (see also Proposition 7.1 below).

EXAMPLES 2.6. (1) The following dual version of Example 2.5(1) provides some examples of pseudo-contraderived categories of modules. Let B be an associative ring and S be a set of strongly finitely presented right B-modules. Denote by $F \subset B = B-\text{mod}$ the full subcategory formed by all the left B-modules F such that $\text{Tor}_i^B(S, F) = 0$ for all $S \in S$ and $i > 0$. Then the full subcategory $F \subset B-\text{mod}$ is a resolving subcategory closed under infinite products (and direct sums). So the induced triangulated functor between the two contraderived categories $D^{\text{ctr}}(F) \rightarrow D^{\text{ctr}}(B-\text{mod})$ is a triangulated equivalence by [44, Proposition A.3.1(b)]. Thus the derived category $D(F)$ of the exact category F is a pseudo-contraderived category of the abelian category $B-\text{mod}$, that is an intermediate

quotient category between the contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$ and the derived category $\mathsf{D}(B\text{–mod})$, as it was explained in Section 1.

(2) In particular, if $\mathsf{S} = \varnothing$, then one has $\mathsf{F} = B\text{–mod}$. On the other hand, if S is the set of all strongly finitely presented right B-modules, then the full subcategory $\mathsf{F} \subset B\text{–mod}$ consists of all the sfp-flat modules. When the ring B is right coherent, all the sfp-flat left B-modules are flat and F is the full subcategory of all flat left B-modules. For any associative ring B, the derived category of the exact category of flat left B-modules is equivalent to the homotopy category $\mathsf{Hot}(B\text{–mod}_{\mathsf{proj}})$ of the additive category of projective left B-modules [38, Proposition 8.1 and Theorem 8.6].

(3) For any associative ring B, the category $\mathsf{Hot}(B\text{–mod}_{\mathsf{proj}})$ can be called the *contraderived category in the sense of Becker* of the category of left B-modules. A complex of left B-modules $Y^\bullet$ is called *contraacyclic in the sense of Becker* if the complex of abelian groups $\mathrm{Hom}_B(P^\bullet, Y^\bullet)$ is acyclic for any complex of projective left B-modules $P^\bullet$. According to [4, Proposition 1.3.6(1)], the full subcategories of contraacyclic complexes in the sense of Becker and complexes of projective modules form a semiorthogonal decomposition of the homotopy category of left B-modules $\mathsf{Hot}(B\text{–mod})$. According to [43, Theorem 3.5(b)], any contraacyclic complex of left B-modules in the sense of [42, Section 4.1], [45, Appendix A] is also contraacyclic in the sense of Becker. Thus $\mathsf{Hot}(B\text{–mod}_{\mathsf{proj}})$ occurs as an intermediate triangulated quotient category between $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$ and $\mathsf{D}(B\text{–mod})$. So the contraderived category in the sense of Becker is a pseudo-contraderived category in our sense.

We do *not* know whether the Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \to \mathsf{Hot}(B\text{–mod}_{\mathsf{proj}})$ is a triangulated equivalence (or, which is the same, the natural fully faithful triangulated functor $\mathsf{Hot}(B\text{–mod}_{\mathsf{proj}}) \to \mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$ is a triangulated equivalence) for an arbitrary associative ring B. A partial result in this direction is provided by [43, Theorem 3.8] (cf. [46, Theorem 4.4]; see also Proposition 7.2 below).

3. Auslander and Bass classes

We recall the definition of a pseudo-dualizing complex of bimodules from Section 0.5 of the introduction. Let A and B be associative rings.

A *pseudo-dualizing complex* $L^\bullet$ for the rings A and B is a finite complex of A-B-bimodules satisfying the following two conditions:

(ii) the complex $L^\bullet$ is strongly finitely presented as a complex of left A-modules and as a complex of right B-modules;

(iii) the homothety maps

$$A \longrightarrow \operatorname{Hom}_{\mathsf{D}^b(\mathsf{mod}-B)}(L^\bullet, L^\bullet[*]) \quad \text{and} \quad B^{\mathrm{op}} \longrightarrow \operatorname{Hom}_{\mathsf{D}^b(A-\mathsf{mod})}(L^\bullet, L^\bullet[*])$$

are isomorphisms of graded rings.

Here the condition (ii) refers to the definition of a strongly finitely presented complex of modules in Section 2. The complex $L^\bullet$ is viewed as an object of the bounded derived category of A-B-bimodules $\mathsf{D}^b(A-\mathsf{mod}-B)$.

We will use the following simplified notation: given two complexes of left A-modules $M^\bullet$ and $N^\bullet$, we denote by $\operatorname{Ext}_A^n(M^\bullet, N^\bullet)$ the groups

$$H^n \mathbb{R}\operatorname{Hom}_A(M^\bullet, N^\bullet) = \operatorname{Hom}_{\mathsf{D}(A-\mathsf{mod})}(M^\bullet, N^\bullet[n]).$$

Given a complex of right B-modules $N^\bullet$ and a complex of left B-modules $M^\bullet$, we denote by $\operatorname{Tor}_n^B(N^\bullet, M^\bullet)$ the groups $H^{-n}(N^\bullet \otimes_B^{\mathbb{L}} M^\bullet)$.

The tensor product functor $L^\bullet \otimes_B -\colon \mathsf{Hot}(B-\mathsf{mod}) \to \mathsf{Hot}(A-\mathsf{mod})$ acting between the unbounded homotopy categories of left B-modules and left A-modules is left adjoint to the functor $\operatorname{Hom}_A(L^\bullet, -)\colon \mathsf{Hot}(A-\mathsf{mod}) \to \mathsf{Hot}(B-\mathsf{mod})$. Using homotopy flat and homotopy injective resolutions of the second arguments, one constructs the derived functors $L^\bullet \otimes_B^{\mathbb{L}} -\colon \mathsf{D}(B-\mathsf{mod}) \to \mathsf{D}(A-\mathsf{mod})$ and $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)\colon \mathsf{D}(A-\mathsf{mod}) \to \mathsf{D}(B-\mathsf{mod})$ acting between the (conventional) unbounded derived categories of left A-modules and left B-modules. As always with the left and right derived functors (e. g., in the sense of Deligne [16, 1.2.1–2]), the functor $L^\bullet \otimes_B^{\mathbb{L}} -$ is left adjoint to the functor $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ [42, Lemma 8.3].

Suppose that the finite complex $L^\bullet$ is situated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$. Then one has $\operatorname{Ext}_A^n(L^\bullet, J) = 0$ for all $n > d_1$ and all sfp-injective left A-modules J. Similarly, one has $\operatorname{Tor}_n^B(L^\bullet, P) = 0$ for all $n > d_1$ and all sfp-flat left B-modules P. Choose an integer $l_1 \geqslant d_1$ and consider the following full subcategories in the abelian categories of left A-modules and left B-modules:

- $\mathsf{E}_{l_1} = \mathsf{E}_{l_1}(L^\bullet) \subset A-\mathsf{mod}$ is the full subcategory consisting of all the A-modules E such that $\operatorname{Ext}_A^n(L^\bullet, E) = 0$ for all $n > l_1$ and the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \to E$ is an isomorphism in $\mathsf{D}^-(A-\mathsf{mod})$;
- $\mathsf{F}_{l_1} = \mathsf{F}_{l_1}(L^\bullet) \subset B-\mathsf{mod}$ is the full subcategory consisting of all the B-modules F such that $\operatorname{Tor}_n^B(L^\bullet, F) = 0$ for all $n > l_1$ and the adjunction morphism $F \to \mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} F)$ is an isomorphism in $\mathsf{D}^+(B-\mathsf{mod})$.

Clearly, for any $l_1'' \geqslant l_1' \geqslant d_1$, one has $\mathsf{E}_{l_1'} \subset \mathsf{E}_{l_1''} \subset A-\mathsf{mod}$ and $\mathsf{F}_{l_1'} \subset \mathsf{F}_{l_1''} \subset B-\mathsf{mod}$.

The category F_{l_1} is our version of what is called the *Auslander class* in [11, 24, 21, 12, 26], while the category E_{l_1} is our version of the *Bass class*. The definition of such classes of modules goes back to Foxby [23, Section 1].

The next three Lemmas 3.1–3.3 are our versions of [26, Lemma 4.1, Proposition 4.2, and Theorem 6.2].

LEMMA 3.1. (a) *The full subcategory* $\mathsf{E}_{l_1} \subset A\text{–mod}$ *is closed under the cokernels of injective morphisms, extensions, and direct summands.*

(b) *The full subcategory* $\mathsf{F}_{l_1} \subset B\text{–mod}$ *is closed under the kernels of surjective morphisms, extensions, and direct summands.*   □

LEMMA 3.2. (a) *The full subcategory* $\mathsf{E}_{l_1} \subset A\text{–mod}$ *contains all the injective left A-modules.*

(b) *The full subcategory* $\mathsf{F}_{l_1} \subset B\text{–mod}$ *contains all the flat left B-modules.*

PROOF. Part (a): let $'L^\bullet$ be a bounded above complex of finitely generated projective right B-modules endowed with a quasi-isomorphism of complexes of right B-modules $'L^\bullet \to L^\bullet$. Then the complex $'L^\bullet \otimes_B \operatorname{Hom}_A(L^\bullet, J)$ computes $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, J)$ as an object of the derived category of abelian groups for any injective left A-module J. Now we have an isomorphism of complexes of abelian groups $'L^\bullet \otimes_B \operatorname{Hom}_A(L^\bullet, J) \simeq \operatorname{Hom}_A(\operatorname{Hom}_{B^{op}}('L^\bullet, L^\bullet), J)$ and a quasi-isomorphism of complexes of left A-modules $A \to \operatorname{Hom}_{B^{op}}('L^\bullet, L^\bullet)$, implying that the natural morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, J) \to J$ is an isomorphism in the derived category of abelian groups, hence also in the derived category of left A-modules.

Part (b): let $''L^\bullet$ be a bounded above complex of finitely generated projective left A-modules endowed with a quasi-isomorphism of complexes of left A-modules $''L^\bullet \to L^\bullet$. Then the complex $\operatorname{Hom}_A(''L^\bullet, L^\bullet \otimes_B P)$ represents the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} P)$ for any flat left B-module P. Now we have an isomorphism of complexes of abelian groups $\operatorname{Hom}_A(''L^\bullet, L^\bullet \otimes_B P) \simeq \operatorname{Hom}_A(''L^\bullet, L^\bullet) \otimes_B P$ and a quasi-isomorphism of complexes of right B-modules $B \to \operatorname{Hom}_A(''L^\bullet, L^\bullet)$.   □

If $L^\bullet$ is finite complex of A-B-bimodules that are strongly finitely presented as left A-modules and as right B-modules, then the class E_{l_1} contains also all the sfp-injective left A-modules and the class F_{l_1} contains all the sfp-flat left B-modules.

LEMMA 3.3. (a) *The full subcategory* $\mathsf{E}_{l_1} \subset A\text{–mod}$ *is closed under infinite direct sums and products.*

(b) *The full subcategory* $\mathsf{F}_{l_1} \subset B\text{–mod}$ *is closed under infinite direct sums and products.*

PROOF. The functor $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)\colon \mathsf{D}(A\text{--mod}) \to \mathsf{D}(B\text{--mod})$ preserves infinite direct sums of uniformly bounded below families of complexes and infinite products of arbitrary families of complexes. The functor $L^\bullet \otimes_B^{\mathbb{L}} -\colon \mathsf{D}(B\text{--mod}) \to \mathsf{D}(A\text{--mod})$ preserves infinite products of uniformly bounded above families of complexes and infinite direct sums of arbitrary families of complexes. These observations imply both the assertions (a) and (b). □

For a further result in the direction of the above three lemmas, see Corollary 10.2 below (claiming that the classes of modules E_{l_1} and F_{l_1} are also closed under filtered inductive limits).

The full subcategories $\mathsf{E}_{l_1} \subset A\text{--mod}$ and $\mathsf{F}_{l_1} \subset B\text{--mod}$ inherit exact category structures from the abelian categories $A\text{--mod}$ and $B\text{--mod}$. It follows from Lemmas 3.1 and 3.2 that the induced triangulated functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{l_1}) \to \mathsf{D}^{\mathsf{b}}(A\text{--mod})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{l_1}) \to \mathsf{D}^{\mathsf{b}}(B\text{--mod})$ are fully faithful. The following lemma describes their essential images.

LEMMA 3.4. (a) *Let $M^\bullet$ be a complex of left A-modules concentrated in the cohomological degrees $-n_1 \leqslant m \leqslant n_2$. Then $M^\bullet$ is quasi-isomorphic to a complex of left A-modules concentrated in the cohomological degrees $-n_1 \leqslant m \leqslant n_2$ with the terms belonging to the full subcategory $\mathsf{E}_{l_1} \subset A\text{--mod}$ if and only if $\operatorname{Ext}_A^n(L^\bullet, M^\bullet) = 0$ for $n > n_2 + l_1$ and the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, M^\bullet) \to M^\bullet$ is an isomorphism in $\mathsf{D}^-(A\text{--mod})$.*

(b) *Let $N^\bullet$ be a complex of left B-modules concentrated in the cohomological degrees $-n_1 \leqslant m \leqslant n_2$. Then $N^\bullet$ is quasi-isomorphic to a complex of left B-modules concentrated in the cohomological degrees $-n_1 \leqslant m \leqslant n_2$ with the terms belonging to the full subcategory $\mathsf{F}_{l_1} \subset B\text{--mod}$ if and only if $\operatorname{Tor}_n^B(L^\bullet, N^\bullet) = 0$ for $n > n_1 + l_1$ and the adjunction morphism $N^\bullet \to \mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} N^\bullet)$ is an isomorphism in $\mathsf{D}^+(B\text{--mod})$.*

PROOF. (a) The "only if" part is obvious. To prove the "if", replace $M^\bullet$ by a quasi-isomorphic complex $'M^\bullet$ concentrated in the same cohomological degrees $-n_1 \leqslant m \leqslant n_2$ with $'M^m \in A\text{--mod}_{\mathsf{inj}}$ for $-n_1 \leqslant m < n_2$, and use Lemma 3.2(a) in order to check that $'M^m \in \mathsf{E}_{l_1}$ for all $-n_1 \leqslant m \leqslant n_2$. Part (b): to prove the "if", replace $N^\bullet$ by a quasi-isomorphic complex $'N^\bullet$ concentrated in the same cohomological degrees $-n_1 \leqslant m \leqslant n_2$ with $'N^m \in B\text{--mod}_{\mathsf{proj}}$ for $-n_1 < m \leqslant n_2$, and use Lemma 3.2(b) in order to check that $'N^m \in \mathsf{F}_{l_1}$ for all $-n_1 \leqslant m \leqslant n_2$. □

Thus the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{l_1}) \subset \mathsf{D}(A\text{--mod})$ consists of all the complexes of left A-modules $M^\bullet$ with bounded cohomology such that the complex $\mathbb{R}\operatorname{Hom}_A(L^\bullet, M^\bullet)$ also has bounded cohomology and the adjunction morphism

$L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, M^\bullet) \to M^\bullet$ is an isomorphism. Similarly, the full subcategory $\mathsf{D}^b(\mathsf{F}_{l_1}) \subset \mathsf{D}(B\text{–mod})$ consists of all the complexes of left B-modules $N^\bullet$ with bounded cohomology such that the complex $L^\bullet \otimes_B^{\mathbb{L}} N^\bullet$ also has bounded cohomology and the adjunction morphism $N^\bullet \to \mathbb{R}\operatorname{Hom}_A(L^\bullet, \, L^\bullet \otimes_B^{\mathbb{L}} N^\bullet)$ is an isomorphism.

These full subcategories are usually called the *derived Bass* and *Auslander classes*. As any pair of adjoint functors restricts to an equivalence between the full subcategories of all objects whose adjunction morphisms are isomorphisms [24, Theorem 1.1], the functors $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ and $L^\bullet \otimes_B^{\mathbb{L}} -$ restrict to a triangulated equivalence between the derived Bass and Auslander classes [2, Theorem 3.2], [11, Theorem 4.6]

$$(10) \qquad\qquad \mathsf{D}^b(\mathsf{E}_{l_1}) \simeq \mathsf{D}^b(\mathsf{F}_{l_1}).$$

LEMMA 3.5. (a) *For any A-module $E \in \mathsf{E}_{l_1}$, the object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \in \mathsf{D}^b(B\text{–mod})$ can be represented by a complex of B-modules concentrated in the cohomological degrees $-d_2 \leqslant m \leqslant l_1$ with the terms belonging to F_{l_1}.*

(b) *For any B-module $F \in \mathsf{F}_{l_1}$, the object $L^\bullet \otimes_B^{\mathbb{L}} F \in \mathsf{D}^b(A\text{–mod})$ can be represented by a complex of A-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant d_2$ with the terms belonging to E_{l_1}.*

PROOF. Part (a) follows from Lemma 3.4(b), as the derived category object $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \simeq E$ has no cohomology in the cohomological degrees $-n < -d_2 - l_1 \leqslant -d_2 - d_1 \leqslant 0$. Part (b) follows from Lemma 3.4(a), as the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, \, L^\bullet \otimes_B^{\mathbb{L}} F) \simeq F$ has no cohomology in the cohomological degrees $n > d_2 + l_1 \geqslant d_2 + d_1 \geqslant 0$. $\qquad\square$

We refer to [59, Section 2] and [44, Section A.5] for discussions of the *coresolution dimension* of objects of an exact category A with respect to its coresolving subcategory E and the *resolution dimension* of objects of an exact category B with respect to its resolving subcategory F (called the *right E-homological dimension* and the *left F-homological dimension* in [44]).

LEMMA 3.6. (a) *For any integers $l_1'' \geqslant l_l' \geqslant d_1$, the full subcategory $\mathsf{E}_{l_1''} \subset A\text{–mod}$ consists precisely of all the left A-modules whose $\mathsf{E}_{l_1'}$-coresolution dimension does not exceed $l_1'' - l_1'$.*

(b) *For any integers $l_1'' \geqslant l_1' \geqslant d_1$, the full subcategory $\mathsf{F}_{l_1''} \subset B\text{–mod}$ consists precisely of all the left B-modules whose $\mathsf{F}_{l_1'}$-resolution dimension does not exceed $l_1'' - l_1'$.*

PROOF. Part (a) is obtained by applying Lemma 3.4(a) to a one-term complex $M^\bullet = E$, concentrated in the cohomological degree 0, with $n_1 = 0$, $n_2 = l_1'' - l_1'$, and $l_1 = l_1'$. Part (b) is obtained by applying Lemma 3.4(b) to a one-term complex $N^\bullet = F$, concentrated in the cohomological degree 0, with $n_2 = 0$, $n_1 = l_1'' - l_1'$, and $l_1 = l_1'$. □

REMARK 3.7. In particular, it follows from Lemmas 3.2 and 3.6 that, for any $n \geqslant 0$, all the left A-modules of injective dimension not exceeding n belong to E_{d_1+n} and all the left B-modules of flat dimension not exceeding n belong to F_{d_1+n}.

We refer to [44, Section A.1], [45, Appendix A], or Section 1 for the definitions of exotic derived categories appearing in the next proposition.

PROPOSITION 3.8. *For any $l_1'' \geqslant l_1' \geqslant d_1$ and any conventional or exotic derived category symbol $\star = $ b, $+$, $-$, $\varnothing$, abs$+$, abs$-$, co, ctr, or abs, the exact embedding functors $\mathsf{E}_{l_1'} \to \mathsf{E}_{l_1''}$ and $\mathsf{F}_{l_1'} \to \mathsf{F}_{l_1''}$ induce triangulated equivalences*

$$\mathsf{D}^\star(\mathsf{E}_{l_1'}) \simeq \mathsf{D}^\star(\mathsf{E}_{l_1''}) \quad and \quad \mathsf{D}^\star(\mathsf{F}_{l_1'}) \simeq \mathsf{D}^\star(\mathsf{F}_{l_1''}).$$

PROOF. The assertions follow from Lemma 3.6 in view of [44, Proposition A.5.6]. □

In particular, the unbounded derived category $\mathsf{D}(\mathsf{E}_{l_1})$ is the same for all $l_1 \geqslant d_1$ and the unbounded derived category $\mathsf{D}(\mathsf{F}_{l_1})$ is the same for all $l_1 \geqslant d_1$.

As it was explained in Section 1, it follows from Lemmas 3.1–3.3 by virtue of [44, Proposition A.3.1(b)] that the natural Verdier quotient functor

$$\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \longrightarrow \mathsf{D}(A\text{–mod})$$

factorizes into two Verdier quotient functors

$$\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \longrightarrow \mathsf{D}(\mathsf{E}_{l_1}) \longrightarrow \mathsf{D}(A\text{–mod}),$$

and the natural Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \to \mathsf{D}(B\text{–mod})$ factorizes into two Verdier quotient functors

$$\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \longrightarrow \mathsf{D}(\mathsf{F}_{l_1}) \longrightarrow \mathsf{D}(B\text{–mod}).$$

In other words, the triangulated category $\mathsf{D}(\mathsf{E}_{l_1})$ is a pseudo-coderived category of the abelian category of left A-modules and the triangulated category $\mathsf{D}(\mathsf{F}_{l_1})$ is a pseudo-contraderived category of the abelian category of left B-modules.

These are called the *lower pseudo-coderived category* of left A-modules and the *lower pseudo-contraderived category* of left B-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The notation is

$$\mathsf{D}'_{L^\bullet}(A\text{–mod}) = \mathsf{D}(\mathsf{E}_{l_1}) \quad \text{and} \quad \mathsf{D}''_{L^\bullet}(B\text{–mod}) = \mathsf{D}(\mathsf{F}_{l_1}).$$

The next theorem provides, in particular, a triangulated equivalence between the lower pseudo-coderived and the lower pseudo-contraderived category,

$$\mathsf{D}'_{L^\bullet}(A\text{–mod}) = \mathsf{D}(\mathsf{E}_{l_1}) \simeq \mathsf{D}(\mathsf{F}_{l_1}) = \mathsf{D}''_{L^\bullet}(B\text{–mod}).$$

THEOREM 3.9. *For any symbol* $\star = $ b, $+$, $-$, $\varnothing$, abs$+$, abs$-$, co, ctr, *or* abs, *there is a triangulated equivalence* $\mathsf{D}^\star(\mathsf{E}_{l_1}) \simeq \mathsf{D}^\star(\mathsf{F}_{l_1})$ *provided by* (*appropriately defined*) *mutually inverse functors* $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ *and* $L^\bullet \otimes^{\mathbb{L}}_B -$.

PROOF. This is a particular case of Theorem 4.2 below. The construction of the derived functors $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ and $L^\bullet \otimes^{\mathbb{L}}_B -$ is explained in the proof of Theorem 4.2 and in the appendix.   □

4. Abstract corresponding classes

More generally, suppose that we are given two full subcategories $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ satisfying the following conditions (for some fixed integers l_1 and l_2):

(I) the full subcategory $\mathsf{E} \subset A\text{–mod}$ is closed under extensions and the cokernels of injective morphisms, and contains all the injective left A-modules;

(II) the full subcategory $\mathsf{F} \subset B\text{–mod}$ is closed under extensions and the kernels of surjective morphisms, and contains all the projective left B-modules;

(III) for any A-module $E \in \mathsf{E}$, the object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \in \mathsf{D}^+(B\text{–mod})$ can be represented by a complex of B-modules concentrated in the cohomological degrees $-l_2 \leqslant m \leqslant l_1$ with the terms belonging to F;

(IV) for any B-module $F \in \mathsf{F}$, the object $L^\bullet \otimes^{\mathbb{L}}_B F \in \mathsf{D}^-(A\text{–mod})$ can be represented by a complex of A-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant l_2$ with the terms belonging to E.

One can see from the conditions (I) and (III), or (II) and (IV), that $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$ if $H^{-d_1}(L^\bullet) \neq 0 \neq H^{d_2}(L^\bullet)$. According to Lemmas 3.1, 3.2, and 3.5, the two classes $\mathsf{E} = \mathsf{E}_{l_1}$ and $\mathsf{F} = \mathsf{F}_{l_1}$ satisfy the conditions (I)–(IV) with $l_2 = d_2$.

The following lemma, providing a kind of converse implication, can be obtained as a byproduct of the proof of Theorem 4.2 below (based on the arguments in the appendix). It is somewhat counterintuitive, claiming that the adjunction isomorphism conditions on the modules in the classes E and F, which were necessary in the context of the previous Section 3, follow from the conditions (I)–(IV) in our present context. So we prefer to present a separate explicit proof.

LEMMA 4.1. (a) *For any A-module $E \in$ E, the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R} \operatorname{Hom}_A(L^\bullet, E) \to E$ is an isomorphism in $\mathsf{D}^b(A\text{–mod})$.*

(b) *For any B-module $F \in$ F, the adjunction morphism $F \to \mathbb{R} \operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} F)$ is an isomorphism in $\mathsf{D}^b(B\text{–mod})$.*

PROOF. We will prove part (a); the proof of part (b) is similar. Specifically, let $0 \to E \to K^0 \to K^1 \to \cdots$ be an exact sequence of left A-modules with $E \in$ E and $K^i \in$ E for all $i \geq 0$. Suppose that the adjunction morphisms $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R} \operatorname{Hom}_A(L^\bullet, K^i) \to K^i$ are isomorphisms in $\mathsf{D}^b(A\text{–mod})$ for all $i \geq 0$. We will show that the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R} \operatorname{Hom}_A(L^\bullet, E) \to E$ is also an isomorphism in this case. As injective left A-modules belong to E by the condition (I), the desired assertion will then follow from Lemma 3.2(a).

Let Z^i denote the kernel of the differential $K^i \to K^{i+1}$; in particular, $Z^0 = E$. The key observation is that, according to the condition (I), one has $Z^i \in$ E for all $i \geq 0$. For every $i \geq 0$, choose a coresolution of the short exact sequence $0 \to Z^i \to K^i \to Z^{i+1}$ by short exact sequences of injective left A-modules $0 \to Y^{i,j} \to J^{i,j} \to Y^{i+1,j} \to 0$, $j \geq 0$. Applying the functor $\operatorname{Hom}_A(L^\bullet, -)$ to the complexes of left A-modules $J^{i,\bullet}$ and $Y^{i,\bullet}$, we obtain short exact sequences of complexes of left B-modules $0 \to G^{i,\bullet} \to F^{i,\bullet} \to G^{i+1,\bullet} \to 0$, where $G^{i,\bullet} = \operatorname{Hom}_A(L^\bullet, Y^{i,\bullet})$ and $F^{i,\bullet} = \operatorname{Hom}_A(L^\bullet, J^{i,\bullet})$. According to the condition (III), each complex $G^{i,\bullet}$ and $F^{i,\bullet}$ is quasi-isomorphic to a complex of left B-modules concentrated in the cohomological degrees $-l_2 \leq m \leq l_1$ with the terms belonging to F.

For every $i \geq 0$, choose complexes of projective left B-modules $Q^{i,\bullet}$ and $P^{i,\bullet}$, concentrated in the cohomological degrees $\leq l_1$ and endowed with quasi-isomorphisms of complexes of left B-modules $Q^{i,\bullet} \to G^{i,\bullet}$ and $P^{i,\bullet} \to F^{i,\bullet}$ so that there are short exact sequences of complexes of left B-modules $0 \to Q^{i,\bullet} \to P^{i,\bullet} \to Q^{i+1,\bullet} \to 0$ and the whole diagram is commutative. Applying the functor $L^\bullet \otimes_B -$ to the complexes of left B-modules $P^{i,\bullet}$ and $Q^{i,\bullet}$, we obtain short exact sequences of complexes of left A-modules $0 \to N^{i,\bullet} \to M^{i,\bullet} \to N^{i+1,\bullet} \to 0$, where $N^{i,\bullet} = L^\bullet \otimes_B Q^{i,\bullet}$ and $M^{i,\bullet} = L^\bullet \otimes_B P^{i,\bullet}$.

It follows from the condition (IV) that each complex $M^{i,\bullet}$ and $N^{i,\bullet}$ is quasi-isomorphic to a complex of left A-modules concentrated in the cohomological degrees $-l_1 - l_2 \leqslant m \leqslant l_1 + l_2$. In particular, the cohomology modules of the complexes $M^{i,\bullet}$ and $N^{i,\bullet}$ are concentrated in the degrees $-l_1 - l_2 \leqslant m \leqslant l_1 + l_2$.

Applying the functors of two-sided canonical truncation $\tau_{\geqslant -l_1 - l_2} \tau_{\leqslant l_1 + l_2}$ to the complexes $M^{i,\bullet}$ and $N^{i,\bullet}$, we obtain short exact sequences $0 \to {}'N^{i,\bullet} \to {}'M^{i,\bullet} \to {}'N^{i+1,\bullet} \to 0$ of complexes whose terms are concentrated in the cohomological degrees $-l_1 - l_2 \leqslant m \leqslant l_1 + l_2$. Similarly, applying the functors of canonical truncation $\tau_{\leqslant l_1 + l_2}$ to the complexes $J^{i,\bullet}$ and $Y^{i,\bullet}$, we obtain short exact sequences $0 \to {}'Y^{i,\bullet} \to {}'J^{i,\bullet} \to {}'Y^{i+1,\bullet} \to 0$ of complexes whose terms are concentrated in the cohomological degrees $0 \leqslant m \leqslant l_1 + l_2$. Now we have two morphisms of bicomplexes ${}'M^{i,j} \to {}'J^{i,j}$ and $K^i \to {}'J^{i,j}$, which are both quasi-isomorphisms of finite complexes along the grading j at every fixed degree i, by assumption. Furthermore, we have two morphisms of bicomplexes ${}'N^{0,j} \to {}'M^{i,j}$ and ${}'Y^{0,j} \to {}'J^{i,j}$, which are both quasi-isomorphisms along the grading i at every fixed degree j, by construction. We also have a quasi-isomorphism $E \to {}'Y^{0,\bullet}$.

Passing to the total complexes, we see that the morphism of complexes ${}'N^{0,\bullet} \to {}'Y^{0,\bullet}$ is a quasi-isomorphism, because so are the morphisms ${}'N^{0,\bullet} \to {}'M^{\bullet,\bullet}$, ${}'M^{\bullet,\bullet} \to {}'J^{\bullet,\bullet}$, and ${}'Y^{0,\bullet} \to {}'J^{\bullet,\bullet}$ in a commutative square. This proves that the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \to E$ is an isomorphism in the derived category. $\square$

Assuming that $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$, it is now clear from the conditions (III)–(IV) and Lemma 4.1 that one has $\mathsf{E} \subset \mathsf{E}_{l_1}$ and $\mathsf{F} \subset \mathsf{F}_{l_1}$ for any two classes of objects $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ satisfying (I)–(IV). Furthermore, it follows from the conditions (I)–(II) that the triangulated functors $\mathsf{D}^b(\mathsf{E}) \to \mathsf{D}^b(A\text{–mod})$ and $\mathsf{D}^b(\mathsf{F}) \to \mathsf{D}^b(B\text{–mod})$ induced by the exact embeddings $\mathsf{E} \to A\text{–mod}$ and $\mathsf{F} \to B\text{–mod}$ are fully faithful. Hence so are the triangulated functors $\mathsf{D}^b(\mathsf{E}) \to \mathsf{D}^b(\mathsf{E}_{l_1})$ and $\mathsf{D}^b(\mathsf{F}) \to \mathsf{D}^b(\mathsf{F}_{l_1})$. In view of the conditions (III)–(IV), we can conclude that equivalence (10) restricts to a triangulated equivalence

$$(11) \qquad\qquad \mathsf{D}^b(\mathsf{E}) \simeq \mathsf{D}^b(\mathsf{F}).$$

The following theorem is the main result of this paper.

THEOREM 4.2. *Let $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ be a pair of full subcategories of modules satisfying the conditions* (I)–(IV) *for a pseudo-dualizing complex of A-B-bimodules $L^\bullet$. Then for any symbol $\star = $ b, $+$, $-$, $\varnothing$, abs+, abs−, co, ctr, or abs, there is a triangulated equivalence $\mathsf{D}^\star(\mathsf{E}) \simeq \mathsf{D}^\star(\mathsf{F})$ provided by (appropriately defined) mutually inverse functors $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ and $L^\bullet \otimes_B^{\mathbb{L}} -$.*

Here, in the case $\star = \mathsf{co}$ it is assumed that the full subcategories $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ are closed under infinite direct sums, while in the case $\star = \mathsf{ctr}$ it is assumed that these two full subcategories are closed under infinite products.

PROOF. The words "appropriately defined" here mean "as defined or constructed in the appendix". Specifically, given a complex of left A-modules $E^\bullet$ with the terms $E^i \in \mathsf{E}$, we choose its termwise injective coresolution, which is a bounded below complex of complexes of injective left A-modules $J^{\bullet,\bullet}$. Applying the functor $\mathrm{Hom}_A(L^\bullet, -)$ to every bounded below complex of injective left A-modules $J^{i,\bullet}$ coresolving the left A-module E^i, we obtain a bicomplex of left B-modules $N^{\bullet,\bullet}$, which can be replaced by a quasi-isomorphic (in the sense explained in the appendix, see (20)) bounded complex of complexes of left B-modules $F^{\bullet,\bullet}$ with the terms $F^{i,j} \in \mathsf{F}$. Totalizing the bicomplex $F^{\bullet,\bullet}$, we obtain the desired complex of left B-modules $\mathbb{R}\,\mathrm{Hom}_A(L^\bullet, E^\bullet)$.

Similarly, given a complex of left B-modules $F^\bullet$ with the terms $F^i \in \mathsf{F}$, we choose its termwise projective resolution, which is a bounded above complex of complexes of projective left B-modules $P^{\bullet,\bullet}$. Applying the functor $L^\bullet \otimes_B -$ to every bounded above complex of projective left B-modules $P^{i,\bullet}$ resolving the left B-module F^i, we obtain a bicomplex of left A-modules $M^{\bullet,\bullet}$, which can be replaced by a quasi-isomorphic bounded complex of complexes of left A-modules $E^{\bullet,\bullet}$ with the terms $E^{i,j} \in \mathsf{E}$. Totalizing the bicomplex $E^{\bullet,\bullet}$, we obtain the desired complex of left A-modules $L^\bullet \otimes_B^{\mathbb{L}} F^\bullet$.

Arguing more formally, the theorem is a straightforward application of the results of the appendix. In the context of the latter, we set

$$\mathsf{A} = A\text{–mod} \supset \mathsf{E} \supset \mathsf{J} = A\text{–mod}_{\mathsf{inj}},$$

$$\mathsf{B} = B\text{–mod} \supset \mathsf{F} \supset \mathsf{P} = B\text{–mod}_{\mathsf{proj}}.$$

Consider the adjoint pair of DG-functors

$$\Psi = \mathrm{Hom}_A(L^\bullet, -)\colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B}),$$

$$\Phi = L^\bullet \otimes_B -\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A}).$$

Then the conditions of Sections A.1 and A.3 are satisfied, so the constructions of Sections A.2–A.3 provide the derived functors $\mathbb{R}\Psi$ and $\mathbb{L}\Phi$. The arguments in Section A.4 show that these two derived functors are naturally adjoint to each other, and the first assertion of Theorem A.5 explains how to deduce the claim that they are mutually inverse triangulated equivalences from the triangulated equivalence (11).

Alternatively, applying the second assertion of Theorem A.5 together with Lemma 3.2 allows to reprove the triangulated equivalence (11) rather than use it, thus obtaining another and more "conceptual" proof of Lemma 4.1.   □

Now suppose that we have two pairs of full subcategories $E_{\prime} \subset E' \subset A-\text{mod}$ and $F_{\prime\prime} \subset F'' \subset B-\text{mod}$ such that both the pairs $(E_{\prime}, F_{\prime\prime})$ and (E', F'') satisfy the conditions (I)–(IV), and both the full subcategories $E_{\prime}$ and E' are closed under infinite direct sums in $A-\text{mod}$, while both the full subcategories $F_{\prime\prime}$ and F'' are closed under infinite products in $B-\text{mod}$. Then, in view of the discussion in Section 1 and according to Theorem 4.2 (for $\star = \varnothing$), we have a diagram of triangulated functors

(12)

$$
\begin{array}{ccc}
\mathsf{D}^{\mathrm{co}}(A-\text{mod}) & & \mathsf{D}^{\mathrm{ctr}}(B-\text{mod}) \\
\downarrow & & \downarrow \\
\mathsf{D}(E_{\prime}) & =\!=\!=\!= & \mathsf{D}(F_{\prime\prime}) \\
\downarrow & & \downarrow \\
\mathsf{D}(E') & =\!=\!=\!= & \mathsf{D}(F'') \\
\downarrow & & \downarrow \\
\mathsf{D}(A-\text{mod}) & & \mathsf{D}(B-\text{mod})
\end{array}
$$

The vertical arrows are Verdier quotient functors, so both the triangulated categories $\mathsf{D}(E_{\prime})$ and $\mathsf{D}(E')$ are pseudo-coderived categories of left A-modules, and both the triangulated categories $\mathsf{D}(F_{\prime\prime})$ and $\mathsf{D}(F'')$ are pseudo-contraderived categories of left B-modules. The horizontal double lines are triangulated equivalences. The inner square in the diagram (12) is commutative, as one can see from the construction of the derived functors in Theorem 4.2.

More generally, for any symbol $\star = b, +, -, \varnothing, \text{abs}+, \text{abs}-$, or abs, there is a commutative diagram of triangulated functors and triangulated equivalences

(13)

$$
\begin{array}{ccc}
\mathsf{D}^{\star}(E_{\prime}) & =\!=\!=\!= & \mathsf{D}^{\star}(F_{\prime\prime}) \\
\downarrow & & \downarrow \\
\mathsf{D}^{\star}(E') & =\!=\!=\!= & \mathsf{D}^{\star}(F'')
\end{array}
$$

When all the four full subcategories $E_{\prime}, E' \subset A-\text{mod}$ and $F_{\prime\prime}, F'' \subset B-\text{mod}$ are closed under infinite direct sums (respectively, infinite products), there is also a commutative diagram of triangulated functors and triangulated equivalences (13) with $\star = \text{co}$ (resp., $\star = \text{ctr}$).

5. Minimal corresponding classes

Let A and B be associative rings, and $L^\bullet$ be a pseudo-dualizing complex of A-B-bimodules.

PROPOSITION 5.1. *Fix* $l_1 = d_1$ *and* $l_2 \geqslant d_2$. *Then there exists a unique minimal pair of full subcategories* $\mathsf{E}^{l_2} = \mathsf{E}^{l_2}(L^\bullet) \subset A$–mod *and* $\mathsf{F}^{l_2} = \mathsf{F}^{l_2}(L^\bullet) \subset B$–mod *satisfying the conditions* (I)–(IV) *together with the additional requirements that* E^{l_2} *is closed under infinite direct sums in* A–mod *and* F^{l_2} *is closed under infinite products in* B–mod. *For any pair of full subcategories* $\mathsf{E} \subset A$–mod *and* $\mathsf{F} \subset B$–mod *satisfying the conditions* (I)–(IV) *such that* E *is closed under infinite direct sums in* A–mod *and* F *is closed under infinite products in* B–mod *one has* $\mathsf{E}^{l_2} \subset \mathsf{E}$ *and* $\mathsf{F}^{l_2} \subset \mathsf{F}$.

PROOF. The full subcategories $\mathsf{E}^{l_2} \subset A$–mod and $\mathsf{F}^{l_2} \subset B$–mod are constructed simultaneously by a kind of generation process. By construction, for any full subcategories $\mathsf{E} \subset A$–mod and $\mathsf{F} \subset B$–mod as above we will have $\mathsf{E}^{l_2} \subset \mathsf{E}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}$. In particular, the pair of full subcategories $\mathsf{E} = \mathsf{E}_{d_1}$ and $\mathsf{F} = \mathsf{F}_{d_1}$ satisfies all the mentioned conditions, so we will have $\mathsf{E}^{l_2} \subset \mathsf{E}_{d_1}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}_{d_1}$.

Firstly, all the injective left A-modules belong to E^{l_2} and all the projective left B-modules belong to F^{l_2}, as dictated by the conditions (I)–(II). Secondly, let E be an A-module belonging to E^{l_2}. Then $E \in \mathsf{E}_{d_1}$, so the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \in \mathsf{D}^b(B$–mod) has cohomology modules concentrated in the degrees $-d_2 \leqslant m \leqslant d_1$. Pick a complex of left B-modules $F^\bullet$ representing $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E)$ such that $F^\bullet$ is concentrated in the degrees $-l_2 \leqslant m \leqslant d_1$ and the B-modules F^m are projective for all $-l_2 + 1 \leqslant m \leqslant d_1$. According to [44, Corollary A.5.2], we have $F^{-l_2} \in \mathsf{F}$. So we say that the B-module F^{-l_2} belongs to F^{l_2}. Similarly, let F be a B-module belonging to F^{l_2}. Then $F \in \mathsf{F}_{d_1}$, so the derived category object $L^\bullet \otimes_B^{\mathbb{L}} F$ has cohomology modules concentrated in the degrees $-d_1 \leqslant m \leqslant d_2$. Pick a complex of left A-modules $E^\bullet$ representing $L^\bullet \otimes_B^{\mathbb{L}} F$ such that $E^\bullet$ is concentrated in the degrees $-d_1 \leqslant m \leqslant l_2$ and the A-modules E^m are injective for all $-d_1 \leqslant m \leqslant l_2 - 1$. According to the dual version of [44, Corollary A.5.2], we have $E^{l_2} \in \mathsf{E}$. So we say that the A-module E^{l_2} belongs to E^{l_2}.

Thirdly and finally, we add to E^{l_2} all the extensions, cokernels of injective morphisms, and infinite direct sums of its objects, and similarly add to F^{l_2} all the extensions, kernels of surjective morphisms, and infinite products of its objects. Then the second and third steps are repeated in transfinite iterations, as it may be necessary, until all the modules that can be obtained in this way have been added

and the full subcategories of all such modules $\mathsf{E}^{l_2} \subset A\text{–mod}$ and $\mathsf{F}^{l_2} \subset B\text{–mod}$ have been formed. $\square$

REMARK 5.2. It is clear from the construction in the proof of Proposition 5.1 that for any two values of the parameters $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$, and any two full subcategories $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ satisfying the conditions (I)–(IV) with the parameters l_1 and l_2 such that E is closed under infinite direct sums in $A\text{–mod}$ and F is closed under infinite products in $B\text{–mod}$, one has $\mathsf{E}^{l_2} \subset \mathsf{E}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}$.

Notice that the conditions (III)–(IV) become weaker as the parameter l_2 increases. It follows that one has $\mathsf{E}^{l_2} \supset \mathsf{E}^{l_2+1}$ and $\mathsf{F}^{l_2} \supset \mathsf{F}^{l_2+1}$ for all $l_2 \geqslant d_2$. So the inclusion relations between our classes of modules have the form

$$\cdots \subset \mathsf{E}^{d_2+2} \subset \mathsf{E}^{d_2+1} \subset \mathsf{E}^{d_2} \subset \mathsf{E}_{d_1} \subset \mathsf{E}_{d_1+1} \subset \mathsf{E}_{d_1+2} \subset \cdots \subset A\text{–mod},$$

$$\cdots \subset \mathsf{F}^{d_2+2} \subset \mathsf{F}^{d_2+1} \subset \mathsf{F}^{d_2} \subset \mathsf{F}_{d_1} \subset \mathsf{F}_{d_1+1} \subset \mathsf{F}_{d_1+2} \subset \cdots \subset B\text{–mod}.$$

LEMMA 5.3. *Let $n \geqslant 0$ and $l_1 \geqslant d_1$, $l_2 \geqslant d_2 + n$ be some integers. Let $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ be a pair of full subcategories satisfying the conditions* (I)–(IV) *with the parameters l_1 and l_2. Denote by $\mathsf{E}(n) \subset A\text{–mod}$ the full subcategory of all left A-modules of E-coresolution dimension not exceeding n and by $\mathsf{F}(n) \subset B\text{–mod}$ the full subcategory of all left B-modules of F-resolution dimension not exceeding n. Then the two classes of modules $\mathsf{E}(n)$ and $\mathsf{F}(n)$ satisfy the conditions* (I)–(IV) *with the parameters $l_1 + n$ and $l_2 - n$.*

PROOF. According to [59, Proposition 2.3(2)] or [44, Lemma A.5.4(a-b)] (and the assertions dual to these), the full subcategories $\mathsf{E}(n) \subset A\text{–mod}$ and $\mathsf{F}(n) \subset B\text{–mod}$ satisfy the conditions (I)–(II). Using [44, Corollary A.5.5(b)], one shows that for any A-module $E \in \mathsf{E}(n)$ the derived category object $\mathbb{R}\,\mathrm{Hom}_A(L^\bullet, E) \in \mathsf{D}^b(B\text{–mod})$ can be represented by a complex concentrated in the cohomological degrees $-l_2 \leqslant m \leqslant l_1 + n$ with the terms belonging to F. Moreover, one has $\mathrm{Ext}_A^m(L^\bullet, E) = 0$ for all $m < -d_2$. It follows that $\mathbb{R}\,\mathrm{Hom}_A(L^\bullet, E)$ can be also represented by a complex concentrated in the cohomological degrees $-l_2 + n \leqslant m \leqslant l_1 + n$ with the terms belonging to $\mathsf{F}(n)$. Similarly one can show that for any B-module $F \in \mathsf{F}(n)$ the derived category object $L^\bullet \otimes_B^{\mathbb{L}} F \in \mathsf{D}^b(A\text{–mod})$ can be represented by a complex concentrated in the cohomological degrees $-l_1 - n \leqslant m \leqslant l_2$ with the terms belonging to E. Moreover, one has $\mathrm{Tor}_{-m}^B(L^\bullet, F) = 0$ for all $m > d_2$. It follows that $L^\bullet \otimes_B^{\mathbb{L}} F$ can be also represented by a complex concentrated in the cohomological degrees $-l_1 - n \leqslant m \leqslant l_2 - n$ with the terms belonging to $\mathsf{E}(n)$. This proves the conditions (III)–(IV). $\square$

PROPOSITION 5.4. *For any $l_2'' \geqslant l_2' \geqslant d_2$ and any conventional or exotic derived category symbol $\star$ = b, +, −, ∅, abs+, abs−, or abs, the exact embedding functors $\mathsf{E}^{l_2''} \to \mathsf{E}^{l_2'}$ and $\mathsf{F}^{l_2''} \to \mathsf{F}^{l_2'}$ induce triangulated equivalences*

$$\mathsf{D}^\star(\mathsf{E}^{l_2''}) \simeq \mathsf{D}^\star(\mathsf{E}^{l_2'}) \quad and \quad \mathsf{D}^\star(\mathsf{F}^{l_2''}) \simeq \mathsf{D}^\star(\mathsf{F}^{l_2'}).$$

The same exact embeddings also induce triangulated equivalences

$$\mathsf{D}^{\mathsf{co}}(\mathsf{E}^{l_2''}) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{E}^{l_2'}) \quad and \quad \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}^{l_2''}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}^{l_2'}).$$

PROOF. As in Proposition 3.8, we check that the $\mathsf{E}^{l_2''}$-coresolution dimension of any object of $\mathsf{E}^{l_2'}$ does not exceed $l_2'' - l_2'$ and the $\mathsf{F}^{l_2''}$-resolution dimension of any object of $\mathsf{F}^{l_2'}$ does not exceed $l_2'' - l_2'$. Indeed, according to Lemma 5.3, the pair of full subcategories $\mathsf{E}^{l_2''}(l_2'' - l_2') \subset A\text{–mod}$ and $\mathsf{F}^{l_2''}(l_2'' - l_2') \subset B\text{–mod}$ satisfies the conditions (I)–(IV) with the parameters $l_1 = d_1 + l_2'' - l_2'$ and $l_2 = l_2'$. Furthermore, since infinite direct sums are exact and the full subcategory $\mathsf{E}^{l_2''}$ is closed under infinite direct sums in $A\text{–mod}$, so is the full subcategory $\mathsf{E}^{l_2''}(l_2'' - l_2')$. Since infinite products are exact and the full subcategory $\mathsf{F}^{l_2''}$ is closed under infinite products in $B\text{–mod}$, so is the full subcategory $\mathsf{F}^{l_2''}(l_2'' - l_2')$. It follows that $\mathsf{E}^{l_2'} \subset \mathsf{E}^{l_2''}(l_2'' - l_2')$ and $\mathsf{F}^{l_2'} \subset \mathsf{F}^{l_2''}(l_2'' - l_2')$. □

In particular, the unbounded derived category $\mathsf{D}(\mathsf{E}^{l_2})$ is the same for all $l_2 \geqslant d_2$ and the unbounded derived category $\mathsf{D}(\mathsf{F}^{l_2})$ is the same for all $l_2 \geqslant d_2$.

As it was explained in Section 1, it follows from the condition (I) together with the condition that E^{l_2} is closed under infinite direct sums in $A\text{–mod}$ that the natural Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \to \mathsf{D}(A\text{–mod})$ factorizes into two Verdier quotient functors

$$\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \longrightarrow \mathsf{D}(\mathsf{E}^{l_2}) \longrightarrow \mathsf{D}(A\text{–mod}).$$

Similarly, it follows from the condition (II) together with the condition that F^{l_2} is closed under infinite products in $B\text{–mod}$ that the natural Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \to \mathsf{D}(B\text{–mod})$ factorizes into two Verdier quotient functors

$$\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \longrightarrow \mathsf{D}(\mathsf{F}^{l_2}) \longrightarrow \mathsf{D}(B\text{–mod}).$$

In other words, the triangulated category $\mathsf{D}(\mathsf{E}^{l_2})$ is a pseudo-coderived category of left A-modules and the triangulated category $\mathsf{D}(\mathsf{F}^{l_2})$ is a pseudo-contraderived category of left B-modules.

These are called the *upper pseudo-coderived category* of left A-modules and the *upper pseudo-contraderived category* of left B-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The notation is

$$\mathsf{D}_r^{L^\bullet}(A\text{–mod}) = \mathsf{D}(\mathsf{E}^{l_2}) \quad and \quad \mathsf{D}_{r\prime}^{L^\bullet}(B\text{–mod}) = \mathsf{D}(\mathsf{F}^{l_2}).$$

The next theorem provides, in particular, a triangulated equivalence between the upper pseudo-coderived and the upper pseudo-contraderived category,

$$\mathsf{D}_r^{L^\bullet}(A\text{–mod}) = \mathsf{D}(\mathsf{E}^{l_2}) \simeq \mathsf{D}(\mathsf{F}^{l_2}) = \mathsf{D}_{r\!r}^{L^\bullet}(B\text{–mod}).$$

THEOREM 5.5. *For any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-,$ *or* abs, *there is a triangulated equivalence* $\mathsf{D}^\star(\mathsf{E}^{l_2}) \simeq \mathsf{D}^\star(\mathsf{F}^{l_2})$ *provided by* (*appropriately defined*) *mutually inverse functors* $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ *and* $L^\bullet \otimes_B^{\mathbb{L}} -$.

PROOF. This is another particular case of Theorem 4.2. It is explained in the proof of that theorem what "appropriately defined" means. □

Substituting $\mathsf{E}' = \mathsf{E}_{l_1}$, $\mathsf{E}_{\!/} = \mathsf{E}^{l_2}$, $\mathsf{F}'' = \mathsf{F}_{l_1}$, and $\mathsf{F}_{\!/\!/} = \mathsf{F}^{l_2}$ (for some $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$) into the commutative diagram of triangulated functors (12) from Section 4, one obtains the commutative diagram of triangulated functors (7) promised in Section 0.5 of the Introduction.

6. Dedualizing complexes

Let A and B be associative rings. A *dedualizing complex* of A-B-bimodules $L^\bullet = T^\bullet$ is a pseudo-dualizing complex (according to the definition in Section 3) satisfying the following additional condition:

(i) As a complex of left A-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of projective A-modules, and as a complex of right B-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of projective B-modules.

Taken together, the conditions (i) and (ii) mean that, as a complex of left A-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of finitely generated projective A-modules, and as a complex of right B-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of finitely generated projective B-modules. In other words, $T^\bullet$ is a perfect complex of left A-modules and a perfect complex of right B-modules.

This definition of a dedualizing complex is slightly less general than that of a *tilting complex* in the sense of [54, Theorem 1.1] and slightly more general than that of a *two-sided tilting complex* in the sense of [54, Definition 3.4].

Let $L^\bullet = T^\bullet$ be a dedualizing complex of A-B-bimodules. We refer to the beginning of Section 3 for the discussion of the pair of adjoint derived functors $\mathbb{R}\operatorname{Hom}_A(T^\bullet, -)\colon \mathsf{D}(A\text{–mod}) \to \mathsf{D}(B\text{–mod})$ and $T^\bullet \otimes_B^{\mathbb{L}} -\colon \mathsf{D}(B\text{–mod}) \to \mathsf{D}(A\text{–mod})$. The following assertion is a version of [54, Proposition 5.1] and [18, Theorem 4.2 (2) $\iff$ (4)].

PROPOSITION 6.1. *The derived functors* $\mathbb{R}\operatorname{Hom}_A(T^\bullet, -)$ *and* $T^\bullet \otimes_B^{\mathbb{L}} -$ *are mutually inverse triangulated equivalences between the conventional unbounded derived categories* $\mathsf{D}(A\text{–mod})$ *and* $\mathsf{D}(B\text{–mod})$.

PROOF. We have to show that the adjunction morphisms are isomorphisms. Let $J^\bullet$ be a homotopy injective complex of left A-modules. Then the complex of left B-modules $\operatorname{Hom}_A(T^\bullet, J^\bullet)$ represents the derived category object $\mathbb{R}\operatorname{Hom}_A(T^\bullet, J^\bullet) \in \mathsf{D}(B\text{–mod})$. Let $'T^\bullet$ be a finite complex of finitely generated projective right B-modules endowed with a quasi-isomorphism of complexes of right B-modules $'T^\bullet \to T^\bullet$. Then the adjunction morphism $T^\bullet \otimes_B^{\mathbb{L}}$ $\mathbb{R}\operatorname{Hom}_A(T^\bullet, J^\bullet) \to J^\bullet$ is represented, as a morphism in the derived category of abelian groups, by the morphism of complexes $'T^\bullet \otimes_B \operatorname{Hom}_A(T^\bullet, J^\bullet) \to J^\bullet$. Now the complex of abelian groups $'T^\bullet \otimes_B \operatorname{Hom}_A(T^\bullet, J^\bullet)$ is naturally isomorphic to $\operatorname{Hom}_A(\operatorname{Hom}_{B^{\mathrm{op}}}('T^\bullet, T^\bullet), J^\bullet)$, and the morphism of complexes of left A-modules $A \to \operatorname{Hom}_{B^{\mathrm{op}}}('T^\bullet, T^\bullet)$ is a quasi-isomorphism by the condition (iii).

Similarly, let $P^\bullet$ be a homotopy flat complex of left B-modules. Then the complex of left A-modules $T^\bullet \otimes_B P^\bullet$ represents the derived category object $T^\bullet \otimes_B^{\mathbb{L}} P^\bullet \in \mathsf{D}(A\text{–mod})$. Let $''T^\bullet$ be a finite complex of finitely generated projective left A-modules endowed with a quasi-isomorphism of complexes of left A-modules $''T^\bullet \to T^\bullet$. Then the adjunction morphism $P^\bullet \to \mathbb{R}\operatorname{Hom}_A(T^\bullet, T^\bullet \otimes_B^{\mathbb{L}} P^\bullet)$ is represented, as a morphism in the derived category of abelian groups, by the morphism of complexes $P^\bullet \to \operatorname{Hom}_A(''T^\bullet, T^\bullet \otimes_B P^\bullet)$. Now the complex of abelian groups $\operatorname{Hom}_A(''T^\bullet, T^\bullet \otimes_B P^\bullet)$ is naturally isomorphic to $\operatorname{Hom}_A(''T^\bullet, T^\bullet) \otimes_B P^\bullet$, and the morphism of complexes of right B-modules $B \to \operatorname{Hom}_A(''T^\bullet, T^\bullet)$ is a quasi-isomorphism by the condition (iii). $\square$

In particular, it follows that the derived Bass and Auslander classes associated with a dedualizing complex $L^\bullet = T^\bullet$ (as discussed in Section 3) coincide with the whole bounded derived categories $\mathsf{D}^{\mathrm{b}}(A\text{–mod})$ and $\mathsf{D}^{\mathrm{b}}(B\text{–mod})$, and the triangulated equivalence (10) takes the form $\mathsf{D}^{\mathrm{b}}(A\text{–mod}) \simeq \mathsf{D}^{\mathrm{b}}(B\text{–mod})$.

Now let us choose the parameter l_1 in such a way that $T^\bullet$ is quasi-isomorphic to a complex of (finitely generated) projective left A-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant d_2$ and to a complex of (finitely generated) projective right B-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant d_2$. Then we have $\mathsf{E}_{l_1}(T^\bullet) = A\text{–mod}$ and $\mathsf{F}_{l_1}(T^\bullet) = B\text{–mod}$.

The next corollary is a (partial) extension of [53, Theorem 6.4], [54, Theorem 3.3 and Proposition 5.1], and [18, Proposition 5.1].

COROLLARY 6.2. *For any symbol* $\star$ = b, +, −, ∅, abs+, abs−, co, ctr, *or* abs, *there is a triangulated equivalence* $D^\star(A-\text{mod}) \simeq D^\star(B-\text{mod})$ *provided by* (*appropriately defined*) *mutually inverse functors* $\mathbb{R}\,\text{Hom}_A(T^\bullet, -)$ *and* $T^\bullet \otimes_B^{\mathbb{L}} -$.

PROOF. In view of the above observations, this is a particular case of Theorem 3.9. □

7. Dualizing complexes

Let A and B be associative rings. Our aim is to work out a generalization of the results of [28, Theorem 4.8] and [46, Sections 2 and 4] falling in line with the exposition in the present paper (with the Noetherianness/coherence assumptions removed).

Firstly we return to the discussion of sfp-injective and sfp-flat modules started in Section 2. Denote the full subcategory of sfp-injective left A-modules by $A-\text{mod}_{\text{sfpin}} \subset A-\text{mod}$ and the full subcategory of sfp-flat left B-modules by $B-\text{mod}_{\text{sfpfl}} \subset B-\text{mod}$. It is clear from Lemma 2.4 that the categories $A-\text{mod}_{\text{sfpin}}$ and $B-\text{mod}_{\text{sfpfl}}$ have exact category structures inherited from the abelian categories $A-\text{mod}$ and $B-\text{mod}$.

PROPOSITION 7.1. (a) *The triangulated functor*

$$D^{\text{co}}(A-\text{mod}_{\text{sfpin}}) \longrightarrow D^{\text{co}}(A-\text{mod})$$

induced by the embedding of exact categories $A-\text{mod}_{\text{sfpin}} \rightarrow A-\text{mod}$ *is an equivalence of triangulated categories.*

(b) *If all sfp-injective left A-modules have finite injective dimensions, then the triangulated functor*

$$\text{Hot}(A-\text{mod}_{\text{inj}}) \longrightarrow D^{\text{co}}(A-\text{mod})$$

induced by the embedding of additive/exact categories $A-\text{mod}_{\text{inj}} \rightarrow A-\text{mod}$ *is an equivalence of triangulated categories.*

PROOF. Part (a) is but an application of the assertion dual to [44, Proposition A.3.1(b)] (cf. [46, Theorem 2.2]). Part (b) was proved in [43, Section 3.7] (for a more general argument, one can use the assertion dual to [44, Corollary A.6.2]). In fact, the assumption in part (b) can be weakened by requiring only that fp-injective left A-modules have finite injective dimensions, as infinite direct sums of fp-injective left A-modules are fp-injective over an arbitrary ring (cf. [46, Theorem 2.4]). □

PROPOSITION 7.2. (a) *The triangulated functor*

$$\mathsf{D}^{\mathrm{ctr}}(B{-}\mathsf{mod}_{\mathsf{sfpfl}}) \longrightarrow \mathsf{D}^{\mathrm{ctr}}(B{-}\mathsf{mod})$$

induced by the embedding of exact categories $B{-}\mathsf{mod}_{\mathsf{sfpfl}} \to B{-}\mathsf{mod}$ *is an equivalence of triangulated categories.*

(b) *If all sfp-flat left B-modules have finite projective dimensions, then the triangulated functor*

$$\mathsf{Hot}(B{-}\mathsf{mod}_{\mathsf{proj}}) \longrightarrow \mathsf{D}^{\mathrm{ctr}}(B{-}\mathsf{mod})$$

induced by the embedding of additive/exact categories $B{-}\mathsf{mod}_{\mathsf{proj}} \to B{-}\mathsf{mod}$ *is an equivalence of triangulated categories.*

PROOF. Part (a) is but an application of [44, Proposition A.3.1(b)] (cf. [46, Theorem 4.4]). Part (b) was proved in [43, Section 3.8] (for a more general argument, see [44, Corollary A.6.2]). □

The following lemma is a version of [46, Lemma 4.1] applicable to arbitrary rings.

LEMMA 7.3. (a) *Let* P *be a flat left B-module and* K *be an A-sfp-injective A-B-bimodule. Then the tensor product* $K \otimes_B P$ *is an sfp-injective left A-module.*

(b) *Let* J *be an injective left A-module and* K *be a B-sfp-injective A-B-bimodule. Then the left B-module* $\mathrm{Hom}_A(K, J)$ *is sfp-flat.*

PROOF. This is a particular case of the next Lemma 7.4. □

LEMMA 7.4. (a) *Let* $P^\bullet$ *be a complex of flat left B-modules concentrated in the cohomological degrees* $-n \leqslant m \leqslant 0$ *and* $K^\bullet$ *be a complex of A-B-bimodules which, as a complex of left A-modules, is quasi-isomorphic to a complex of sfp-injective A-modules concentrated in the cohomological degrees* $-d \leqslant m \leqslant l$. *Then the tensor product* $K^\bullet \otimes_B P^\bullet$ *is a complex of left A-modules quasi-isomorphic to a complex of sfp-injective left A-modules concentrated in the cohomological degrees* $-n - d \leqslant m \leqslant l$.

(b) *Let* $J^\bullet$ *be a complex of injective left A-modules concentrated in the cohomological degrees* $0 \leqslant m \leqslant n$ *and* $K^\bullet$ *be a complex of A-B-bimodules which, as a complex of right B-modules, is quasi-isomorphic to a complex of sfp-injective right B-modules concentrated in the cohomological degrees* $-d \leqslant m \leqslant l$. *Then the complex of left B-modules* $\mathrm{Hom}_A(K^\bullet, J^\bullet)$ *is quasi-isomorphic to a complex of sfp-flat B-modules concentrated in the cohomological degrees* $-l \leqslant m \leqslant n + d$.

PROOF. (a) Clearly, the tensor product $K^\bullet \otimes_B P^\bullet$ is quasi-isomorphic to a complex of left A-modules concentrated in the cohomological degrees $-n - d \leqslant m \leqslant l$; the nontrivial aspect is to show that there is such a complex with sfp-injective terms. Equivalently, this means that $\mathrm{Ext}_A^i(M, K^\bullet \otimes_B P^\bullet) = 0$ for all strongly finitely presented left A-modules M and all $i > l$. Indeed, let $R^\bullet$ be a resolution of M by finitely generated projective left A-modules. Without loss of generality, we can assume that $K^\bullet$ is a finite complex of A-B-bimodules. Then the complex $\mathrm{Hom}_A(R^\bullet, K^\bullet \otimes_B P^\bullet)$ is isomorphic to $\mathrm{Hom}_A(R^\bullet, K^\bullet) \otimes_B P^\bullet$ and the cohomology modules of the complex $\mathrm{Hom}_A(R^\bullet, K^\bullet)$ are concentrated in the degrees $-d \leqslant m \leqslant l$.

(b) Clearly, the complex $\mathrm{Hom}_A(K^\bullet, J^\bullet)$ is quasi-isomorphic to a complex of left B-modules concentrated in the cohomological degrees $-l \leqslant m \leqslant n + d$; we have to show that there is such a complex with sfp-flat terms. Equivalently, this means that $\mathrm{Tor}_i^B(N, \mathrm{Hom}_A(K^\bullet, J^\bullet)) = 0$ for all strongly finitely presented right B-modules N and all $i > l$. Indeed, let $Q^\bullet$ be a resolution of N by finitely generated projective right B-modules. Without loss of generality, we can assume that $K^\bullet$ is a finite complex of A-B-bimodules. Then the complex $Q^\bullet \otimes_B \mathrm{Hom}_A(K^\bullet, J^\bullet)$ is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_{B^\mathrm{op}}(Q^\bullet, K^\bullet), J^\bullet)$ and the cohomology modules of the complex $\mathrm{Hom}_{B^\mathrm{op}}(Q^\bullet, K^\bullet)$ are concentrated in the degrees $-d \leqslant m \leqslant l$. $\square$

A *dualizing complex* of A-B-bimodules $L^\bullet = D^\bullet$ is a pseudo-dualizing complex (according to the definition in Section 3) satisfying the following additional condition:

(i) As a complex of left A-modules, $D^\bullet$ is quasi-isomorphic to a finite complex of sfp-injective A-modules, and as a complex of right B-modules, $D^\bullet$ is quasi-isomorphic to a finite complex of sfp-injective B-modules.

This definition of a dualizing complex is a version of the definition of a "cotilting bimodule complex" in [34, Section 2], reproduced as the definition of a "weak dualizing complex" in [46, Section 3] (cf. the definition of a "dualizing complex" in [46, Section 4]), extended from the case of coherent rings to arbitrary rings A and B in the spirit of [26, Definition 2.1] and [44, Section B.4]. (Other versions of the definition of a dualizing complex of bimodules known in the literature can be found in [65, Definition 1.1] and [12, Definition 1.1].) In order to prove the results below, we will have to impose some homological dimension conditions on the rings A and B, bringing our definition of a dualizing complex even closer to the definition in [34] and the definition of a weak dualizing complex in [46].

Specifically, we will have to assume that all sfp-injective left A-modules have finite injective dimensions. This assumption always holds when the ring A is left coherent and there exists an integer $n \geqslant 0$ such that every left ideal in A is generated by at most $\aleph_n$ elements [46, Proposition 2.3].

We will also have to assume that all sfp-flat left B-modules have finite projective dimensions. For a right coherent ring B, this would simply mean that all flat left B-modules have finite projective dimensions. The class of rings satisfying the latter condition was discussed, under the name of "left n-perfect rings," in the paper [21]. We refer to [46, Proposition 4.3], the discussions in [28, Section 3] and [43, Section 3.8], and the references therein, for further sufficient conditions.

Let us choose the parameter l_2 in such a way that $D^\bullet$ is quasi-isomorphic to a complex of sfp-injective left A-modules concentrated in the cohomological degrees $-d_1 \leqslant m \leqslant l_2$ and to a compex of sfp-injective right B-modules concentrated in the cohomological degrees $-d_1 \leqslant m \leqslant l_2$.

PROPOSITION 7.5. *Let A and B be associative rings such that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions. Let $L^\bullet = D^\bullet$ be a dualizing complex of A-B-bimodules, and let the parameter l_2 be chosen as stated above. Then the related minimal corresponding classes $\mathsf{E}^{l_2} = \mathsf{E}^{l_2}(D^\bullet)$ and $\mathsf{F}^{l_2} = \mathsf{F}^{l_2}(D^\bullet)$ are contained in the classes of sfp-injective A-modules and spf-flat B-modules, $\mathsf{E}^{l_2} \subset A\text{--mod}_{\mathsf{sfpin}}$ and $\mathsf{F}^{l_2} \subset B\text{--mod}_{\mathsf{sfpfl}}$.*

Moreover, let $n \geqslant 0$ be an integer such that the injective dimensions of sfp-injective left A-modules do not exceed n and the projective dimensions of sfp-flat left B-modules do not exceed n. Then the classes of modules $\mathsf{E} = A\text{--mod}_{\mathsf{sfpin}}$ and $\mathsf{F} = B\text{--mod}_{\mathsf{sfpfl}}$ satisfy the conditions (I)–(IV) with the parameters $l_1 = n + d_1$ and l_2.

PROOF. The second assertion is true, as the conditions (I)–(II) are satisfied by Lemma 2.4 and the conditions (III)–(IV) hold by Lemma 7.4. The first assertion follows from the second one together with Lemma 2.4. □

Let $B\text{--mod}_{\mathsf{flat}} \subset B\text{--mod}$ denote the full subcategory of flat left B-modules. It inherits the exact category structure of the abelian category $B\text{--mod}$.

Corollary 7.6. *Let A and B be associative rings such that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions. Let $L^\bullet = D^\bullet$ be a dualizing complex of A-B-bimodules, and let the parameter l_2 be chosen as above. Then there is a triangulated equivalence $\mathsf{D}^{\mathrm{co}}(A\text{–mod}) \simeq \mathsf{D}^{\mathrm{ctr}}(B\text{–mod})$ provided by (appropriately defined) mutually inverse functors $\mathbb{R}\operatorname{Hom}_A(D^\bullet, -)$ and $D^\bullet \otimes_B^{\mathbb{L}} -$.*

Furthermore, there is a chain of triangulated equivalences

$$\mathsf{D}^{\mathrm{co}}(A\text{–mod}) \simeq \mathsf{D}^{\mathrm{abs}=\varnothing}(A\text{–mod}_{\mathrm{sfpin}})$$

$$\simeq \mathsf{D}^{\mathrm{abs}=\varnothing}(\mathsf{E}^{l_2})$$

$$\simeq \operatorname{Hot}(A\text{–mod}_{\mathrm{inj}})$$

$$\simeq \operatorname{Hot}(B\text{–mod}_{\mathrm{proj}})$$

$$\simeq \mathsf{D}^{\mathrm{abs}=\varnothing}(\mathsf{F}^{l_2})$$

$$\simeq \mathsf{D}^{\mathrm{abs}=\varnothing}(B\text{–mod}_{\mathrm{flat}})$$

$$\simeq \mathsf{D}^{\mathrm{abs}=\varnothing}(B\text{–mod}_{\mathrm{sfpfl}})$$

$$\simeq \mathsf{D}^{\mathrm{ctr}}(B\text{–mod}),$$

where the notation $\mathsf{D}^{\mathrm{abs}=\varnothing}(\mathsf{C})$ is a shorthand for an identity isomorphism of triangulated categories $\mathsf{D}^{\mathrm{abs}}(\mathsf{C}) = \mathsf{D}(\mathsf{C})$ between the absolute derived category and the conventional derived category of an exact category C. Moreover, for any symbol $\star = \mathsf{b}, +, -, or \varnothing$, there are triangulated equivalences

$$\mathsf{D}^\star(A\text{–mod}_{\mathrm{sfpin}}) \simeq \mathsf{D}^\star(\mathsf{E}^{l_2})$$

$$\simeq \operatorname{Hot}^\star(A\text{–mod}_{\mathrm{inj}})$$

$$\simeq \operatorname{Hot}^\star(B\text{–mod}_{\mathrm{proj}})$$

$$\simeq \mathsf{D}^\star(\mathsf{F}^{l_2})$$

$$\simeq \mathsf{D}^\star(B\text{–mod}_{\mathrm{flat}})$$

$$\simeq \mathsf{D}^\star(B\text{–mod}_{\mathrm{sfpfl}}).$$

Proof. The exact categories A–mod$_{\mathrm{sfpin}}$ and B–mod$_{\mathrm{sfpfl}}$ have finite homological dimensions by assumption. Hence so do their full subcategories E^{l_2}, F^{l_2}, and B–mod$_{\mathrm{flat}}$ satisfying the condition (I) or (II). It follows easily (see, e. g., [42, Remark 2.1] and [44, Proposition A.5.6]) that a complex in any one of these exact categories is acyclic if and only if it is absolutely acyclic, and that their (conventional or absolute) derived categories are equivalent to the homotopy categories

of complexes of injective or projective objects. The same, of course, applies to the coderived and/or contraderived categories of those of these exact categories that happen to be closed under the infinite direct sums or infinite products in their respective abelian module categories. The same also applies to the bounded versions of the conventional or absolute derived categories and bounded versions of the homotopy categories.

Propositions 7.1 and 7.2 provide the equivalences with the coderived category $\mathsf{D}^{\mathsf{co}}(A\text{–mod})$ or the contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$. Thus we have shown in all the cases that the mentioned triangulated categories of complexes of A-modules are equivalent to each other and the mentioned triangulated categories of complexes of B-modules are equivalent to each other. It remains to construct the equivalences connecting complexes of A-modules with complexes of B-modules.

Specifically, the equivalence $\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$ can be obtained in the same way as in [46, Theorem 4.5], using the equivalence $\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \simeq \mathsf{Hot}(A\text{–mod}_{\mathsf{inj}})$ in order to construct the derived functor $\mathbb{R}\operatorname{Hom}_A(D^\bullet, -)$ and the equivalence $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \simeq \mathsf{D}^{\mathsf{abs}}(B\text{–mod}_{\mathsf{flat}})$ or $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \simeq \mathsf{Hot}(B\text{–mod}_{\mathsf{proj}})$ in order to construct the derived functor $D^\bullet \otimes_B^{\mathbb{L}} -$. More generally, the equivalence $\mathsf{D}^\star(\mathsf{E}^{l_2}) \simeq \mathsf{D}^\star(\mathsf{F}^{l_2})$ can be produced as a particular case of Theorem 5.5 above. $\square$

8. Base change

The aim of this section and the next one is to formulate a generalization of the definitions and results of [46, Section 5] that would fit naturally in our present context. Our exposition is informed by that in [11, Section 5].

Let $A \to R$ and $B \to S$ be two homomorphisms of associative rings. Let $\mathsf{E} \subset A\text{–mod}$ be a full subcategory satisfying the condition (I), and let $\mathsf{F} \subset B\text{–mod}$ be a full subcategory satisfying the condition (II). We denote by $\mathsf{G} = \mathsf{G}_{\mathsf{E}} \subset R\text{–mod}$ the full subcategory formed by all the left R-modules whose underlying left A-modules belong to E, and by $\mathsf{H} = \mathsf{H}_{\mathsf{F}} \subset S\text{–mod}$ the full subcategory formed by all the left S-modules whose underlying left B-modules belong to F.

LEMMA 8.1. (a) *The full subcategory* $\mathsf{G}_{\mathsf{E}} \subset R\text{–mod}$ *satisfies the condition* (I) *if and only if the underlying A-modules of all the injective left R-modules belong to* E.

(b) *The full subcategory* $\mathsf{H}_{\mathsf{F}} \subset S\text{–mod}$ *satisfies the condition* (II) *if and only if the underlying B-modules of all the projective left S-modules belong to* F. $\square$

Assume further that the equivalent conditions of Lemma 8.1(a) and (b) hold, and additionally that the full subcategory $E \subset A$–mod is closed under infinite direct sums and the full subcategory $F \subset B$–mod is closed under infinite products. Then we get two commutative diagrams of triangulated functors, where the vertical arrows are Verdier quotient functors described in Section 1, and the horizontal arrows are the forgetful functors:

$$
\begin{array}{ccc}
\mathsf{D}^{\mathrm{co}}(R\text{--mod}) \longrightarrow \mathsf{D}^{\mathrm{co}}(A\text{--mod}) \\
\downarrow \qquad\qquad \downarrow \\
\mathsf{D}(\mathsf{G_E}) \longrightarrow \mathsf{D}(\mathsf{E}) \\
\downarrow \qquad\qquad \downarrow \\
\mathsf{D}(R\text{--mod}) \longrightarrow \mathsf{D}(A\text{--mod})
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{D}^{\mathrm{ctr}}(S\text{--mod}) \longrightarrow \mathsf{D}^{\mathrm{ctr}}(B\text{--mod}) \\
\downarrow \qquad\qquad \downarrow \\
\mathsf{D}(\mathsf{H_F}) \longrightarrow \mathsf{D}(\mathsf{F}) \\
\downarrow \qquad\qquad \downarrow \\
\mathsf{D}(S\text{--mod}) \longrightarrow \mathsf{D}(B\text{--mod})
\end{array}
$$

We recall that a triangulated functor is called *conservative* if it reflects isomorphisms, or equivalently, takes nonzero objects to nonzero objects. For example, the forgetful functors $\mathsf{D}(R\text{--mod}) \to \mathsf{D}(A\text{--mod})$ and $\mathsf{D}(S\text{--mod}) \to \mathsf{D}(B\text{--mod})$ are conservative, while the forgetful functors $\mathsf{D}^{\mathrm{co}}(R\text{--mod}) \to \mathsf{D}^{\mathrm{co}}(A\text{--mod})$ and $\mathsf{D}^{\mathrm{ctr}}(S\text{--mod}) \to \mathsf{D}^{\mathrm{ctr}}(B\text{--mod})$ are *not*, in general.

LEMMA 8.2. *The forgetful functors* $\mathsf{D}(\mathsf{G_E}) \to \mathsf{D}(\mathsf{E})$ *and* $\mathsf{D}(\mathsf{H_F}) \to \mathsf{D}(\mathsf{F})$ *are conservative.*

PROOF. Follows from the definition of the derived category of an exact category.   □

One can say that a complex of left A-modules is E-*pseudo-coacyclic* if its image under the Verdier quotient functor $\mathsf{D}^{\mathrm{co}}(A\text{--mod}) \to \mathsf{D}(\mathsf{E})$ vanishes. All coacyclic complexes are pseudo-coacyclic, and all pseudo-coacyclic complexes are acyclic.

Similarly, one can say that a complex of left B-modules is F-*pseudo-contraacyclic* if its image under the Verdier quotient functor $\mathsf{D}^{\mathrm{ctr}}(B\text{--mod}) \to \mathsf{D}(\mathsf{F})$ vanishes. All contraacyclic complexes are pseudo-contraacyclic, and all pseudo-contraacyclic complexes are acyclic.

LEMMA 8.3. (a) *Let* $\mathsf{E} \subset A$–mod *be a full subcategory satisfying the condition* (I), *closed under infinite direct sums, and containing the underlying A-modules of injective left R-modules. Then a complex of left R-modules is $\mathsf{G_E}$-pseudo-coacyclic if and only if it is E-pseudo-coacyclic as a complex of left A-modules.*

(b) *Let* $\mathsf{F} \subset B\text{--mod}$ *be a full subcategory satisfying the condition* (II), *closed under infinite products, and containing the underlying B-modules of projective left S-modules. Then a complex of left S-modules is* H_{F}-*pseudo-contraacyclic if and only if it is* F-*pseudo-contraacyclic as a complex of left B-modules.*

Proof. This is a restatement of Lemma 8.2.   □

The terminology in the following definition follows that in [46, Section 5], where "relative dualizing complexes" are discussed. In [11, Section 5], a related phenomenon is called "base change".

A *relative pseudo-dualizing complex* for a pair of associative ring homomorphisms $A \to R$ and $B \to S$ is a set of data consisting of a pseudo-dualizing complex of A-B-bimodules $L^\bullet$, a pseudo-dualizing complex of R-S-bimodules $U^\bullet$, and a morphism of complexes of A-B-bimodules $L^\bullet \to U^\bullet$ satisfying the following condition:

(iv) the induced morphism $R \otimes_A^{\mathbb{L}} L^\bullet \to U^\bullet$ is an isomorphism in the derived category of left R-modules $\mathsf{D}^-(R\text{--mod})$, and the induced morphism $L^\bullet \otimes_B^{\mathbb{L}} S \to U^\bullet$ is an isomorphism in the derived category of right S-modules $\mathsf{D}^-(\text{mod--}S)$.

Notice that the condition (ii) in the definition of a pseudo-dualizing complex in Section 3 holds for the complex $U^\bullet$ whenever it holds for the complex $L^\bullet$ and the above condition (iv) is satisfied. The following result, which is our version of [11, Theorem 5.1], explains what happens with the condition (iii). We will assume that the complex $L^\bullet$ is concentrated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$ and the complex $U^\bullet$ is concentrated in the cohomological degrees $-t_1 \leqslant m \leqslant t_2$. Let $L^{\bullet\,\mathrm{op}}$ denote the complex $L^\bullet$ viewed as a complex of B^{op}-A^{op}-bimodules.

Proposition 8.4. *Let* $L^\bullet$ *be a pseudo-dualizing complex of A-B-bimodules,* $U^\bullet$ *be a finite complex of R-S-bimodules, and* $L^\bullet \to U^\bullet$ *be a morphism of complexes of A-B-bimodules satisfying the condition* (iv). *Then* $U^\bullet$ *is a pseudo-dualizing complex of R-S-bimodules if and only if there exists an integer* $l_1 \geqslant d_1$ *such that the right A-module R belongs to the class* $\mathsf{F}_{l_1}(L^{\bullet\,\mathrm{op}}) \subset A^{\mathrm{op}}\text{--mod}$ *and the left B-module S belongs to the class* $\mathsf{F}_{l_1}(L^\bullet) \subset B\text{--mod}$.

Proof. The key observation is that the natural isomorphism

$$\mathbb{R}\operatorname{Hom}_R(U^\bullet, U^\bullet) \simeq \mathbb{R}\operatorname{Hom}_R(R \otimes_A^{\mathbb{L}} L^\bullet, \, U^\bullet)$$

$$\simeq \mathbb{R}\operatorname{Hom}_A(L^\bullet, U^\bullet)$$

$$\simeq \mathbb{R}\operatorname{Hom}_A(L^\bullet, \, L^\bullet \otimes_B^{\mathbb{L}} S)$$

identifies the homothety morphism $S^{\mathrm{op}} \to \mathbb{R}\operatorname{Hom}_R(U^\bullet, U^\bullet)$ with the adjunction morphism $S \to \mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} S)$. Similarly, the natural isomorphism $\mathbb{R}\operatorname{Hom}_{S^{\mathrm{op}}}(U^\bullet, U^\bullet) \simeq \mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(L^\bullet, R \otimes_A^{\mathbb{L}} L^\bullet)$ identifies the homothety morphism $R \to \mathbb{R}\operatorname{Hom}_{S^{\mathrm{op}}}(U^\bullet, U^\bullet)$ with the adjunction morphism $R \to \mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(L^\bullet, R \otimes_A^{\mathbb{L}} L^\bullet)$. It remains to say that one can take any integer l_1 such that $l_1 \geqslant d_1$ and $l_1 \geqslant t_1$. □

The next proposition is our version of [11, Proposition 5.3].

PROPOSITION 8.5. *Let $L^\bullet \to U^\bullet$ be a relative pseudo-dualizing complex for a pair of ring homomorphisms $A \to R$ and $B \to S$. Let l_1 be an integer such that $l_1 \geqslant d_1$ and $l_1 \geqslant t_1$. Then*

(a) *a left R-module belongs to the full subcategory $\mathsf{E}_{l_1}(U^\bullet) \subset R\text{–mod}$ if and only if its underlying A-module belongs to the full subcategory $\mathsf{E}_{l_1}(L^\bullet) \subset A\text{–mod}$;*

(b) *a left S-module belongs to the full subcategory $\mathsf{F}_{l_1}(U^\bullet) \subset S\text{–mod}$ if and only if its underlying B-module belongs to the full subcategory $\mathsf{F}_{l_1}(L^\bullet) \subset B\text{–mod}$.*

PROOF. The assertions follow from the commutative diagrams of the pairs of adjoint functors and the forgetful functors

$$
\begin{array}{ccc}
\mathsf{D}(R\text{–mod}) & \xrightarrow{\ \mathbb{R}\operatorname{Hom}_R(U^\bullet,-)\ } & \mathsf{D}(S\text{–mod}) \\
\downarrow & & \downarrow \\
\mathsf{D}(A\text{–mod}) & \xrightarrow{\ \mathbb{R}\operatorname{Hom}_A(L^\bullet,-)\ } & \mathsf{D}(B\text{–mod})
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{D}(R\text{–mod}) & \xleftarrow{\ U^\bullet \otimes_S^{\mathbb{L}} -\ } & \mathsf{D}(S\text{–mod}) \\
\downarrow & & \downarrow \\
\mathsf{D}(A\text{–mod}) & \xleftarrow{\ L^\bullet \otimes_B^{\mathbb{L}} -\ } & \mathsf{D}(B\text{–mod})
\end{array}
$$

together with the compatibility of the adjunctions with the forgetful functors and conservativity of the forgetful functors. □

PROPOSITION 8.6. *Let $L^\bullet \to U^\bullet$ be a relative pseudo-dualizing complex for a pair of ring homomorphisms $A \to R$ and $B \to S$, and let $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ be a pair of full subcategories satisfying the conditions (I)–(IV) with respect to the pseudo-dualizing complex $L^\bullet$ with some parameters l_1 and l_2 such that $l_1 \geqslant d_1$, $l_1 \geqslant t_1$, $l_2 \geqslant d_2$, and $l_2 \geqslant t_2$. Suppose that the underlying A-modules of all the injective left R-modules belong to E and the underlying B-modules of all the projective left S-modules belong to F. Then the pair of full subcategories $\mathsf{G}_\mathsf{E} \subset R\text{–mod}$ and $\mathsf{H}_\mathsf{F} \subset S\text{–mod}$ satisfies the conditions (I)–(IV) with respect to the pseudo-dualizing complex $U^\bullet$ with the same parameters l_1 and l_2.*

PROOF. The conditions (I)–(II) hold by Lemma 8.1, and (III)–(IV) are easy to check using the standard properties of the (co)resolution dimensions [44, Corollary A.5.2]. □

COROLLARY 8.7. *In the context and assumptions of Proposition 8.6, for any symbol $\star$ = b, +, −, ∅, abs+, abs−, co, ctr, or abs, there is a triangulated equivalence* $\mathsf{D}^\star(\mathsf{G_E}) \simeq \mathsf{D}^\star(\mathsf{H_F})$ *provided by* (*appropriately defined*) *mutually inverse functors* $\mathbb{R}\operatorname{Hom}_R(U^\bullet, -)$ *and* $U^\bullet \otimes_S^{\mathbb{L}} -$.

Here, in the case $\star$ = co it is assumed that the full subcategories $\mathsf{E} \subset A\text{–mod}$ *and* $\mathsf{F} \subset B\text{–mod}$ *are closed under infinite direct sums, while in the case $\star$ = ctr it is assumed that these two full subcategories are closed under infinite products.*

PROOF. This is a particular case of Theorem 4.2. □

In the situation of Corollary 8.7 the triangulated equivalences $\mathsf{D}^\star(\mathsf{G_E}) \simeq \mathsf{D}^\star(\mathsf{H_F})$ and $\mathsf{D}^\star(\mathsf{E}) \simeq \mathsf{D}^\star(\mathsf{F})$ form a commutative diagram with the triangulated forgetful functors

(14)

$$
\begin{array}{ccc}
\mathsf{D}^\star(\mathsf{G_E}) & =\!=\!=\!= & \mathsf{D}^\star(\mathsf{H_F}) \\
\Big\downarrow & & \Big\downarrow \\
\mathsf{D}^\star(\mathsf{E}) & =\!=\!=\!= & \mathsf{D}^\star(\mathsf{F})
\end{array}
$$

9. Relative dualizing complexes

Let A be an associative ring. The *sfp-injective dimension* of an A-module is the minimal length of its coresolution by sfp-injective A-modules. The sfp-injective dimension of a left A-module E is equal to the supremum of all the integers $n \geqslant 0$ for which there exists a strongly finitely presented left A-module M such that $\operatorname{Ext}_A^n(M, E) \neq 0$. The *sfp-flat dimension* of an A-module is the minimal length of its resolution by sfp-flat A-modules. The sfp-flat dimension of a left A-module F is equal to the supremum of all the integers $n \geqslant 0$ for which there exists a strongly finitely presented right A-module N such that $\operatorname{Tor}_n^A(N, F) \neq 0$.

LEMMA 9.1. *The sfp-flat dimension of a right A-module G is equal to the sfp-injective dimension of the left A-module* $\operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. □

Let $A \to R$ and $B \to S$ be homomorphisms of associative rings.

Lemma 9.2. (a) *The supremum of sfp-injective dimensions of the underlying left A-modules of injective left R-modules is equal to the sfp-flat dimension of the right A-module R.*

(b) *The supremum of sfp-flat dimensions of the underlying left B-modules of projective left S-modules is equal to the sfp-flat dimension of the left B-module S.*

Proof. In (a), one notices that the injective left R-modules are precisely the direct summands of infinite products of copies of the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, and takes into account Lemma 9.1. Part (b) is easy (cf. Lemma 2.4).   □

Assume that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions, as in Section 7. Fix an integer $n \geqslant 0$, and set $\mathsf{E} = A\text{–mod}_{\mathsf{sfpin}}(n) \subset A\text{–mod}$ to be the full subcategory of all left A-modules whose sfp-injective dimension does not exceed n. Similarly, set $\mathsf{F} = B\text{–mod}_{\mathsf{sfpfl}}(n) \subset B\text{–mod}$ to be the full subcategory of all left B-modules whose sfp-flat dimension does not exceed n.

Proposition 9.3. (a) *The embedding of exact/abelian categories* $\mathsf{E} \to A\text{–mod}$ *induces an equivalence of triangulated categories* $\mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{E}) \simeq \mathsf{D}^{\mathsf{co}}(A\text{–mod})$.

(b) *The embedding of exact/abelian categories* $\mathsf{F} \to B\text{–mod}$ *induces an equivalence of triangulated categories* $\mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{F}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$.

Proof. Follows from [42, Remark 2.1], Propositions 7.1–7.2, and [44, Proposition A.5.6] (cf. the proof of Corollary 7.6).   □

In other words, in the terminology of Section 8, one can say that the class of E-pseudo-coacyclic complexes coincides with that of coacyclic complexes of left A-modules, while the class of F-pseudo-contraacyclic complexes coincides with that of contraacyclic complex of left B-modules.

The following definitions were given in the beginning of [46, Section 5]. The *R/A-semicoderived category* of left R-modules $\mathsf{D}_A^{\mathsf{sico}}(R\text{–mod})$ is defined as the quotient category of the homotopy category of complexes of left R-modules $\mathsf{Hot}(R\text{–mod})$ by its thick subcategory of complexes of R-modules *that are coacyclic as complexes of A-modules*. Similarly, the *S/B-semicontraderived category* of left S-modules $\mathsf{D}_B^{\mathsf{sictr}}(S\text{–mod})$ is defined as the quotient category of the homotopy category of complexes of left S-modules $\mathsf{Hot}(S\text{–mod})$ by its thick subcategory of complexes of S-modules *that are contraacyclic as complexes of B-modules*.

As in Section 8, we denote by $\mathsf{G}_\mathsf{E} \subset R\text{–mod}$ the full subcategory of all left R-modules whose underlying A-modules belong to E, and by $\mathsf{H}_\mathsf{F} \subset S\text{–mod}$ the

full subcategory of all left S-modules whose underlying B-modules belong to F. The next proposition is our version of [46, Theorems 5.1 and 5.2].

PROPOSITION 9.4. (a) *Assume that all sfp-injective left A-modules have finite injective dimensions and the sfp-flat dimension of the right A-module R does not exceed n. Then the embedding of exact/abelian categories $\mathsf{G_E} \to R\text{–}\mathrm{mod}$ induces an equivalence of triangulated categories $\mathsf{D}(\mathsf{G_E}) \simeq \mathsf{D}_A^{\mathsf{sico}}(R\text{–}\mathrm{mod})$.*

(b) *Assume that all sfp-flat left B-modules have finite projective dimensions and the sfp-flat dimension of the left B-module S does not exceed n. Then the embedding of exact/abelian categories $\mathsf{H_F} \to S\text{–}\mathrm{mod}$ induces an equivalence of triangulated categories $\mathsf{D}(\mathsf{H_F}) \simeq \mathsf{D}_B^{\mathsf{sictr}}(S\text{–}\mathrm{mod})$.*

PROOF. The assumptions of Lemma 8.3(a) or (b) hold by Lemma 9.2, so its conclusion is applicable, and it remains to recall Proposition 9.3.   □

So, in the assumptions of Proposition 9.4, the R/A-semicoderived category of left R-modules is a pseudo-coderived category of left R-modules and the S/B-semicontraderived category of left S-modules is a pseudo-contraderived category of left S-modules, in the sense of Section 1.

A *relative dualizing complex* for a pair of associative ring homomorphisms $A \to R$ and $B \to S$ is a relative pseudo-dualizing complex $L^\bullet \to U^\bullet$ in the sense of the definition in Section 8 such that $L^\bullet = D^\bullet$ is a dualizing complex of A-B-bimodules in the sense of the definition in Section 7. In other words, the condition (i) of Section 7 and the conditions (ii)–(iii) of Section 3 have to be satisfied for $D^\bullet$, the condition (iii) of Section 3 has to be satisfied for $U^\bullet$, and the condition (iv) of Section 8 has to be satisfied for the morphism $D^\bullet \to U^\bullet$.

Notice that, in the assumption of finiteness of flat dimensions of the right A-module R and the left B-module S, the condition (iii) for the complex $U^\bullet$ follows from the similar condition for the complex $L^\bullet$ together with the condition (iv), by Proposition 8.4 and Remark 3.7.

The following corollary is our generalization of [46, Theorem 5.6].

COROLLARY 9.5. *Let A and B be associative rings such that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions. Let $A \to R$ and $B \to S$ be associative ring homomorphisms such that the ring R is a right A-module of finite flat dimension and the ring S is a left B-module of finite flat dimension. Let $D^\bullet \to U^\bullet$ be a relative dualizing complex for $A \to R$ and $B \to S$. Then there is a triangulated equivalence $\mathsf{D}_A^{\mathsf{sico}}(R\text{–}\mathrm{mod}) \simeq \mathsf{D}_B^{\mathsf{sictr}}(S\text{–}\mathrm{mod})$ provided by mutually inverse functors $\mathbb{R}\operatorname{Hom}_R(U^\bullet, -)$ and $U^\bullet \otimes_S^{\mathbb{L}} -$.*

PROOF. Combine Corollary 8.7 (for $\star = \varnothing$) with Proposition 9.4.   □

The triangulated equivalences provided by Corollaries 7.6 and 9.5 form a commutative diagram with the (conservative) triangulated forgetful functors

(15)
$$
\begin{array}{ccc}
\mathsf{D}_A^{\mathsf{sico}}(R\text{--mod}) & =\!=\!=\!=\!=\!= & \mathsf{D}_B^{\mathsf{sictr}}(S\text{--mod}) \\
\Big\downarrow & & \Big\downarrow \\
\mathsf{D}^{\mathsf{co}}(A\text{--mod}) & =\!=\!=\!=\!=\!= & \mathsf{D}^{\mathsf{ctr}}(B\text{--mod})
\end{array}
$$

This is the particular case of the commutative diagram (14) that occurs in the situation of Corollary 9.5.

10. Deconstructibility of the Auslander and Bass classes

Let $L^\bullet$ be a pseudo-dualizing complex for associative rings A and B. Assume that the finite complex $L^\bullet$ is situated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$, and choose an integer $l_1 \geqslant d_1$. As in Section 3, we consider the Auslander class of left B-modules $\mathsf{F}_{l_1} = \mathsf{F}_{l_1}(L^\bullet) \subset B\text{--mod}$ and the Bass class of left A-modules $\mathsf{E}_{l_1} = \mathsf{E}_{l_1}(L^\bullet) \subset A\text{--mod}$. In this section we will show, generalizing the result of [22, Proposition 3.10], that F_{l_1} and E_{l_1} are *deconstructible classes* of modules.

We recall the notation $\tau_{\leqslant n}$ and $\tau_{\geqslant n}$ for the functors of canonical truncation of complexes of modules.

LEMMA 10.1. (a) *For every $n \in \mathbb{Z}$, the functor assigning to a left B-module N the left A-module $\operatorname{Tor}_n^B(L^\bullet, N)$ preserves filtered inductive limits.*

(b) *For every $n \in \mathbb{Z}$ and $i \in \mathbb{Z}$, the functor assigning to a left B-module N the left B-module $\operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -n}(L^\bullet \otimes_B^{\mathbb{L}} N))$ preserves filtered inductive limits.*

(c) *For every $n \in \mathbb{Z}$, the functor assigning to a left A-module M the left B-module $\operatorname{Ext}_A^n(L^\bullet, M)$ preserves filtered inductive limits.*

(d) *For every $i \in \mathbb{Z}$, the functor assigning to a left A-module M the left A-module $\operatorname{Tor}_i^B(L^\bullet, \mathbb{R}\operatorname{Hom}_A(L^\bullet, M))$ preserves filtered inductive limits.*

PROOF. First of all we notice that in all the four cases it suffices to check that the functor in question preserves filtered inductive limits when viewed as a functor taking values in the category of abelian groups. Part (a) holds for any complex of A-B-bimodules $L^\bullet$ (it suffices to replace $L^\bullet$, viewed as a complex of right B-modules, by its homotopy flat resolution). Part (c) holds because the

complex $L^\bullet$, viewed as a complex of left A-modules, is quasi-isomorphic to a bounded above complex of finitely generated projective left A-modules.

To prove part (b), it is convenient to use the existence of a (nonadditive) functor assigning to a module its free resolution. Given a filtered diagram of left B-modules (N_α), this allows to construct a filtered diagram of nonpositively cohomologically graded complexes of free left B-modules $(Q_\alpha^\bullet)$ endowed with a quasi-isomorphism of diagrams of complexes of left B-modules $Q_\alpha^\bullet \to N_\alpha$. Then the inductive limit of such free resolutions $\varinjlim_\alpha Q_\alpha^\bullet$ is a flat resolution of the left B-module $\varinjlim_\alpha N_\alpha$.

We have a filtered diagram of complexes of left A-modules $\tau_{\geqslant -n}(L^\bullet \otimes_B Q_\alpha^\bullet)$ representing the derived category objects $\tau_{\geqslant -n}(L^\bullet \otimes_B^{\mathbb{L}} N_\alpha)$, and the complex of left A-modules $\tau_{\geqslant -n}(L^\bullet \otimes_B \varinjlim_\alpha Q_\alpha^\bullet)$ representing the derived category object $\tau_{\geqslant -n}(L^\bullet \otimes_B^{\mathbb{L}} \varinjlim_\alpha N_\alpha)$ is the inductive limit of this diagram. Denote by $'L^\bullet$ a bounded above complex of finitely generated projective left A-modules endowed with a quasi-isomorphism of complexes of left A-modules $'L^\bullet \to L^\bullet$. It remains to observe the functor $\mathrm{Hom}_A('L^\bullet, -)$ preserves inductive limits of diagrams of uniformly bounded below complexes of left A-modules.

To prove part (d), we use the existence of an (also nonadditive) functor assigning to a module its injective resolution. Given a filtered diagram of left A-modules (M_α), this allows to produce a filtered diagram of nonnegatively cohomologically graded complexes of injective left A-modules $(J_\alpha^\bullet)$ endowed with a quasi-isomorphism of diagrams of complexes of left A-modules $M_\alpha \to J_\alpha^\bullet$. By Lemma 2.4(a), the inductive limit $\varinjlim_\alpha J_\alpha^\bullet$ is a complex of sfp-injective left A-modules (but we will not need to use this fact). Let $H^\bullet$ be a nonnegatively cohomologically graded complex of injective left A-modules endowed with a quasi-isomorphism of complexes of left A-modules $\varinjlim_\alpha J_\alpha^\bullet \to H^\bullet$. Then $H^\bullet$ is an injective resolution of the left A-module $\varinjlim_\alpha M_\alpha$.

Let us show that the induced morphism of complexes of left B-modules $\varinjlim_\alpha \mathrm{Hom}_A(L^\bullet, J_\alpha^\bullet) \to \mathrm{Hom}_A(L^\bullet, H^\bullet)$ is a quasi-isomorphism. It is enough to check that this is a quasi-isomorphism of complexes of abelian groups. As above, let $'L^\bullet$ be a bounded above complex of finitely generated projective left A-modules endowed with a quasi-isomorphism of complexes of left A-modules $'L^\bullet \to L^\bullet$. Then the morphisms of complexes of abelian groups $\mathrm{Hom}_A(L^\bullet, J_\alpha^\bullet) \to \mathrm{Hom}_A('L^\bullet, J_\alpha^\bullet)$ and $\mathrm{Hom}_A(L^\bullet, H^\bullet) \to \mathrm{Hom}_A('L^\bullet, H^\bullet)$ are quasi-isomorphisms, since $J_\alpha^\bullet$ and $H^\bullet$ are bounded below complexes of injective left A-modules. Hence it suffices to check that the morphism of complexes of abelian groups $\varinjlim_\alpha \mathrm{Hom}_A('L^\bullet, J_\alpha^\bullet) \to \mathrm{Hom}_A('L^\bullet, H^\bullet)$ is a quasi-isomorphism.

The natural map $\varinjlim_\alpha \operatorname{Hom}_A(\prime L^\bullet, J_\alpha^\bullet) \to \operatorname{Hom}_A(\prime L^\bullet, \varinjlim_\alpha J_\alpha^\bullet)$ is an isomorphism of complexes of abelian groups, since $\prime L^\bullet$ is a bounded above complex of finitely presented left A-modules and $J_\alpha^\bullet$ are uniformly bounded below complexes of left A-modules. So it remains to check that applying the functor $\operatorname{Hom}_A(\prime L^\bullet, -)$ to the morphism of complexes of left A-modules $\varinjlim_\alpha J_\alpha^\bullet \to H^\bullet$ produces a quasi-isomorphism of complexes of abelian groups. Let $K^\bullet$ denote the cone of the latter morphism of complexes of left A-modules. Then $K^\bullet$ is a bounded below acyclic complex of left A-modules. Since $\prime L^\bullet$ is a bounded above complex of projective left A-modules, the complex of abelian groups $\operatorname{Hom}_A(\prime L^\bullet, K^\bullet)$ is acyclic.

Therefore the morphism of complexes of left B-modules $\varinjlim_\alpha \operatorname{Hom}_A(L^\bullet, J_\alpha^\bullet)$ $\to \operatorname{Hom}_A(L^\bullet, H^\bullet)$ is a quasi-isomorphism. Now the complex $\operatorname{Hom}_A(L^\bullet, J_\alpha^\bullet)$ represents the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, M_\alpha)$, while the complex $\operatorname{Hom}_A(L^\bullet, H^\bullet)$ represents the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, \varinjlim_\alpha M_\alpha) \in$ $\mathsf{D}^+(B\text{–mod})$. Denote by $\prime\prime L^\bullet$ a bounded above compelex of flat right B-modules endowed with a quasi-isomorphism of complexes of right B-modules $\prime\prime L^\bullet \to L^\bullet$. It remains to observe that the functor $\prime\prime L^\bullet \otimes_B -$ takes inductive limits of complexes of left B-modules to inductive limits of complexes of abelian groups, and quasi-isomorphisms of complexes of left B-modules to quasi-isomorphisms of complexes of abelian groups.   □

COROLLARY 10.2. (a) *The full subcategory* $\mathsf{F}_{l_1} \subset B\text{–mod}$ *is closed under filtered inductive limits.*

(b) *The full subcategory* $\mathsf{E}_{l_1} \subset A\text{–mod}$ *is closed under filtered inductive limits.*

PROOF. (a) let (F_α) be a filtered diagram of left B-modules $F_\alpha \in \mathsf{F}_{l_1}$. Set $F = \varinjlim_\alpha F_\alpha$. Then, by the definition, $\operatorname{Tor}_n^B(L^\bullet, F_\alpha) = 0$ for $n > l_1$, and by Lemma 10.1(a) it follows that $\operatorname{Tor}_n^B(L^\bullet, F) = 0$ for $n > l_1$. Hence $L^\bullet \otimes_B^{\mathbb{L}} F_\alpha = \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} F_\alpha)$ and $L^\bullet \otimes_B^{\mathbb{L}} F = \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} F)$. By the definition, the adjunction morphisms $F_\alpha \to \mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} F_\alpha)$ are isomorphisms in $\mathsf{D}(B\text{–mod})$, and by Lemma 10.1(b) we can conclude that the adjunction morphism $F \to \mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} F)$ is an isomorphism in $\mathsf{D}(B\text{–mod})$, too.

The proof of part (b) is similar, based on Lemma 10.1(c-d).   □

Let R be an associative ring and $\mathsf{G} \subset R\text{–mod}$ be a class of left R-modules. Following the papers [22, 56], we say that G is a *Kaplansky class* if there exists a cardinal κ such that for every left R-module $G \in \mathsf{G}$ and every element $x \in G$ there is an R-submodule $F \subset G$ with less than κ generators for which $F, G/F \in \mathsf{G}$ and $x \in F$.

Lemma 10.3. *Let λ be an infinite cardinal not smaller than the cardinalities of the rings A and B. Then*

(a) *for every complex $N^\bullet$ of left B-modules with the cohomology modules $H^i(N^\bullet)$, $i \in \mathbb{Z}$, of the cardinalities not exceeding λ, the cardinalities of the left A-modules $\mathrm{Tor}_j^B(L^\bullet, N^\bullet)$, $j \in \mathbb{Z}$, do not exceed λ;*

(b) *for every **bounded below** complex $M^\bullet$ of left A-modules with the cohomology modules $H^i(M^\bullet)$, $i \in \mathbb{Z}$, of the cardinalities not exceeding λ, the cardinalities of the left B-modules $\mathrm{Ext}_A^j(L^\bullet, N^\bullet)$, $j \in \mathbb{Z}$, do not exceed λ.*

Proof. (a) Representing the complex $N^\bullet$ as the inductive limit of its subcomplexes of canonical truncation $\tau_{\leqslant n}(N^\bullet)$, one can assume $N^\bullet$ to be bounded above (since the Tor commutes with filtered inductive limits and the class of all modules of the cardinality not exceeding λ is preserved by countable inductive limits). Then there is a bounded above complex of left B-modules $'N^\bullet$ with the terms $'N^i$ of the cardinality not exceeding λ, endowed with a quasi-isomorphism of complexes of left B-modules $'N^\bullet \to N^\bullet$. One can also replace $L^\bullet$ by a quasi-isomorphic bounded above complex of finitely generated projective right B-modules, and the desired assertion follows.

(b) For every given degree j, the left B-module $\mathrm{Ext}_A^j(L^\bullet, M^\bullet)$ only depends on a large enough finite truncation $\tau_{\leqslant n} M^\bullet$ of the complex $M^\bullet$ (since the complex $L^\bullet$ is bounded above). So one can assume $M^\bullet$ to be a finite complex; and then there is a finite complex of left A-modules $'M^\bullet$ with the terms $'M^i$ of the cardinality not exceeding λ, endowed with a quasi-isomorphism of complexes of left A-modules $'M^\bullet \to M^\bullet$. It remains to replace $L^\bullet$ by a quasi-isomorphic bounded above complex of finitely generated projective left A-modules. $\square$

The following lemma is our version of [22, Proposition 3.10].

Lemma 10.4. (a) *The class of left B-modules $\mathsf{F}_{l_1} \subset B\mathrm{-mod}$ is a Kaplansky class.*
(b) *The class of left A-modules $\mathsf{E}_{l_1} \subset A\mathrm{-mod}$ is a Kaplansky class.*

Proof. Let λ be an infinite cardinal not smaller than the cardinalities of the rings A and B. We will show that for every left B-module $G \in \mathsf{F}_{l_1}$ and any B-submodule $N \subset G$ with at most λ generators there exists a B-submodule $F \subset G$ with at most λ generators such that $F, G/F \in \mathsf{F}_{l_1}$ and $N \subset F \subset G$. The class of left A-modules E_{l_1} has the same property (with respect to the same cardinals λ).

Indeed, by Lemma 10.3, for any left B-module N of the cardinality not exceeding λ and every i, $n \in \mathbb{Z}$, the cardinalities of the modules $\operatorname{Tor}_i^B(L^\bullet, N)$ do not exceed λ; and it follows that the cardinalities of the modules $\operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -n}(L^\bullet \otimes_B^{\mathbb{L}} N))$ do not exceed λ, either. (Similarly, for any left A-module M of the cardinality not exceeding λ, the cardinalities of the modules $\operatorname{Ext}_A^i(L^\bullet, M)$ do not exceed λ, and consequently the cardinalities of the modules $\operatorname{Tor}_i^B(L^\bullet, \mathbb{R}\operatorname{Hom}_A(L^\bullet, M))$ do not exceed λ as well.)

Furthermore, for any left B-module $G \in \mathsf{F}_{l_1}$ and a submodule $N \subset G$ of the cardinality not exceeding λ, from the homology long exact sequence induced by the short exact sequence of left B-modules $0 \to N \to G \to G/N \to 0$ one observes that the cardinalities of the modules $\operatorname{Tor}_i^B(L^\bullet, G/N)$ do not exceed λ for all $i > l_1$. The same applies to the cardinalities of the modules $\operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} G/N))$, $i \in \mathbb{Z} \setminus \{0\}$, and the cardinalities of the kernel and the cokernel of the natural map $G/N \to \operatorname{Ext}_A^0(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} G/N))$.

Let N_α denote the filtered diagram of all B-submodules $N \subset N_\alpha \subset G$ such that the left B-module N_α/N is finitely generated. Then $G = \varinjlim_\alpha N_\alpha$. By Lemma 10.1(a), for every $n > l_1$ $\varinjlim_\alpha \operatorname{Tor}_n^B(L^\bullet, N_\alpha) = \operatorname{Tor}_n^B(L^\bullet, G) = 0$. Hence for every element $\xi \in \operatorname{Tor}_n^B(L^\bullet, N)$ there exists an index $\alpha = \alpha(\xi)$ such that the image of ξ in $\operatorname{Tor}_n^B(L^\bullet, N_\alpha)$ vanishes. Similarly, $\varinjlim_\alpha \operatorname{Tor}_n^B(L^\bullet, G/N_\alpha) = \operatorname{Tor}_n^B(L^\bullet, 0) = 0$. Hence for every element $\xi' \in \operatorname{Tor}_n^B(L^\bullet, G/N)$, $n > l_1$, there exists $\alpha = \alpha(\xi')$ such that the image of ξ' in $\operatorname{Tor}_n^B(L^\bullet, G/N_\alpha)$ vanishes.

In the same fashion, using Lemma 10.1(b), one finds an index $\alpha(\eta)$ for every element η of $\operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} N))$, $i \in \mathbb{Z} \setminus \{0\}$, an index $\alpha(\eta')$ for every element η' of $\operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} G/N))$, $i \in \mathbb{Z} \setminus \{0\}$, an index $\alpha(\zeta)$ for every element ζ of the kernel or cokernel of the map $N \to \operatorname{Ext}_A^0(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} N))$, and an index $\alpha(\zeta')$ for every element ζ' of the kernel or cokernel of the map $G/N \to \operatorname{Ext}_A^0(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} G/N))$. Set $N' \subset G$ to be the sum of all the submodules $N_\alpha \subset G$ over the chosen indices $\alpha = \alpha(\xi)$, $\alpha(\xi')$, $\alpha(\eta)$, etc. The cardinality of the set of all chosen indices does not exceed λ, hence the cardinality of left B-module N' does not exceed λ, either.

By construction, the map $\operatorname{Tor}_n^B(L^\bullet, N) \to \operatorname{Tor}_n^B(L^\bullet, N')$ induced by the embedding of left B-modules $N \to N'$ vanishes for all $n > l_1$, and so does the map $\operatorname{Tor}_n^B(L^\bullet, G/N) \to \operatorname{Tor}_n^B(L^\bullet, G/N')$, the map $\operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} N)) \to \operatorname{Ext}_A^i(L^\bullet, \tau_{\geqslant -l_1}(L^\bullet \otimes_B^{\mathbb{L}} N'))$ for $i \in \mathbb{Z} \setminus \{0\}$, etc. It remains to iterate our construction over the well-ordered set of nonnegative integers, producing an increasing chain of B-submodules $N \subset N' \subset N'' \subset \cdots \subset G$, and put $F = \bigcup_{m \geqslant 0} N^{(m)} \subset G$. By Lemma 10.1(a-b), we have $F \in \mathsf{F}_{l_1}$ and $G/F \in \mathsf{F}_{l_1}$. This finishes the proof of part (a), and the proof of part (b) is similar. $\square$

Let R be an associative ring and $\mathsf{S} \subset R\text{--mod}$ be a class of left R-modules. A left R-module G is said to be S-*filtered* if there exists a ordinal α and an increasing chain of R-submodules $G_i \subset G$ indexed by $0 \leqslant i \leqslant \alpha$, such that $G_0 = 0$, $G_\alpha = G$, $G_j = \bigcup_{i<j} G_i$ for all limit ordinals $j \leqslant \alpha$, and the quotient module G_{i+1}/G_i is isomorphic to a left R-module from S for every $0 \leqslant i < \alpha$. A class of left R-modules $\mathsf{G} \subset R\text{--mod}$ is said to be *deconstructible* if there is a set (rather than a proper class) of left R-modules S such that G consists precisely of all the S-filtered left R-modules.

COROLLARY 10.5. (a) *The class of left B-modules* $\mathsf{F}_{l_1} \subset B\text{--mod}$ *is deconstructible.*

(b) *The class of left A-modules* $\mathsf{E}_{l_1} \subset A\text{--mod}$ *is deconstructible.*

PROOF. It is not difficult to see that any Kaplansky class of modules G closed under extensions and filtered inductive limits is deconstructible (one take S to be a set of representatives of the isomorphism classes of left R-modules from G with less than κ generators). Conversely, any deconstructible class is Kaplansky (see [25, Lemmas 6.7 and 6.9] or [56, Lemma 2.5]). Thus the assertions of the corollary follow from Lemmas 3.1, 10.2, and 10.4.   □

In view of the Eklof–Trlifaj theorem [20, Theorem 10], it follows from Lemma 3.1(b) and Corollary 10.5(a) that the Auslander class $\mathsf{F}_{l_1}(L^\bullet) \subset B\text{--mod}$ is the left-hand part of a *hereditary complete cotorsion pair* in $B\text{--mod}$ (cf. [22, Theorem 3.11]). It would be interesting to know whether the Bass class $\mathsf{E}_{l_1}(L^\bullet) \subset A\text{--mod}$ is always the *right-hand* part of a hereditary complete cotorsion pair (cf. the recent paper [55], where the authors prove that Gorenstein injective modules over an arbitrary ring form the right-hand part of a hereditary complete cotorsion pair).

11. Derived deconstructible classes have Hom sets

The notions of a Kaplansky class and a deconstructible class of objects are applicable not only to the categories of modules, but more generally to Grothendieck abelian categories [58] (and even more generally to locally presentable abelian categories [50, Section 4]). For the purposes of this section, it is important that one can speak about Kaplansky or deconstructible classes of *complexes of modules*.

Specifically, let $\mathsf{G} \subset R\text{--mod}$ be a deconstructible class of modules (e. g., a Kaplansky class of modules closed under extensions and filtered inductive limits). Then the full subcategory G inherits the exact category structure from

the abelian category R–mod, so one can speak about exact sequences in G. It is important for us that the class of all exact sequences in G is a Kaplansky class in the abelian category $C(R\text{–mod})$ of complexes of left R-modules. The following lemma provides a precise formulation suitable for our purposes.

LEMMA 11.1. *Let* $G \subset R$–mod *be a deconstructible class of left R-modules. Then there is a proper class of cardinals κ with the following property. For every exact complex $G^\bullet$ in G and any subcomplex of left R-modules $N^\bullet \subset G^\bullet$ whose every term N^i, $i \in \mathbb{Z}$, is an R-module with less than κ generators, there exists a subcomplex of left R-modules $F^\bullet \subset G^\bullet$ for which $N^\bullet \subset F^\bullet \subset G^\bullet$, both $F^\bullet$ and $G^\bullet/F^\bullet$ are exact complexes in G, and every term F^i, $i \in \mathbb{Z}$, of the complex $F^\bullet$ is an R-module with less than κ generators.*

PROOF. This is a particular case of [58, Theorem 4.2(2)]. □

For the terminological discussion of "existence of Hom sets in Verdier quotient categories", see Section 0.10 in the introduction.

THEOREM 11.2. *Let R be an associative ring and $G \subset R$–mod be a deconstructible class of left R-modules. Then, for any conventional derived category symbol $\star = b$, $+$, $-$, or $\varnothing$, the derived category $D^\star(G)$ has Hom sets.*

PROOF. The derived category $D^\star(G)$ is obtained from the (similarly bounded or unbounded) homotopy category $Hot^\star(G)$ by inverting all the morphisms whose cones are homotopy equivalent to exact complexes in G [36, 30, 10]. In order to show that the resulting category has Hom sets, we check that the class of all morphisms we are inverting is locally small in $Hot^\star(G)$ in the sense of [63, Set-Theoretic Considerations 10.3.6]. Indeed, given any ($\star$-bounded) complex $X^\bullet$ in G, morphisms $X_1^\bullet \to X^\bullet$ in $Hot^\star(G)$ with an exact cone are in bijection with morphisms $X^\bullet \to G^\bullet$ in $Hot^\star(G)$ with $G^\bullet$ an exact complex. Given a fixed complex $X^\bullet$, let λ be the maximal cardinality of generator sets of its terms X^i, $i \in \mathbb{Z}$. Let $\kappa > \lambda$ be a cardinal with the property described in Lemma 11.1. Then, for any complex $Y^\bullet$ in G, it suffices, for the purposes of constructing morphisms $X^\bullet \to Y^\bullet$ in $D^\star(G)$, to consider cocones $X_1^\bullet \to X^\bullet$ of morphisms $X^\bullet \to F^\bullet$, where $F^\bullet$ ranges over the ($\star$-bounded) exact complexes in G whose terms are left R-modules with less than κ generators. □

Let $L^\bullet$ be a pseudo-dualizing complex for associative rings A and B. As in Sections 3 and 10, we assume that the finite complex $L^\bullet$ is situated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$, and choose an integer $l_1 \geqslant d_1$.

CoROLLARY 11.3. *For any pseudo-dualizing complex $L^\bullet$, the lower pseudo-coderived category of left A-modules $\mathsf{D}'_{L^\bullet}(A\text{–mod}) = \mathsf{D}(\mathsf{E}_{l_1})$ and the lower pseudo-contraderived category of left B-modules $\mathsf{D}''_{L^\bullet}(B\text{–mod}) = \mathsf{D}(\mathsf{F}_{l_1})$ have Hom sets.*

PROOF. Follows from Theorem 11.2 and Corollary 10.5.   □

12. Existence of fully faithful adjoints

The aim of this section is to prove existence of the adjoint functors promised in Section 0.8. We start with the adjoint functors on the diagram (8) before passing to the ones on the diagram (9).
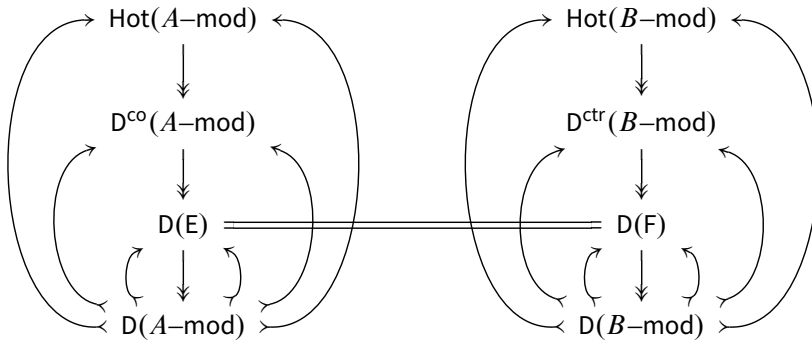
LEMMA 12.1. *Let $q\colon \mathsf{D}_2 \to \mathsf{D}_1$ and $p\colon \mathsf{D}_1 \to \mathsf{D}_0$ be Verdier quotient functors between triangulated categories. Let $r\colon \mathsf{D}_0 \to \mathsf{D}_2$ be a right adjoint functor to the composition $pq\colon \mathsf{D}_2 \to \mathsf{D}_0$. Then the composition $qr\colon \mathsf{D}_0 \to \mathsf{D}_1$ is a right adjoint functor to the functor $p\colon \mathsf{D}_1 \to \mathsf{D}_0$.*

PROOF. Let $\mathsf{K}^0 \subset \mathsf{D}_2$ denote the kernel of the functor pq and $\mathsf{K}^1 \subset \mathsf{D}_2$ the kernel of the functor q; so $\mathsf{K}^1 \subset \mathsf{K}^0$. Then $\mathsf{K}^0/\mathsf{K}^1 \subset \mathsf{D}_2/\mathsf{K}^1 = \mathsf{D}_1$ is the kernel of the functor p. The functor $r\colon \mathsf{D}_0 \to \mathsf{D}_2$ is fully faithful (as an adjoint to a Verdier quotient functor) and its essential image $r(\mathsf{D}_0)$ together with the full subcategory K^0 form a semi-orthogonal decomposition of the triangulated category D_2. The composition $(pq)r$ is the identity functor $\mathsf{D}_0 \to \mathsf{D}_0$.

Passing to the quotient category by the full subcategory $\mathsf{K}^1 \subset \mathsf{K}^0 \subset \mathsf{D}_2$, we conclude that the functor qr is fully faithful, the two full subcategories $\mathsf{K}^0/\mathsf{K}^1$ and $qr(\mathsf{D}_0)$ form a semi-orthogonal decomposition of the triangulated category D_1, and the composition $p(qr)$ is the identity functor. Thus the functor qr is right adjoint to p.   □

Now we can construct the adjoint functors shown on the diagram (8). More generally, let $\mathsf{E} \subset A\text{–mod}$ and $\mathsf{F} \subset B\text{–mod}$ be a pair of full subcategories satisfying the conditions (I)–(IV) of Section 4 for a given pseudo-dualizing complex $L^\bullet$ and two integers l_1 and l_2. Assume that the full subcategory E is closed under infinite direct sums in $A\text{–mod}$, while the full subcategory F is closed under infinite products in $B\text{–mod}$. Then there exist fully faithful adjoint functors shown by curvilinear arrows on the diagram

(16)

Indeed, for any associative ring R the natural Verdier quotient functor $K \colon \mathsf{Hot}(R\text{–mod}) \to \mathsf{D}(R\text{–mod})$ has a right and a left adjoint. The fully faithful left adjoint functor $K_\lambda \colon \mathsf{D}(R\text{–mod}) \to \mathsf{Hot}(R\text{–mod})$ to the functor K assigns to a complex of left R-modules its homotopy projective resolution. The fully faithful right adjoint functor $K_\rho \colon \mathsf{D}(R\text{–mod}) \to \mathsf{Hot}(R\text{–mod})$ to the functor K assigns to a complex of right R-modules its homotopy injective resolution.

Lemma 12.1 tells that the other fully faithful adjoint functors on the diagram (16) can be obtained as the compositions of the functors K_λ and K_ρ with the Verdier quotient functors on the diagram. We refer to the discussion in Section 1, based on [44, Proposition A.3.1(b)], for the constructions of the Verdier quotient functors $\mathsf{D}^{\mathsf{co}}(A\text{–mod}) \to \mathsf{D}(\mathsf{E})$ and $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) \to \mathsf{D}(\mathsf{F})$.

In particular, the right adjoint functor $P_\rho \colon \mathsf{D}(A\text{–mod}) \to \mathsf{D}(\mathsf{E})$ to the Verdier quotient functor $P \colon \mathsf{D}(\mathsf{E}) \to \mathsf{D}(A\text{–mod})$ assigns to a complex of left A-modules $M^\bullet$ a homotopy injective complex of injective left A-modules $J^\bullet$ quasi-isomorphic to $M^\bullet$, viewed as an object $P_\rho(M^\bullet) = J^\bullet \in \mathsf{D}(\mathsf{E})$ of the conventional derived category of the exact category E. Here it is important that, according to the condition (I), all the injective left A-modules belong to E.

To construct the image $P_\lambda(M^\bullet)$ of the complex $M^\bullet$ under the left adjoint functor $P_\lambda \colon \mathsf{D}(A\text{–mod}) \to \mathsf{D}(\mathsf{E})$ to the functor P, consider a homotopy projective complex of left A-modules $G^\bullet$ quasi-isomorphic to $M^\bullet$. Let $G^\bullet \to E^\bullet$ be a morphism of complexes of left A-modules with a coacyclic cone acting from the complex $G^\bullet$ to a complex $E^\bullet$ with the terms E^i belonging to the full subcategory $\mathsf{E} \subset A\text{–mod}$. Then $P_\lambda(M^\bullet) = E^\bullet \in \mathsf{D}(\mathsf{E})$.

Similarly, the left adjoint functor $Q_\lambda \colon \mathsf{D}(B\text{–mod}) \to \mathsf{D}(\mathsf{F})$ to the Verdier quotient functor $Q \colon \mathsf{D}(\mathsf{F}) \to \mathsf{D}(B\text{–mod})$ assigns to a complex of left B-modules $N^\bullet$ a homotopy projective complex of projective left B-modules $G^\bullet$ quasi-isomorphic to $N^\bullet$, viewed as an object $Q_\lambda(N^\bullet) = G^\bullet \in \mathsf{D}(\mathsf{F})$ of the conventional derived

category of the exact category F. Here it is important that, according to the condition (II), all the projective left B-modules belong to F.

To construct the image $Q_\rho(N^\bullet)$ of the complex $N^\bullet$ under the right adjoint functor $Q_\rho : D(B\text{–mod}) \to D(F)$ to the functor Q, consider a homotopy injective complex of left B-modules $J^\bullet$ quasi-isomorphic to $N^\bullet$. Let $F^\bullet \to J^\bullet$ be a morphism of complexes of left B-modules with a contraacyclic cone acting into the complex $J^\bullet$ from a complex $F^\bullet$ with the terms F^i belonging to the full subcategory $F \subset B\text{–mod}$. Then $Q_\rho(N^\bullet) = F^\bullet \in D(F)$.

To end, let us prove existence of the adjoint functors on the diagram (9) from Section 0.8.

THEOREM 12.2. *Let $L^\bullet$ be a pseudo-dualizing complex for associative rings A and B. Then*

(a) *assuming that all sfp-injective left A-modules have finite injective dimensions, the natural Verdier quotient functor*

$$D^{co}(A\text{–mod}) \longrightarrow D'_{L^\bullet}(A\text{–mod})$$

has a right adjoint;

(b) *assuming that the ring A is left coherent and all fp-injective left A-modules have finite injective dimensions, the natural Verdier quotient functor*

$$D^{co}(A\text{–mod}) \longrightarrow D'_{L^\bullet}(A\text{–mod})$$

has a left adjoint;

(c) *assuming that all sfp-flat left B-modules have finite projective dimensions, the natural Verdier quotient functor*

$$D^{ctr}(B\text{–mod}) \longrightarrow D''_{L^\bullet}(B\text{–mod})$$

has a right adjoint;

(d) *assuming that the ring B is right coherent and all flat left B-modules have finite projective dimensions, the natural Verdier quotient functor*

$$D^{ctr}(B\text{–mod}) \longrightarrow D''_{L^\bullet}(B\text{–mod})$$

has a left adjoint.

Proof. First of all we show that all the four exotic derived categories in question have infinite direct sums and products, and both the Verdier quotient functors preserve both the infinite direct sums and products. It is helpful to keep in mind that the Verdier quotient category of a triangulated category with infinite direct sums by a triangulated subcategory closed under infinite direct sums has infinite sums, and the Verdier quotient functor in such a situation preserves infinite direct sums [37, Lemma 3.2.10]. Dually, the same assertions apply to infinite products.

Assume that the finite complex $L^\bullet$ is situated in the cohomologiclal degrees $-d_1 \leqslant m \leqslant d_2$, and choose an integer $l_1 \geqslant d_1$. By Lemma 3.3, both the full subcategories $\mathsf{E}_{l_1} \subset A\text{–mod}$ and $\mathsf{F}_{l_1} \subset B\text{–mod}$ are closed under both the infinite direct sums and products. Thus E_{l_1} and F_{l_1} are exact categories with exact functors of infinite direct sums and products, and it follows that their derived categories $\mathsf{D}'_{L^\bullet}(A\text{–mod}) = \mathsf{D}(\mathsf{E}_{l_1})$ and $\mathsf{D}''_{L^\bullet}(B\text{–mod}) = \mathsf{D}(\mathsf{F}_{l_1})$ have infinite direct sums and products.

The coderived category $\mathsf{D}^{\mathsf{co}}(A\text{–mod})$ is the Verdier quotient category of the homotopy category $\mathsf{Hot}(A\text{–mod})$ by the full subcategory of coacyclic complexes, which is, by the definition, closed under infinite direct sums in $\mathsf{Hot}(A\text{–mod})$. Hence the triangulated category $\mathsf{D}^{\mathsf{co}}(A\text{–mod})$ has infinite direct sums. Similarly, the contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$ is the Verdier quotient category of the homotopy category $\mathsf{Hot}(B\text{–mod})$ by the full subcategory of contraacyclic complexes, which is, by the definition, closed under infinite products in $\mathsf{Hot}(B\text{–mod})$. Hence the triangulated category $\mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$ has infinite products.

The Verdier quotient functor

$$\mathsf{D}^{\mathsf{co}}(A\text{–mod}) = \mathsf{D}^{\mathsf{co}}(\mathsf{E}_{l_1}) \longrightarrow \mathsf{D}(\mathsf{E}_{l_1}) = \mathsf{D}'_{L^\bullet}(A\text{–mod})$$

preserves infinite direct sums, since E_{l_1} is an exact category with exact functors of infinite direct sums. The Verdier quotient functor

$$\mathsf{D}^{\mathsf{ctr}}(B\text{–mod}) = \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}_{l_1}) \longrightarrow \mathsf{D}(\mathsf{F}_{l_1}) = \mathsf{D}''_{L^\bullet}(B\text{–mod})$$

preserves infinite products, since F_{l_1} is an exact category with exact fuctors of infinite products.

Assuming that all sfp-injective left A-modules have finite injective dimensions, the triangulated functor $\mathsf{Hot}(A\text{–mod}_{\mathsf{inj}}) \to \mathsf{D}^{\mathsf{co}}(A\text{–mod})$ induced by the inclusion $A\text{–mod}_{\mathsf{inj}} \to A\text{–mod}$ is an equivalence by Proposition 7.1(b). The additive category of injective left A-modules $A\text{–mod}_{\mathsf{inj}}$ has infinite products, hence so does its homotopy category. Similarly, assuming that all sfp-flat left B-modules have finite projective dimensions, the triangulated functor $\mathsf{Hot}(B\text{–mod}_{\mathsf{proj}}) \to \mathsf{D}^{\mathsf{ctr}}(B\text{–mod})$

induced by the inclusion $B\text{-mod}_{proj} \to B\text{-mod}$ is an equivalence by Proposition 7.2(b). The additive category of projective left B-modules $B\text{-mod}_{proj}$ has infinite direct sums, hence so does its homotopy category.

The functor $\mathsf{D}^{co}(A\text{-mod}) = \mathsf{Hot}(A\text{-mod}_{inj}) \to \mathsf{D}(\mathsf{E}_{l_1})$ preserves infinite products, since E_{l_1} is an exact category with exact functors of infinite products and $A\text{-mod}_{inj} \subset \mathsf{E}_{l_1}$ is a split exact subcategory closed under infinite products. The functor $\mathsf{D}^{ctr}(B\text{-mod}) = \mathsf{Hot}(B\text{-mod}_{proj}) \to \mathsf{D}(\mathsf{F}_{l_1})$ preserves infinite direct sums, since F_{l_1} is an exact category with exact functors of infinite direct sums and $B\text{-mod}_{proj} \subset \mathsf{F}_{l_1}$ is a split exact subcategory closed under infinite direct sums.

For the rest of our argument, it is important that the lower pseudo-derived categories $\mathsf{D}'_{L\bullet}(A\text{-mod})$ and $\mathsf{D}''_{L\bullet}(B\text{-mod})$ have Hom sets by Corollary 11.3.

By [38, Theorem 5.9], for any associative ring B the homotopy category of projective modules $\mathsf{Hot}(B\text{-mod}_{proj})$ is *well-generated* in the sense of [37, 31]. By [39, Theorem 3.13], for any associative ring A the homotopy category of injective modules $\mathsf{Hot}(A\text{-mod}_{inj})$ is well-generated. According to the Brown representability theorem for well-generated triangulated categories (see [37, Proposition 8.4.2] or [33, Theorem 5.1.1]), a triangulated functor from a well-generated triangulated category to a triangulated category with small Hom sets has a right adjoint if and only if it preserves infinite direct sums. This proves parts (a) and (c).

By [38, Proposition 7.14], for any right coherent ring B the homotopy category of projective left B-modules $\mathsf{Hot}(B\text{-mod}_{proj})$ is compactly generated. By [60, Corollary 6.13] (see also [46, Corollary 2.6(b)]), for any left coherent ring A the homotopy category of injective left A-modules $\mathsf{Hot}(A\text{-mod}_{inj})$ is compactly generated. According to the covariant Brown representability theorem for compactly generated triangulated categories (see [37, Theorem 8.6.1], [32, Section 2], or [33, Proposition 5.3.1(2)]), a triangulated functor from a compactly triangulated category to a triangulated category with small Hom sets has a left adjoint if and only if it preserves infinite products. This proves parts (b) and (d). $\qquad\square$

Appendix A. Derived functors of finite homological dimension II

The aim of this appendix is to work out a generalization of the constructions of [45, Appendix B] that is needed for the purposes of the present paper. We use an idea borrowed from [19, Appendix A] in order to simplify and clarify the exposition.

A.1 – *Posing the problem*

First we need to recall some notation from [45]. Given an additive category A, we denote by $\mathsf{C}^+(\mathsf{A})$ the category of bounded below complexes in A, viewed

either as a DG-category (with complexes of morphisms), or simply as an additive category, with closed morphisms of degree 0. When A is an exact category, the full subcategory $C^{\geq 0}(A) \subset C^+(A)$ of nonnegatively cohomologically graded complexes in A and closed morphisms of degree 0 between them has a natural exact category structure, with termwise exact short exact sequences of complexes.

Let E be an exact category and $J \subset E$ be a coresolving subcategory (in the sense of Section 1), endowed with the exact category structure inherited from E. As it was pointed out in [45], a closed morphism in $C^+(J)$ is a quasi-isomorphism of complexes in J if and only if it is a quasi-isomorphism of complexes in E. A short sequence in $C^{\geq 0}(J)$ is exact in $C^{\geq 0}(J)$ if and only if it is exact in $C^{\geq 0}(E)$.

Modifying slightly the notation in [45], we denote by $_EC^{\geq 0}(J)$ the full subcategory in the exact category $C^{\geq 0}(J)$ consisting of all the complexes $0 \to J^0 \to J^1 \to J^2 \to \cdots$ in J for which there exists an object $E \in E$ together with a morphism $E \to J^0$ such that the sequence $0 \to E \to J^0 \to J^1 \to \cdots$ is exact in E. By the definition, one has $_EC^{\geq 0}(J) = C^{\geq 0}(J) \cap {}_EC^{\geq 0}(E) \subset C^{\geq 0}(E)$. The full subcategory $_EC^{\geq 0}(J)$ is closed under extensions and the cokernels of admissible monomorphisms in $C^{\geq 0}(J)$; so it inherits an exact category structure.

Let B be another exact category and $F \subset B$ be a resolving subcategory. We will suppose that the additive category B contains the images of idempotent endomorphisms of its objects. Let $-l_2 \leq l_1$ be two integers. Denote by $C^{\geq -l_2}(B)$ the exact category $C^{\geq 0}(B)[l_2] \subset C^+(B)$ of complexes in B concentrated in the cohomological degrees $\geq -l_2$, and by $C^{\geq -l_2}(B)^{\leq l_1} \subset C^{\geq -l_2}(B)$ the full subcategory consisting of all complexes $0 \to B^{-l_2} \to \cdots \to B^{l_1} \to \cdots$ such that the sequence $B^{l_1} \to B^{l_1+1} \to B^{l_1+2} \to \cdots$ is exact in B. Furthermore, let $C_F^{\geq -l_2}(B)^{\leq l_1} \subset C^{\geq -l_2}(B)^{\leq l_1}$ be the full subcategory of all complexes that are isomorphic in the derived category D(B) to complexes of the form $0 \to F^{-l_2} \to \cdots \to F^{l_1} \to 0$, with the terms belonging to F and concentrated in the cohomological degrees $-l_2 \leq m \leq l_1$.

For example, one has $C^{\geq 0}(B)^{\leq 0} = {}_BC^{\geq 0}(B)$. The full subcategory $C^{\geq -l_2}(B)^{\leq l_1}$ is closed under extensions and the cokernels of admissible monomorphisms in the exact category $C^{\geq -l_2}(B)$, while (essentially by [59, Proposition 2.3(2)] or [44, Lemma A.5.4(a-b)]) the full subcategory $C_F^{\geq -l_2}(B)^{\leq l_1}$ is closed under extensions and the kernels of admissible epimorphisms in $C^{\geq -l_2}(B)^{\leq l_1}$. So the full subcategory $C_F^{\geq -l_2}(B)^{\leq l_1}$ inherits an exact category structure from $C^{\geq -l_2}(B)$.

Suppose that we are given a DG-functor $\Psi: C^+(J) \to C^+(B)$ taking acyclic complexes in the exact category J to acyclic complexes in the exact category B. Suppose further that the restriction of Ψ to the subcategory $_EC^{\geq 0}(J) \subset C^+(J)$ is

an exact functor between exact categories

$$(17) \qquad \Psi\colon {}_\mathsf{E}\mathsf{C}^{\geq 0}(\mathsf{J}) \longrightarrow \mathsf{C}_\mathsf{F}^{\geq -l_2}(\mathsf{B})^{\leq l_1}.$$

Our aim is to construct the right derived functor

$$(18) \qquad \mathbb{R}\Psi\colon \mathsf{D}^\star(\mathsf{E}) \longrightarrow \mathsf{D}^\star(\mathsf{F})$$

acting between any bounded or unbounded, conventional or absolute derived categories $\mathsf{D}^\star$ with the symbols $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-,$ or abs.

Under certain conditions, one can also have the derived functor $\mathbb{R}\Psi$ acting between the coderived or contraderived categories, $\star = \mathsf{co}$ or ctr, of the exact categories E and F. When the exact categories E and B have exact functors of infinite product, the full subcategories $\mathsf{J} \subset \mathsf{E}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under infinite products, and the functor Ψ preserves infinite products, there will be the derived functor $\mathbb{R}\Psi$ acting between the contraderived categories, $\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{ctr}}(\mathsf{E}) \to \mathsf{D}^{\mathsf{ctr}}(\mathsf{F})$.

When the exact categories E and B have exact functors of infinite direct sum, the full subcategory $\mathsf{F} \subset \mathsf{B}$ is closed under infinite direct sums, and for any family of complexes $J_\alpha^\bullet \in \mathsf{C}^{\geq 0}(\mathsf{J})$ and a complex $I^\bullet \in \mathsf{C}^{\geq 0}(\mathsf{J})$ endowed with a quasi-isomorphism $\bigoplus_\alpha J_\alpha^\bullet \to I^\bullet$ of complexes in the exact category E, the induced morphism

$$\bigoplus_\alpha \Psi(J_\alpha^\bullet) \longrightarrow \Psi(I^\bullet)$$

is a quasi-isomorphism of complexes in the exact category B, there will be the derived functor $\mathbb{R}\Psi$ acting between the coderived categories, $\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathsf{E}) \to \mathsf{D}^{\mathsf{co}}(\mathsf{F})$.

The construction of the derived functor $\mathbb{R}\Psi$ in [45, Appendix B] is the particular case of the construction below corresponding to the situation with $\mathsf{F} = \mathsf{B}$.

A.2 – *The construction of derived functor*

The following construction of the derived functor (18) is based on a version of the result of [19, Proposition A.3].

Since the DG-functor $\Psi\colon \mathsf{C}^+(\mathsf{J}) \to \mathsf{C}^+(\mathsf{B})$ preserves quasi-isomorphisms, it induces a triangulated functor

$$\Psi\colon \mathsf{D}^+(\mathsf{J}) \longrightarrow \mathsf{D}^+(\mathsf{B}).$$

Taking into account the triangulated equivalence $\mathsf{D}^+(\mathsf{J}) \simeq \mathsf{D}^+(\mathsf{E})$ (provided by the dual version of [44, Proposition A.3.1(a)]), we obtain the derived functor

$$\mathbb{R}\Psi\colon \mathsf{D}^+(\mathsf{E}) \longrightarrow \mathsf{D}^+(\mathsf{B}).$$

Now our assumptions on Ψ imply that the functor $\mathbb{R}\Psi$ takes the full subcategory $\mathsf{D}^b(\mathsf{E}) \subset \mathsf{D}^+(\mathsf{E})$ into the full subcategory $\mathsf{D}^b(\mathsf{F}) \subset \mathsf{D}^b(\mathsf{B}) \subset \mathsf{D}^+(\mathsf{B})$; hence the triangulated functor

$$(19) \qquad \mathbb{R}\Psi \colon \mathsf{D}^b(\mathsf{E}) \longrightarrow \mathsf{D}^b(\mathsf{F}).$$

For any exact category A, we denote by $\mathsf{C}(\mathsf{A})$ the exact category of unbounded complexes in A, with termwise exact short exact sequences of complexes. In order to construct the derived functor $\mathbb{R}\Psi$ for the derived categories with the symbols other than $\star = \mathsf{b}$, we are going to substitute into (19) the exact category $\mathsf{C}(\mathsf{E})$ in place of E and the exact category $\mathsf{C}(\mathsf{F})$ in place of F.

For any category Γ and DG-category DG, there is a DG-category whose objects are all the functors $\Gamma \to \mathsf{DG}$ taking morphisms in Γ to closed morphisms of degree 0 in DG, and whose complexes of morphisms are constructed as the complexes of morphisms of functors. We denote this DG-category by DG^Γ. So diagrams of any fixed shape in a given DG-category form a DG-category. Given a DG-functor $F \colon {}'\mathsf{DG} \to {}''\mathsf{DG}$, there is the induced DG-functor between the categories of diagrams $F^\Gamma \colon {}'\mathsf{DG}^\Gamma \to {}''\mathsf{DG}^\Gamma$. In particular, the DG-category of complexes $\mathsf{C}(\mathsf{DG})$ in a given DG-category DG can be constructed as a full DG-subcategory of the DG-category of diagrams of the corresponding shape in DG.

Applying this construction to the DG-functor Ψ and restricting to the full DG-subcategories of bicomplexes that are uniformly bounded on the relevant side, we obtain a DG-functor

$$\Psi_\mathsf{C} \colon \mathsf{C}^+(\mathsf{C}(\mathsf{J})) \longrightarrow \mathsf{C}^+(\mathsf{C}(\mathsf{B})).$$

Here the categories of unbounded complexes $\mathsf{C}(\mathsf{J})$ and $\mathsf{C}(\mathsf{B})$ are simply viewed as additive/exact categories of complexes and closed morphisms of degree 0 between them. The DG-structures come from the differentials raising the degree in which the bicomplexes are bounded below.

The functor Ψ_C takes acyclic complexes in the exact category $\mathsf{C}(\mathsf{J})$ to acyclic complexes in the exact category $\mathsf{C}(\mathsf{B})$. In view of the standard properties of the resolution dimension [44, Corollary A.5.2], the functor Ψ_C takes the full subcategory ${}_{\mathsf{C}(\mathsf{E})}\mathsf{C}^{\geq 0}(\mathsf{C}(\mathsf{J})) \subset \mathsf{C}^+(\mathsf{C}(\mathsf{J}))$ into the full subcategory $\mathsf{C}^{\geq -l_2}_{\mathsf{C}(\mathsf{F})}(\mathsf{C}(\mathsf{B}))^{\leq l_1} \subset \mathsf{C}^+(\mathsf{C}(\mathsf{B}))$,

$$\Psi_\mathsf{C} \colon {}_{\mathsf{C}(\mathsf{E})}\mathsf{C}^{\geq 0}(\mathsf{C}(\mathsf{J})) \longrightarrow \mathsf{C}^{\geq -l_2}_{\mathsf{C}(\mathsf{F})}(\mathsf{C}(\mathsf{B}))^{\leq l_1}.$$

Finally, the functor Ψ_C is exact in restriction to the exact category ${}_{\mathsf{C}(\mathsf{E})}\mathsf{C}^{\geq 0}(\mathsf{C}(\mathsf{J}))$, since the functor Ψ is exact in restriction to the exact category ${}_\mathsf{E}\mathsf{C}^{\geq 0}(\mathsf{J})$.

Applying the construction of the derived functor (19) to the DG-functor Ψ_C in place of Ψ, we obtain a triangulated functor

$$(20) \qquad \mathbb{R}\Psi_\mathsf{C} \colon \mathsf{D}^b(\mathsf{C}(\mathsf{E})) \longrightarrow \mathsf{D}^b(\mathsf{C}(\mathsf{F})).$$

Similarly one can construct the derived functors $\mathbb{R}\Psi_{C^{\leq 0}}\colon \mathsf{D}^b(C^{\leq 0}(E)) \to \mathsf{D}^b(C^{\leq 0}(F))$ and $\mathbb{R}\Psi_{C^{\geq 0}}\colon \mathsf{D}^b(C^{\geq 0}(E)) \to \mathsf{D}^b(C^{\geq 0}(F))$ acting between the bounded derived categories of the exact categories of nonpositively or nonnegatively cohomologically graded complexes. Shifting and passing to the direct limits of fully faithful embeddings, one can obtain the derived functors $\mathbb{R}\Psi_{C^-}\colon \mathsf{D}^b(C^-(E)) \to \mathsf{D}^b(C^-(F))$ and $\mathbb{R}\Psi_{C^+}\colon \mathsf{D}^b(C^+(E)) \to \mathsf{D}^b(C^+(F))$ acting between the bounded derived categories of the exact categories of bounded above or bounded below complexes, etc.

In order to pass from (20) to (18) with $\star = \mathsf{abs}$, we will apply the following version of [19, Proposition A.3(2)]. Clearly, for any exact category A the totalization of bounded complexes of complexes in A is a triangulated functor

$$(21) \qquad\qquad \mathsf{D}^b(C(A)) \longrightarrow \mathsf{D}^{\mathsf{abs}}(A).$$

PROPOSITION A.1. *For any exact category* A*, the totalization functor* (21) *is a Verdier quotient functor. Its kernel is the thick subcategory generated by the contractible complexes in* A*, viewed as objects of* $C(A)$*.*

PROOF. Denote by $\mathsf{A}_{\mathsf{spl}}$ the additive category A endowed with the split exact category structure (i. e., all the short exact sequences are split). Following [19], one first checks the assertion of proposition for the exact category $\mathsf{A}_{\mathsf{spl}}$.

In this case, $C(\mathsf{A}_{\mathsf{spl}})$ is a Frobenius exact category whose projective-injective objects are the contractible complexes, and $\mathsf{D}^{\mathsf{abs}}(\mathsf{A}_{\mathsf{spl}}) = \mathsf{Hot}(\mathsf{A}_{\mathsf{spl}})$ is the stable category of the Frobenius exact category $C(\mathsf{A}_{\mathsf{spl}})$. The quotient category of the bounded derived category $\mathsf{D}^b(C(\mathsf{A}_{\mathsf{spl}}))$ by the bounded homotopy category of complexes of projective-injective objects in $C(\mathsf{A}_{\mathsf{spl}})$ is just another construction of the stable category of a Frobenius exact category, and the totalization functor is the inverse equivalence to the comparison functor between the two constructions of the stable category.

Then, in order to pass from the functor (21) for the exact category $\mathsf{A}_{\mathsf{spl}}$ to the similar functor for the exact category A, one takes the quotient category by the acyclic bounded complexes of complexes on the left-hand side, transforming $\mathsf{D}^b(C(\mathsf{A}_{\mathsf{spl}}))$ into $\mathsf{D}^b(C(A))$, and the quotient category by the totalizations of such bicomplexes on the right-hand side, transforming $\mathsf{Hot}(A)$ into $\mathsf{D}^{\mathsf{abs}}(A)$. □

It remains to notice that the contractible complexes in A are the direct summands of the cones of identity endomorphisms of complexes in A, and the functor (20) obviously takes the cones of identity endomorphisms of complexes in E (viewed as objects of $C(E)$) to bicomplexes whose totalizations are contractible complexes in F. This provides the desired derived functor (18) for $\star = \mathsf{abs}$.

In order to pass from (20) to (18) with $\star = \varnothing$, the following corollary of Proposition A.1 can be applied. Consider the totalization functor

$$(22) \qquad\qquad \mathsf{D}^b(\mathsf{C}(\mathsf{A})) \longrightarrow \mathsf{D}(\mathsf{A}).$$

COROLLARY A.2. *For any exact category* A*, the totalization functor* (22) *is a Verdier quotient functor. Its kernel is the thick subcategory generated by the acyclic complexes in* A*, viewed as objects of* $\mathsf{C}(\mathsf{A})$*.*   □

Using the condition that the functor (17) takes short exact sequences to short exact sequences together with [45, Lemma B.2(e)], one shows that the functor (20) takes acyclic complexes in E (viewed as objects of $\mathsf{C}(\mathsf{E})$) to bicomplexes with acyclic totalizations. This provides the derived functor (18) for $\star = \varnothing$.

To construct the derived functors $\mathbb{R}\Psi$ acting between the bounded above and bounded below versions of the conventional and absolute derived categories (with $\star = +, -,$ abs+, or abs−), one can notice that the functors $\mathbb{R}\Psi$ for $\star = \varnothing$ or abs take bounded above/below complexes to (objects representable by) bounded above/below complexes, and use the fact that the embedding functors from the bounded above/below conventional/absolute derived categories into the unbounded ones are fully faithful [44, Lemma A.1.1]. Alternatively, one can repeat the above arguments with the categories of unbounded complexes $\mathsf{C}(\mathsf{A})$ replaced with the bounded above/below ones $\mathsf{C}^-(\mathsf{A})$ or $\mathsf{C}^+(\mathsf{A})$. The derived functor $\mathbb{R}\Psi$ with $\star = b$ constructed in such a way agrees with the functor (19).

To construct the derived functor $\mathbb{R}\Psi$ acting between the coderived or contraderived categories (under the respective assumptions in Section A.1), one considers the derived functor $\mathbb{R}\Psi$ for $\star = $ abs, and checks that the kernel of the composition $\mathsf{C}(\mathsf{E}) \to \mathsf{D}^{abs}(\mathsf{E}) \to \mathsf{D}^{abs}(\mathsf{F}) \to \mathsf{D}^{co}(\mathsf{F})$ or $\mathsf{C}(\mathsf{E}) \to \mathsf{D}^{abs}(\mathsf{E}) \to \mathsf{D}^{abs}(\mathsf{F}) \to \mathsf{D}^{ctr}(\mathsf{F})$ is closed under the infinite direct sums or infinite products, respectively. The facts that the kernels of the additive functors $\mathsf{C}(\mathsf{F}) \to \mathsf{D}^{co/ctr}(\mathsf{F})$ are closed under the infinite direct sums/products and the total complex of a finite acyclic complex of unbounded complexes in F is absolutely acyclic need to be used.

A.3 – *The dual setting*

The notation $\mathsf{C}^{\leq 0}(\mathsf{B}) \subset \mathsf{C}^-(\mathsf{B})$ for an additive or exact category B has the similar or dual meaning to the one in Section A.1.

Let F be an exact category and $\mathsf{P} \subset \mathsf{F}$ be a resolving subcategory, endowed with the inherited exact category structure. A closed morphism in $\mathsf{C}^-(\mathsf{P})$ is a quasi-isomorphism of complexes in P if and only if it is a quasi-isomorphism of

complexes in F. A short sequence in $C^{\leq 0}(P)$ is exact in $C^{\leq 0}(P)$ if and only if it is exact in $C^{\leq 0}(F)$.

Following the notation in Section A.1, denote by $_FC^{\leq 0}(P)$ the full subcategory in the exact category $C^{\leq 0}(P)$ consisting of all the complexes $\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0$ in P for which there exists an object $F \in F$ together with a morphism $P^0 \to F$ such that the sequence $\cdots \to P^{-1} \to P^0 \to F \to 0$ is exact in F. By the definition, one has $_FC^{\leq 0}(P) = C^{\leq 0}(P) \cap {}_FC^{\leq 0}(F) \subset C^{\leq 0}(F)$. The full subcategory $_FC^{\leq 0}(P)$ is closed under extensions and the kernels of admissible epimorphisms in $C^{\leq 0}(P)$; so it inherits an exact category structure.

Let A be another exact category and $E \subset A$ be a coresolving subcategory. Suppose that the additive category A contains the images of idempotent endo-morphisms of its objects. Let $-l_1 \leq l_2$ be two integers. Denote by $C^{\leq l_2}(A)$ the exact category $C^{\leq 0}(A)[-l_2] \subset C^-(A)$ of complexes in A concentrated in the cohomological degrees $\leq l_2$, and by $C^{\leq l_2}(A)^{\geq -l_1} \subset C^{\leq l_2}(A)$ the full subcat-egory consisting of all complexes $\cdots \to A^{-l_1} \to \cdots \to A^{l_2} \to 0$ such that the sequence $\cdots \to A^{-l_1-2} \to A^{-l_1-1} \to A^{-l_1}$ is exact in A. Fur-thermore, let $C_E^{\leq l_2}(A)^{\geq -l_1} \subset C^{\leq l_2}(A)^{\geq -l_1}$ be the full subcategory of all com-plexes that are isomorphic in the derived category $D(A)$ to complexes of the form $0 \to E^{-l_1} \to \cdots \to E^{l_2} \to 0$, with the terms belonging to E and concentrated in the cohomological degrees $-l_1 \leq m \leq l_2$.

For example, one has $C^{\leq 0}(A)^{\geq 0} = {}_AC^{\leq 0}(A)$. The full subcategory $C^{\leq l_2}(A)^{\geq -l_1}$ is closed under extensions and the kernels of admissible epimorphisms in the exact category $C^{\leq l_2}(A)$, while the full subcategory $C_E^{\leq l_2}(A)^{\geq -l_1}$ is closed under extension and the cokernels of admissible monomorphisms in $C^{\leq l_2}(A)^{\geq -l_1}$. So the full subcategory $C_E^{\leq l_2}(A)^{\geq -l_1}$ inherits an exact category structure from $C^{\leq l_2}(A)$.

Suppose that we are given a DG-functor $\Phi: C^-(P) \to C^-(A)$ taking acyclic complexes in the exact category P to acyclic complexes in the exact category A. Suppose further that the restriction of Φ to the subcategory $_FC^{\leq 0}(P) \subset C^-(P)$ is an exact functor between exact categories

$$(23) \qquad\qquad {}_FC^{\leq 0}(P) \longrightarrow C_E^{\leq l_2}(A)^{\geq -l_1}.$$

Then the construction dual to that in Section A.2 provides the left derived functor

$$(24) \qquad\qquad \mathbb{L}\Phi: D^\star(F) \longrightarrow D^\star(E)$$

acting between any bounded or unbounded, conventional or absolute derived categories $D^\star$ with the symbols $\star = b, +, -, \varnothing, abs+, abs-,$ or abs.

Under certain conditions, one can also have the derived functor $\mathbb{L}\Phi$ acting between the coderived or contraderived categories. When the exact categories F and A have exact functors of infinite direct sum, the full subcategories $\mathsf{P} \subset \mathsf{F}$ and $\mathsf{E} \subset \mathsf{A}$ are closed under infinite direct sums, and the functor Φ preserves infinite direct sums, there is the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^{co}(\mathsf{F}) \to \mathsf{D}^{co}(\mathsf{E})$.

When the exact categories F and A have exact functors of infinite product, the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under infinite products, and for any family of complexes $P_\alpha^\bullet \in \mathsf{C}^{\leq 0}(\mathsf{P})$ and a complex $Q^\bullet \in \mathsf{C}^{\leq 0}(\mathsf{P})$ endowed with a quasi-isomorphism $Q^\bullet \to \prod_\alpha P_\alpha^\bullet$ of complexes in the exact category F, the induced morphism

$$\Phi(Q^\bullet) \longrightarrow \prod_\alpha \Phi(P_\alpha^\bullet)$$

is a quasi-isomorphism of complexes in the exact category A, there is the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^{ctr}(\mathsf{F}) \to \mathsf{D}^{ctr}(\mathsf{E})$.

Let us spell out the major steps of the construction of the derived functor (24). Since the DG-functor $\Phi\colon \mathsf{C}^-(\mathsf{P}) \to \mathsf{C}^-(\mathsf{A})$ preserves quasi-isomorphisms, it induces a triangulated functor $\Phi\colon \mathsf{D}^-(\mathsf{P}) \to \mathsf{D}^-(\mathsf{A})$. Taking into account the triangulated equivalence $\mathsf{D}^-(\mathsf{P}) \simeq \mathsf{D}^-(\mathsf{F})$ provided by [44, Proposition A.3.1(a)], we obtain the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^-(\mathsf{F}) \to \mathsf{D}^-(\mathsf{A})$. Our assumptions on Φ imply that this functor $\mathbb{L}\Phi$ takes the full subcategory $\mathsf{D}^b(\mathsf{F}) \subset \mathsf{D}^-(\mathsf{F})$ into the full subcategory $\mathsf{D}^b(\mathsf{E}) \subset \mathsf{D}^b(\mathsf{A}) \subset \mathsf{D}^-(\mathsf{A})$; hence the triangulated functor

$$(25) \qquad\qquad \mathbb{L}\Phi\colon \mathsf{D}^b(\mathsf{F}) \longrightarrow \mathsf{D}^b(\mathsf{E}).$$

Passing from the DG-functor $\Phi\colon \mathsf{C}^-(\mathsf{P}) \to \mathsf{C}^-(\mathsf{A})$ to the induced DG-functor between the DG-categories of unbounded complexes in the given DG-categories, as explained in Section A.2, and restricting to the full DG-subcategories of uniformly bounded bicomplexes, one obtains the DG-functor

$$\Phi_\mathsf{C}\colon \mathsf{C}^-(\mathsf{C}(\mathsf{P})) \longrightarrow \mathsf{C}^-(\mathsf{C}(\mathsf{A})).$$

The functor Φ_C takes acyclic complexes in the exact category $\mathsf{C}(\mathsf{P})$ to acyclic complexes in the exact category $\mathsf{C}(\mathsf{A})$. It also takes the full subcategory $_{\mathsf{C}(\mathsf{F})}\mathsf{C}^{\leq 0}(\mathsf{C}(\mathsf{P})) \subset \mathsf{C}^-(\mathsf{C}(\mathsf{P}))$ into the full subcategory $\mathsf{C}_{\mathsf{C}(\mathsf{E})}^{\leq l_2}(\mathsf{C}(\mathsf{A}))^{\geq -l_1} \subset \mathsf{C}^-(\mathsf{C}(\mathsf{A}))$. So we can apply the construction of the derived functor (25) to the DG-functor Φ_C in place of Φ, and produce a triangulated functor

$$(26) \qquad\qquad \mathbb{L}\Phi_\mathsf{C}\colon \mathsf{D}^b(\mathsf{C}(\mathsf{F})) \longrightarrow \mathsf{D}^b(\mathsf{C}(\mathsf{E})).$$

Using Proposition A.1 and Corollary A.2, one shows that the triangulated functor (26) descends to a triangulated functor (24) between the absolute or conventional derived categories, $\star = $ abs or $\varnothing$. The cases of bounded above or below absolute or conventional derived categories, $\star = +, -,$ abs+, or abs$-$ can be treated as explained in Section A.2. Under the respective assumptions, one can also descend from the absolute derived categories to the coderived or contraderived categories, producing the derived functor (24) for $\star = $ co or ctr.

A.4 – *Deriving adjoint functors*

Let A and B be exact categories containing the images of idempotent endomorphisms of its objects, let $J \subset E \subset A$ be coresolving subcategories in A, and let $P \subset F \subset B$ be resolving subcategories in B.

Let $\Psi \colon C^+(J) \to C^+(B)$ be a DG-functor satisfying the conditions of Section A.1, and let $\Phi \colon C^-(P) \to C^-(A)$ be a DG-functor satisfying the conditions of Section A.3. Suppose that the DG-functors Φ and Ψ are partially adjoint, in the sense that for any two complexes $J^\bullet \in C^+(J)$ and $P^\bullet \in C^-(P)$ there is a natural isomorphism of complexes of abelian groups

(27) $$\mathrm{Hom}_A(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}_B(P^\bullet, \Psi(J^\bullet)),$$

where Hom_A and Hom_B denote the complexes of morphisms in the DG-categories of unbounded complexes C(A) and C(B).

Our aim is to show that the triangulated functor $\mathbb{L}\Phi$ (24) is left adjoint to the triangulated functor $\mathbb{R}\Phi$ (18), for any symbol $\star = $ b, +, -, $\varnothing$, abs+, abs$-$, or abs. When the functors $\mathbb{L}\Phi$ and $\mathbb{R}\Psi$ acting between the categories D^{co} or D^{ctr} are defined (i. e., the related conditions in Sections A.1 and A.3 are satisfied), the former of them is also left adjoint to the latter one.

Our first step is the following lemma.

LEMMA A.3. *In the assumptions above, the induced triangulated functors* $\Phi \colon D^-(P) \to D^-(A)$ *and* $\Psi \colon D^+(J) \to D^+(B)$ *are partially adjoint, in the sense that for any complexes* $J^\bullet \in C^+(J)$ *and* $P^\bullet \in C^-(P)$ *there is a natural isomorphism of abelian groups of morphisms in the unbounded derived categories*

$$\mathrm{Hom}_{D(A)}(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}_{D(B)}(P^\bullet, \Psi(J^\bullet)).$$

PROOF. Passing to the cohomology groups in the DG-adjunction isomorphism (27), one obtains an isomorphism of the groups of morphisms in the homotopy categories

$$\mathrm{Hom}_{Hot(A)}(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}_{Hot(B)}(P^\bullet, \Psi(J^\bullet)).$$

In order to pass from this to the desired isomorphism of the groups of morphisms in the unbounded derived categories, one can notice that for any (unbounded) complex $A^\bullet \in \mathsf{C}(\mathsf{A})$ endowed with a quasi-isomorphism $J^\bullet \to A^\bullet$ of complexes in A there exists a bounded below complex $I^\bullet \in \mathsf{C}^+(\mathsf{J})$ together with a quasi-isomorphism $A^\bullet \to I^\bullet$ of complexes in A. The composition $J^\bullet \to A^\bullet \to I^\bullet$ is then a quasi-isomorphism of bounded below complexes in J. Similarly, for any (unbounded) complex $B^\bullet \in \mathsf{C}(\mathsf{B})$ endowed with a quasi-isomorphism $B^\bullet \to P^\bullet$ of complexes in B there exists a bounded above complex $Q^\bullet \in \mathsf{C}^-(\mathsf{P})$ together with a quasi-isomorphism $Q^\bullet \to B^\bullet$ of complexes in B. The composition $Q^\bullet \to B^\bullet \to P^\bullet$ is then a quasi-isomorphism of bounded above complexes in P.   □

Restricting to the full subcategories $\mathsf{D}^b(\mathsf{E}) \subset \mathsf{D}^+(\mathsf{J}) \subset \mathsf{D}(\mathsf{A})$ and $\mathsf{D}^b(\mathsf{F}) \subset \mathsf{D}^-(\mathsf{P}) \subset \mathsf{D}(\mathsf{B})$, we conclude that the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^b(\mathsf{F}) \to \mathsf{D}^b(\mathsf{E})$ (25) is left adjoint to the derived functor $\mathbb{R}\Psi\colon \mathsf{D}^b(\mathsf{E}) \to \mathsf{D}^b(\mathsf{F})$ (19). Replacing all the exact categories with the categories of unbounded complexes in them, we see that the derived functor $\mathbb{L}\Phi_{\mathsf{C}}\colon \mathsf{D}^b(\mathsf{C}(\mathsf{F})) \to \mathsf{D}^b(\mathsf{C}(\mathsf{E}))$ (26) is left adjoint to the derived functor $\mathbb{R}\Psi_{\mathsf{C}}\colon \mathsf{D}^b(\mathsf{C}(\mathsf{E})) \to \mathsf{D}^b(\mathsf{C}(\mathsf{F}))$ (20).

In order to pass to the desired adjunction between the derived functors $\mathbb{R}\Psi\colon \mathsf{D}^\star(\mathsf{E}) \to \mathsf{D}^\star(\mathsf{F})$ (18) and $\mathbb{L}\Phi\colon \mathsf{D}^\star(\mathsf{F}) \to \mathsf{D}^\star(\mathsf{E})$ (24), it remains to apply the next (well-known) lemma.

LEMMA A.4. *Suppose that we are given two commutative diagrams of triangulated functors*

where the vertical arrows are Verdier quotient functors. Suppose further that the functor $F\colon \mathsf{D}_2 \to \mathsf{D}_1$ is left adjoint to the functor $G\colon \mathsf{D}_1 \to \mathsf{D}_2$. Then the functor $\bar{F}\colon \bar{\mathsf{D}}_2 \to \bar{\mathsf{D}}_1$ is also naturally left adjoint to the functor $\bar{G}\colon \bar{\mathsf{D}}_1 \to \bar{\mathsf{D}}_2$.

PROOF. The adjunction morphisms $F \circ G \to \mathrm{Id}_{\mathsf{D}_1}$ and $\mathrm{Id}_{\mathsf{D}_2} \to G \circ F$ induce adjunction morphisms $\bar{F} \circ \bar{G} \to \mathrm{Id}_{\bar{\mathsf{D}}_1}$ and $\mathrm{Id}_{\bar{\mathsf{D}}_2} \to \bar{G} \circ \bar{F}$.   □

A.5 – *Triangulated equivalences*

The following theorem describes the situation in which the adjoint triangulated functors $\mathbb{R}\Psi$ and $\mathbb{L}\Phi$ turn out to be triangulated equivalences (cf. the proofs of [45, Theorems 4.9 and 5.10], [49, Theorems 3.6 and 4.3], and [48, Theorem 7.6], where this technique was used).

THEOREM A.5. *In the context of Section* A.4, *suppose that the adjoint derived functors* $\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{b}}(\mathsf{E}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{F})$ (19) *and* $\mathbb{L}\Phi\colon \mathsf{D}^{\mathsf{b}}(\mathsf{F}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{E})$ (25) *are mutually inverse triangulated equivalences. Then so are the adjoint derived functors* $\mathbb{R}\Psi\colon \mathsf{D}^{\star}(\mathsf{E}) \to \mathsf{D}^{\star}(\mathsf{F})$ (18) *and* $\mathbb{L}\Phi\colon \mathsf{D}^{\star}(\mathsf{F}) \to \mathsf{D}^{\star}(\mathsf{E})$ (24) *for all the symbols* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs, *and also for any one of the symbols* $\star = \mathsf{co}$ *or* ctr *for which these two functors are defined by the constructions of Sections* A.2–A.3.

Moreover, assume that the adjunction morphisms $\mathbb{L}\Phi(\Psi(J)) \to J$ *and* $P \to \mathbb{R}\Psi(\Phi(P))$ *are isomorphisms in* $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathsf{F})$ *for all objects* $J \in \mathsf{J}$ *and* $P \in \mathsf{P}$. *Then the adjoint derived functors* (18) *and* (24) *are mutually inverse triangulated equivalences for all the symbols* $\star$ *for which they are defined.*

PROOF. A complex of complexes in an exact category G is acyclic if and only if it is termwise acyclic. In other words, one can consider the family of functors $\Theta_{\mathsf{G}}^n \colon \mathsf{C}(\mathsf{G})) \to \mathsf{G}$, indexed by the integers n, assigning to a complex $G^{\bullet}$ its n-th term G^n. Then the family of induced triangulated functors $\Theta_{\mathsf{G}}^n \colon \mathsf{D}(\mathsf{C}(\mathsf{G})) \to \mathsf{D}(\mathsf{G})$ is conservative in total. This means that for any nonzero object $G^{\bullet,\bullet} \in \mathsf{D}(\mathsf{C}(\mathsf{G}))$ there exists $n \in \mathbb{Z}$ such that $\Theta_{\mathsf{G}}^n(G^{\bullet,\bullet}) \neq 0$ in $\mathsf{D}(\mathsf{G})$.

Now the two such functors $\Theta_{\mathsf{E}}^n \colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{E})) \to \mathsf{D}^{\mathsf{b}}(\mathsf{E})$ and $\Theta_{\mathsf{F}}^n \colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{F})) \to \mathsf{D}^{\mathsf{b}}(\mathsf{F})$ form commutative diagrams with the adjoint derived functors (19)–(20) and (25)–(26). Therefore, the adjoint functors (20) and (26) are mutually inverse equivalences whenever so are the adjoint functors (19) and (25). It remains to point out that, in the context of Lemma A.4, the two adjoint functors $\bar{F}$ and $\bar{G}$ are mutually inverse equivalences whenever so are the two adjoint functors F and G.

This proves the first assertion of the theorem, and in fact somewhat more than that. We have shown that the adjunction morphism $\mathbb{L}\Phi(\mathbb{R}\Psi(E^{\bullet})) \to E^{\bullet}$ is an isomorphism in $\mathsf{D}^{\star}(\mathsf{E})$ whenever for every $n \in \mathbb{Z}$ the adjunction morphism $\mathbb{L}\Phi(\mathbb{R}\Psi(E^n)) \to E^n$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$. Now, replacing an object $E \in \mathsf{E}$ by its coresolution $J^{\bullet}$ by objects from J, viewed as an object in $\mathsf{D}^{\star}(\mathsf{E})$ with $\star = +$, we see that it suffices to check that the adjunction morphism is an isomorphism for an object $J \in \mathsf{J}$. Similarly, the adjunction morphism $F^{\bullet} \to \mathbb{R}\Psi(\mathbb{L}\Phi(F^{\bullet}))$ is an isomorphism in $\mathsf{D}^{\star}(\mathsf{F})$ whenever for every $n \in \mathbb{Z}$ the adjunction morphism $F^n \to \mathbb{R}\Psi(\mathbb{L}\Phi(F^n))$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathsf{F})$. Replacing an object $F \in \mathsf{F}$ by

its resolution $P^\bullet$ by objects from P, viewed as an object in $\mathsf{D}^\star(\mathsf{F})$ with $\star = -$, we see that it suffices to check that the adjunction morphism is an isomorphism for an object $P \in \mathsf{P}$. □

References

[1] L. Alonso – A. Jeremías – J. Lipman, *Duality and flat base change on formal schemes* In L. Alonso Tarrío, A. Jeremías López, and J. Lipman (eds.), *Studies in duality on Noetherian formal schemes and non-Noetherian ordinary schemes,* Contemporary Mathematics 244. American Mathematical Society, Providence, R.I., 1999, pp. 3–90. Correction: Proc. Amer. Math. Soc. 131, no. #2 (2003), pp. 351–357.

[2] L. L. Avramov – H.-B. Foxby, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. 75 (1997), no. 2, pp. 241–270.

[3] S. Bazzoni – M. Tarantino, *Recollements from cotorsion pairs,* J. Pure Appl. Algebra 223 (2019), no. 5, pp. 1833–1855.

[4] H. Becker, *Models for singularity categories, Adv. Math.* 254 (2014), pp. 187–232.

[5] R. Bieri, *Homological dimension of discrete groups,* Queen Mary College Mathematical Notes, Mathematics Department, Queen Mary College, London, 1976.

[6] N. Bourbaki, *Elements of mathematics,* Commutative algebra, Addison-Wesley, Reading, MA, 1972; translated from *Éléments de mathématique,* Algèbre commutative, Hermann, Paris, 1964–1969.

[7] D. Bravo – J. Gillespie, *Absolutely clean, level, and Gorenstein AC-injective complexes,* Comm. Algebra 44 (2016), no. 5, pp. 2213–2233.

[8] D. Bravo – J. Gillespie – M. Hovey, *The stable module category of a general ring,* preprint, 2014. arXiv:1405.5768 [math.RA]

[9] D. Bravo – M. A. Pérez, *Finiteness conditions and cotorsion pairs,* J. Pure Appl. Algebra 221 (2017), no. 6, pp. 1249–1267.

[10] T. Bühler, *Exact categories,* Expo. Math. 28 (2010), no. 1, pp. 1–69.

[11] L. W. Christensen, *Semi-dualizing complexes and their Auslander categories,* Trans. Amer. Math. Soc. 353 (2001), no. 5, pp. 1839–1883.

[12] L. W. Christensen – A. Frankild – H. Holm, *On Gorenstein projective, injective, and flat dimensions—A functorial description with applications,* J. Algebra 302 (2006), no. 1, pp. 231–279.

[13] E. Cline – B. Parshall – L. Scott, *Derived categories and Morita theory.* J. Algebra 104 (1986), no. 2, pp. 397–409.

[14] R. Colpi, *Tilting in Grothendieck categories,* Forum Math. 11 (1999), no. 6, pp. 735–759.

[15] R. Colpi – J. Trlifaj, *Tilting modules and tilting torsion theories, J. Algebra* 178 (1995), no. 2, pp. 614–634.

[16] P. Deligne, *Cohomologie à supports propres,* in *Séminaire de Géométrie Algébrique du Bois Marie* 4, Tome 3, Lecture Notes in Mathematics 305, Springer-Verlag, Berlin etc., 1973, pp. 250–480.

[17] G. C. Drummond-Cole – J. Hirsh, *Model structures for coalgebras,* Proc. Amer. Math. Soc. 144 (2016), no. 4, pp. 1467–1481.

[18] D. Dugger – B. Shipley, *K-theory and derived equivalences,* Duke Math. J. 124 (2004), no. 3, pp. 587–617.

[19] A. I. Efimov, *Homotopy finiteness of some DG categories from algebraic geometry,* to appear in J. Eur. Math. Soc.; preprint, 2013. arXiv:1308.0135 [math.AG]

[20] P. C. Eklof – J. Trlifaj, *How to make* Ext *vanish,* Bull. of the London Math. Soc. 33 (2001), no. 1, pp. 41–51.

[21] E. E. Enochs – O. M. G. Jenda – J. A. López-Ramos, *Dualizing modules and n-perfect rings,* Proc. Edinb. Math. Soc. 48 (2005), no. 1, pp. 75–90.

[22] E. E. Enochs – H. Holm, *Cotorsion pairs associated with Auslander categories.* Israel J. Math. 174 (2009), no. 1, pp. 253–268.

[23] H.-B. Foxby, *Gorenstein modules and related modules,* Math. Scand. 31 (1972), pp. 267–284.

[24] A. Frankild – P. Jørgensen, *Foxby equivalence, complete modules, and torsion modules,* J. Pure Appl. Algebra 174 (2002), no. 2, pp. 135–147.

[25] D. Herbera – J. Trlifaj, *Almost free modules and Mittag-Leffler conditions,* Adv. Math. 229 (2012), no. 6, pp. 3436–3467.

[26] H. Holm – D. White, *Foxby equivalence over associative rings,* J. Math. Kyoto Univ. 47 (2007), no. 4, pp. 781–808.

[27] L. Illusie, *Généralités sur les conditions de finitude dans les catégories dérivées,* in *Séminaire de Géométrie Algébrique du Bois Marie* 6, Lecture Notes in Mathematics 225, Springer-Verlag, Berlin etc., 1971, pp. 78–159.

[28] S. Iyengar – H. Krause, *Acyclicity versus total acyclicity for complexes over noetherian rings,* Doc. Math. 11 (2006), pp. 207–240.

[29] A. J. de Jong et al, The Stacks Project. https://stacks.math.columbia.edu/

[30] B. Keller, *Derived categories and their uses,* in M. Hazewinkel (ed.), *Handbook of algebra,* vol. 1, Elsevier/North-Holland, Amsterdam, 1996, pp. 671–701.

[31] H. Krause, *On Neeman's well-generated triangulated categories,* Doc. Math. 6 (2001), pp. 121–126.

[32] H. Krause, *A Brown representability theorem via coherent functors,* Topology 41, no. 4 (2002), pp. 853–861.

[33] H. Krause, *Localization theory for triangulated categories,* in Th. Holm, P. Jørgensen, and R. Rouquier (eds.), *Triangulated categories,* London Mathematical Society Lecture Note Series, 375. Cambridge University Press, Cambridge, 2010, pp. 161–235.

[34] J. Miyachi, *Derived categories and Morita duality theory,* J. Pure Appl. Algebra 128 (1998), no. 2, pp. 153–170.

[35] Y. Miyashita, *Tilting modules of finite projective dimension,* Math. Z. 193 (1986), no. 1, pp. 113–146.

[36] A. Neeman, *The derived category of an exact category,* J. Algebra 135 (1990), no. 2, pp. 388–394.

[37] A. Neeman, *Triangulated categories,* Annals of Mathematics Studies 148, Princeton University Press, Princeton, N.J., 2001.

[38] A. Neeman, *The homotopy category of flat modules, and Grothendieck duality.* Invent. Math. 174 (2008), no. 2, pp. 225–308.

[39] A. Neeman, *The homotopy category of injectives.* Algebra Number Theory 8 (2014), no. 2, pp. 429–456.

[40] M. Porta – L. Shaul – A. Yekutieli, *On the homology of completion and torsion,* Algebr. Represent. Theory 17 (2014), no. 1, pp. 31–67; erratum, ibid. 18 (2015), no. 5, pp. 1401–1405.

[41] M. Porta – L. Shaul – A. Yekutieli, *Cohomologically cofinite complexes,* Comm. Algebra 43 (2015), no. 2, pp. 597–615.

[42] L. Positselski, *Homological algebra of semimodules and semicontramodules,* Semi-infinite homological algebra of associative algebraic structures, Appendix C in collaboration with D. Rumynin, Appendix D in collaboration with S. Arkhipov, Monografie Matematyczne 70, Birkhäuser/Springer Basel, Basel, 2010.

[43] L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence, Mem. Amer. Math. Soc.* 212 (2011), no. 996.

[44] L. Positselski, *Contraherent cosheaves.* Preprint 2012. arXiv:1209.2995 [math.CT]

[45] L. Positselski, *Dedualizing complexes and MGM duality. J. Pure Appl. Algebra* 220 (2016), no. 12, pp. 3866–3909.

[46] L. Positselski, *Coherent rings, fp-injective modules, dualizing complexes, and covariant Serre–Grothendieck duality,* Selecta Math. (New Ser.) 23 (2017), no. 2, pp. 1279–1307.

[47] L. Positselski, *Contraadjusted modules, contramodules, and reduced cotorsion modules,* Moscow Math. J. 17 (2017), no. 3, pp. 385–455.

[48] L. Positselski, *Triangulated Matlis equivalence,* J. Algebra Appl. 17 (2018), no. 4, article ID 1850067, 44 pp.

[49] L. Positselski, *Dedualizing complexes of bicomodules and MGM duality over coalgebras.* Algebr. Represent. Theory 21 (2018), no. 4, pp. 737–767.

[50] L. Positselski – J. Rosický, *Covers, envelopes, and cotorsion theories in locally presentable abelian categories and contramodule categories,* J. Algebra 483 (2017), pp. 83–128.

[51] L. Positselski – J. Šťovíček, *The tilting-cotilting correspondence,* to appear in Internat. Math. Research Notices; preprint, 2017. arXiv:1710.02230 [math.CT]

[52] L. Positselski – J. Šťovíček, ∞-*tilting theory,* Pacific J. Math. 301 (2019), no. 1, pp. 297–334.

[53] J. Rickard, *Morita theory for derived categories,* J. London Math. Soc. 39 (1989), no. 3, pp. 436–456.

[54] J. Rickard, *Derived equivalences as derived functors,* J. London Math. Soc. 43 (1991), no. 1, pp. 37–48.

[55] J. Šaroch – J. Šťovíček, *Singular compactness and definability for* Σ-*cotorsion and Gorenstein modules,* Selecta Math. New Ser. 26 (2020), no. 2, article 23, 40 pp.

[56] J. Šaroch – J. Trlifaj, *Kaplansky classes, finite character and* $\aleph_1$-*projectivity,* Forum Math. 24 (2012), no. 5, pp. 1091–1109.

[57] P. Schenzel, *Proregular sequences, local cohomology, and completion,* Math. Scand. 92 (2003), no. 2, pp. 161–180.

[58] J. Šťovíček, *Deconstructibility and the Hill Lemma in Grothendieck categories,* Forum Math. 25 (2013), no. 1, pp. 193–219.

[59] J. Šťovíček, *Derived equivalences induced by big cotilting modules,* Adv. Math. 263 (2014), pp. 45–87.

[60] J. Šťovíček, *On purity and applications to coderived and singularity categories,* preprint, 2014. arXiv:1412.1615 [math.CT]

[61] T. Wakamatsu, *On modules with trivial self-extensions,* J. Algebra 114 (1988), no. 1, pp. 106–114.

[62] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules, J. Algebra* 134 (1990), no. 2, pp. 298–325.

[63] C. A. Weibel, *An introduction to homological algebra,* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

[64] A. Yekutieli, *A separated cohomologically complete module is complete,* Comm. Algebra 43 (2015), no. 2, pp. 616–622.

[65] A. Yekutieli – J. J. Zhang, *Rings with Auslander dualizing complexes,* J. Algebra 213 (1999), no. 1, pp. 1–51.