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The p-adic Measure on the Orbit of an Element of $\mathbb{C}_p$.

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ABSTRACT - Given a prime number p and the Galois orbit $O(x)$ of an element x of $\mathbb{C}_p$, the topological completion of the algebraic closure of the field of p-adic numbers, we study functionals on the algebra $\mathcal{C}(O(x), \mathbb{C}_p)$ with values in a subfield of $\mathbb{C}_p$.

Introduction.

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic valuation. Let $O(x)$ denote the orbit of an element $x \in \mathbb{C}_p$, with respect to the Galois group $G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$. We are interested in the behavior of rigid analytic functions defined on $E(x) = (\mathbb{C}_p \cup \{\infty\}) \setminus O(x)$, the complement of $O(x)$. In the present paper we provide several results concerned with functionals defined on the algebra $\mathcal{C}(O(x), \mathbb{C}_p)$ with values in a suitable subfield of $\mathbb{C}_p$. This investigation is needed in the more general attack on the problem of explicit description of rigid analytic functions on $E(x)$, since, as we shall see below, there is a close relationship between these functionals and certain classes of rigid analytic functions. The paper consists of seven sections. The first one contains notations and some basic results. The

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second section is concerned with linear functions and functionals on various fields. Section 4 studies p-adic measures and p-adic equivariant measures on $O(x)$. Theorem 1 shows that the study of all functionals on $\mathcal{C}(O(x), \mathbb{C}_p)$ can be reduced to the study of all functionals on $\mathcal{C}_G(O(x), \mathbb{C}_p)$. Then, using results from the previous sections, the functionals on $\mathcal{C}(O(x), \mathbb{C}_p)$ are closely related to the trace. Theorems 2 and 3 are useful complements to Theorem 1. In order to further clarify the relation between functionals and rigid analytic functions, in Section 5 we investigate the Cauchy transform of a function with respect to a measure. In the last section we present an analogue at an important theorem of Barsky [B], which relates the measures on $O(x)$ with a suitable class of rigid analytic functions on the complement of $O(x)$. We remark that some of the results of this paper can be extended to a wider class of compact subsets of $\mathbb{C}_p$.

1. Notations and basic results.

1. Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ (see [Ar], [APZ1], [APZ2]). Denote by G the Galois group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. One knows that G is canonically isomorphic to $G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, the group of all continuous automorphisms of $\mathbb{C}_p$. We shall identify these two groups.

For any closed subgroup H of G denote $Fix(H) = \{x \in \mathbb{C}_p : \sigma(x) = x$ for all $\sigma \in H\}$. Then $Fix(H)$ is a closed subfield of $\mathbb{C}_p$. If $x \in \mathbb{C}_p$, denote $H(x) = \{\sigma \in G : \sigma(x) = x\}$. Then $H(x)$ is a subgroup of G, and $Fix(H(x)) = \widetilde{\mathbb{Q}_p[x]}$ is the closure of the polynomial ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$. We say that x is a *topological generic element* of $\widetilde{\mathbb{Q}_p[x]}$. Moreover, by [APZ1] one knows that any closed subfield K of $\mathbb{C}_p$ has a topological generic element, i.e. there exists $x \in K$ such that $K = \widetilde{\mathbb{Q}_p[x]}$.

2. Let $x \in \mathbb{C}_p$. Denote $O(x) = \{\sigma(x) : \sigma \in G\}$ the orbit of x. The map $\sigma \rightsquigarrow \sigma(x)$ from G to $O(x)$ is continuous, and it defines a homeomorphism from $G/H(x)$ (endowed with the quotient topology) to $O(x)$ (endowed with the induced topology from $\mathbb{C}_p$) (see [APZ1]). In such a way, $O(x)$ is a closed compact and totally disconnected subspace of $\mathbb{C}_p$, and the group G acts continuously on $O(x)$: if $\sigma \in G$, $\tau(x) \in O(x)$ then $\sigma \star \tau(x) = (\sigma\tau)(x)$. One has the following result:

PROPOSITION 1. 1) *The subfield $\widetilde{\mathbb{Q}_p[x]}$ is canonically isomorphic to the set of all equivariant continuous functions $f : O(x) \to \mathbb{C}_p$, i.e. the contin-*

uous functions which verify the condition: $f(\sigma \star y) = \sigma(f(y))$ *for all* $\sigma \in G$ *and* $y \in O(x)$.

2) *There exists a family* $\{M_n(x)\}_{n\geq 0}$ *of polynomials in* $\mathbb{Q}_p[x]$ *such that*

i) $\deg M_n(x) = n$ *for all* $n \geq 0$,

ii) $\dfrac{1}{p} < |M_n(x)| \leq 1$,

iii) *Any element* $f \in \widetilde{\mathbb{Q}_p[x]}$ *can be written uniquely in the form:* $f = \sum_{n\geq 0} a_n M_n(x)$ *where* $\{a_n\}_n$ *is a sequence of elements in* $\mathbb{Q}_p$ *such that* $\lim_n a_n = 0$. *Moreover one has:* $|f| = \sup_{n\geq 0} |a_n M_n(x)|$.

3) *If* $K_x = \widetilde{\mathbb{Q}_p[x]} \cap \overline{\mathbb{Q}}_p$, *then* $\widetilde{K_x} = \widetilde{\mathbb{Q}_p[x]}$ *and* $Gal(\overline{\mathbb{Q}}_p/K_x)$ *is canonically isomorphic to* $H(x)$.

2. Linear functions and functionals.

1. Let K/k be a (not necessarily finite) algebraic extension, and $\{K_n\}_{n\geq 0}$, $K_0 = k$, $K_n \subset K_{n+1}$ for any n, a family of subfields of K, finite over k such that $\cup_n K_n = K$. Denote by $\mathcal{L}(K/k)$ the set of all k-linear maps of K into k. Then $\mathcal{L}(K/k)$ is in a canonical way a K-vector space. Namely if $\varphi \in \mathcal{L}(K/k)$ and $x \in K$ then $x\varphi$ is the linear map defined by $(x\varphi)(a) = \varphi(xa)$ for all $a \in K$.

Now assume that k is of zero characteristic and for all $a \in K$ denote by $Tr_{K/k}(a)$ or simply by $Tr(a)$ the element of k defined as follows: if $k \subseteq K' \subset K$ is a finite intermediate extension and $a \in K'$, then $Tr(a) = \dfrac{1}{[K' : k]} tr_{K'/k}(a)$, where $tr_{K'/k}(a)$ denotes the usual relative trace of a over k. It is clear that $Tr(a)$ is independent of the choice of K'.

If K/k is a finite extension, then for any $\varphi \in \mathcal{L}(K/k)$ there exists a unique $\alpha \in K$ such that $\varphi = \alpha Tr$. Now assume $[K : k] = \infty$ and let φ_n be the restriction of φ to K_n. Then $\varphi_n \in \mathcal{L}(K_n/k)$ and so there exists a unique element $\alpha_n \in K_n$ such that for all $a \in K_n$ one has:

$$\varphi_n(a) = Tr(\alpha_n a). \tag{1}$$

Thus for any $a \in K_n$, one has: $\varphi_{n+1}(a) = \varphi_n(a)$, and so by (1) one has: $Tr(\alpha_{n+1} a) = Tr(\alpha_n a)$. This means that

$$\frac{1}{[K_{n+1} : K_n]} tr_{K_{n+1}/K_n}(\alpha_{n+1}) = \alpha_n, \quad n \geq 0. \tag{2}$$

Conversely, let $A = \{\alpha_n\}_n$ be a sequence of elements of K such that $\alpha_n \in K_n$ for all $n \geq 0$, and that condition (2) is accomplished. Then for any $n \geq 0$ denote by $\varphi_n^A : K_n \to k$ the map $\varphi_n^A(a) = Tr(\alpha_n a)$, $a \in K_n$. By (2) there results that one can define a linear map $\varphi^A : K \to k$ such that the restriction of φ^A to K_n is just φ_n^A. Denote $P = \lim \left(K_n, \frac{1}{[K_{n+1} : K_n]} tr_{K_{n+1}/K_n} \right)_{n \geq 0}$. Then by the above considerations there results that the map $A \rightsquigarrow \varphi^A$ defines a k-isomorphism between the k-vector space P and $\mathcal{L}(K/k)$.

2. Now assume that K is endowed with an ultrametric absolute value $|\cdot| : K \to \mathbb{R}$. We consider the functionals $\varphi : K \to k$, i.e. all k-linear maps which are continuous.

A linear map $\varphi : K \to k$ is continuous if and only if there exists a positive real number M such that $|\varphi(a)| \leq M|a|$, for any $a \in K$. Since φ is defined uniquely by a sequence $A = \{\alpha_n\}_n$, $\alpha_n \in K$ for all $n \geq 0$, which verify condition (2), then φ is continuous if and only if there exists a positive real number M such that

$$|Tr(\alpha_n y)| \leq M|y|,$$

for all n, and all $y \in K_n$. Now one can write $y = \frac{x}{\alpha_n}$, where $x \in K_n$, and so the above condition can be written as:

$$|Tr(x)| \cdot |\alpha_n| \leq M|x|,$$

for all $n \geq 0$ and any $x \in K_n$.

Let $L = K_n$, be fixed. We want to relate $\sup_{x \in K_n} \frac{tr(x)}{[L : k]|x|}$ and $\frac{\mathcal{D}_L}{[L : k]}$, where $\mathcal{D}_L$ is the different of L with respect to k (see [Ar]).

REMARK 1. Any $x \in K_n^\times$ can be represented as $x = p^{l_x}\alpha$, where $1 \geq |\alpha| > \frac{1}{p}$ and l_x is integer. Then one has: $\max_{x \in K_n} \frac{|tr(x)|}{|x|} = \max_{x \in K_n, \frac{1}{p} < |x| \leq 1} \frac{|tr(x)|}{|x|}$, and the quotient between this number and $M_{K_n} = \max\left\{ |tr(x)| : x \in \in K_n, \frac{1}{p} < |x| \leq 1 \right\}$ is a real number whose module belongs to the real interval $\left[1, \frac{1}{p}\right)$.

If $\alpha \in K_n$ then $|\alpha| \leq |\mathcal{D}_{K_n}|^{-1}$ if and only if $|tr(\alpha O_{K_n})| \leq 1$. (Here $|tr(\alpha O_{K_n})| = \max\{|tr(\alpha x)| : x \in O_{K_n}$, the ring of integers of $K_n\}$.) Let us denote by m_{K_n} the integer part of $\log_{1/p}|\mathcal{D}_{K_n}|$. If $\alpha = p^{-m_{K_n}}$, then $|tr(p^{-m_{K_n}} O_{K_n})| \leq 1$, or equivalently $|tr(O_{K_n})| \leq (1/p)^{m_{K_n}}$. This means that $M_{K_n} \leq (1/p)^{m_{K_n}}$.

Let $\alpha = p^{-(m_{K_n}+1)}$. Then there exists $y \in O_{K_n}$ such that $|trp^{-(m_{K_n}+1)}y| > 1$ or equivalently $\log_{1/p}|try| < m_{K_n} + 1$. Therefore one obtains

$$\log_{1/p}(M_{K_n}) < m_{K_n} + 1 \text{ or } \left(\frac{1}{p}\right)^{m_{K_n}+1} < M_{K_n}.$$

Thus we have $m_{K_n} \leq \log_{1/p}(M_{K_n}) < m_{K_n} + 1$. In conclusion, one has the following result:

PROPOSITION 2. *The linear mapping $\varphi : K \to k$, $\varphi = \varphi^A$, is continuous if and only if there exists a positive real number M such that:*

$$\frac{|\mathcal{D}_{K_n}||\alpha_n|}{[K_n : k]} \leq M, \text{ for all } n \geq 1. \tag{3}$$

REMARK 2. Let $x \in \mathbb{C}_p$ such that $Tr(x)$ is defined (see [APZ2]). One has the following result. Let $K = \widetilde{\mathbb{Q}_p[x]} \cap \overline{\mathbb{Q}}_p$, and $K = \cup_n K_n$ be the union of a filtered family of finite extensions of $\mathbb{Q}_p$. The following assertions are equivalent:

1) The sequence $\{|Tr(M_n(x))|\}_n$ is bounded (see Proposition 1).

2) The sequence $\left\{\frac{|\mathcal{D}_{K_n/\mathbb{Q}_p}|}{|[K_n : \mathbb{Q}_p]|}\right\}_n$ is bounded.

3) The linear map $Tr : K \to \mathbb{Q}_p$ is continuous.

These results are related to some results from [APP].

3. Denote by $\mathcal{A}$ the set of all elements $A \in P$ which verify condition (3) for a suitable $M > 0$. It is easy to see that $\mathcal{A}$ is a k-vector subspace of P. Denote by K' the k-vector space of all k-functionals on K. By the above considerations one has:

PROPOSITION 3. *The mapping $A \rightsquigarrow \varphi^A$ defines an isomorphism of k-vector spaces between $\mathcal{A}$ and K'.*

Now let $A = \{\alpha_n\}_n$ be an element of $\mathcal{A}$ such that $\alpha_n \neq 0$ for all $n \geq 0$. Let $\psi \in K'$, $\psi = \varphi^B$, $B = \{\beta_n\}_n \in \mathcal{A}$. Denote $u_n = \dfrac{\beta_n}{\alpha_n}$, $n \geq 0$. We assert that the sequence $\{u_n\}_n$ is bounded, i.e. there exists a real number $M > 0$ such that $|u_n| \leq M$ for all $n \geq 0$. Indeed, assume that there exists a subsequence $\{u_q n\}$ such that $|u_{q_n}| \to \infty$. Then $|1/u_{q_n}| \to 0$ and so $\lim_n \psi(1/u_{q_n}) = 0$. But $\psi(1/u_{q_n}) = Tr(\alpha_{q_n})$, and thus $\varphi^A(1) = 0$, a contradiction.

If for any $n \geq 0$, one denotes $\varphi_n = u_n \varphi^A$, then one has: $\psi(x) = \lim_n \varphi_n(x)$ for any $x \in K$. Indeed, for n large enough one has: $\varphi_n(x) = (u_n \varphi^A)(x) = = \varphi^A(u_n x) = Tr(\alpha_n u_n x) = Tr(\beta_n x) = \psi(x)$. This shows that the sequence $\{\varphi_n\}_n$ converges pointwise to ψ in K' and so by the Banach-Steinhaus Theorem (see [R]) one has $\psi = \lim_n \varphi_n = \lim_n (u_n \varphi^A)$.

PROPOSITION 4. *Notations and hypotheses are as above. Let $\varphi = \varphi^A$. Denote by S_A the family of all sequences $u = \{u_n\}_n$ such that:*

1) $u_n \in K_n$ *for all* $n \geq 1$.
2) *The sequence of real numbers* $|u_n|$ *is bounded.*
3) *One has:* $\dfrac{1}{[K_{n+1} : K_n]} tr_{K_{n+1}/K_n}(u_{n+1}) = u_n$, $n \geq 0$.

For any $u = \{u_n\}_n \in S_A$, denote by u_φ the mapping defined by: $(u_\varphi)(x) = \lim_n \varphi(u_n x)$. Then $u_\varphi \in K'$ and if $\varphi \neq 0$, then for any $\psi \in K'$ there exists a unique $u \in S_A$ such that $u_\varphi = \psi$.

We leave the details to the reader.

3. Integration on $C(X, K)$.

1. Let K be a field which is complete with respect to an ultrametric absolute value $|\cdot|$, and let X be a compact ultrametric space. Denote by $C(X, K)$ the K-algebra of all continuous functions from X to K (see [Sch]). Also, denote by $\Omega(X)$ the collection of all the open compact subspaces of X. By a *K-valued measure* on X we mean a function $\mu : \Omega(X) \to K$ such that:

(i) If $U, V \in \Omega(X)$, $U \cap V = \emptyset$, then $\mu(U \cup V) = \mu(U) + \mu(V)$ (additivity).

(ii) $\|\mu\| = \sup\{|\mu(U)| : U \in \Omega(X)\} < \infty$ (boundedness).

The K-valued measures on K form a normed vector space $M(X, K)$ under the obvious operations and with the norm $\|\cdot\|$ defined by (ii).

The following statement (see [Sch]) can be viewed as the ultrametric analog of Riesz representation theorem:

PROPOSITION 5. *For each $\varphi \in C(X, K)'$ (the space of all functionals on $C(X, K)$), the mapping $U \rightsquigarrow \varphi(\xi_U) = \mu_\varphi(U)$, $U \in \Omega(X)$, is a measure μ_φ on X (here ξ_U denotes the characteristic function of U). The mapping $\varphi \rightsquigarrow \mu_\varphi$ is a K-linear isometry of $C(X, K)'$ onto $M(X, K)$.*

4. Equivariant measures on $O(x)$.

1. Let $x \in \mathbb{C}_p$. The subset of $C(O(x), \mathbb{C}_p)$ consisting of all equivariant elements (see Proposition 1) is denoted by $C_G(O(x), \mathbb{C}_p)$. The mapping $f \rightsquigarrow f(x)$ defines an isomorphism between $C_G(O(x), \mathbb{C}_p)$ and $\widetilde{\mathbb{Q}_p[x]}$. We shall identify these $\mathbb{Q}_p$-algebras via this isomorphism.

Let $f \in C(O(x), \mathbb{C}_p)$ and $\sigma \in G$. Denote by $\sigma \star f$ the function defined by: $(\sigma \star f)(y) = f(\sigma^{-1}(y))$ for all $y \in O(x)$. In this way the group G acts continuously on the algebra $C(O(x), \mathbb{C}_p)$.

2. Let K be a closed and normal subfield of $\mathbb{C}_p$ (i.e. for any $\sigma \in G$, one has $\sigma(K) = K$). By an *equivariant K-functional* on $C(O(x), \mathbb{C}_p)$ we mean a linear and continuous map $\varphi : C(O(x), \mathbb{C}_p) \to K$ such that $\varphi(\sigma \star f) = \sigma(\varphi(f))$ for all $\sigma \in G$ and all $f \in C(O(x), \mathbb{C}_p)$. It is easy to see that by the correspondence stated in Proposition 5, the equivariant K-functionals on $C(O(x), \mathbb{C}_p)$ are in one to one correspondence with the so called *equivariant measures* on $O(x)$, i.e. the elements $\mu \in M(O(x), \mathbb{C}_p)$, such that $\mu(\sigma U) = \sigma(\mu(U))$ for all $U \in \Omega(O(x))$ and $\sigma \in G$. In this paper we consider mainly the case $K = \mathbb{Q}_p$ and shall investigate the above correspondence between functionals and measures.

THEOREM 1. *Any $\mathbb{Q}_p$-functional $\varphi : C_G(O(x), \mathbb{C}_p) \to \mathbb{Q}_p$ can be uniquely extended to a $\mathbb{C}_p$-functional $\widetilde{\varphi} : C(O(x), \mathbb{C}_p) \to \mathbb{C}_p$ and this functional provides us with a measure μ_φ on $O(x)$ with values in $\overline{\mathbb{Q}}_p$. The measure μ_φ is equivariant.*

Next, we recall an important criterion for the existence of a measure with given properties.

PROPOSITION 6 (The abstract Kummer congruences, [Ka]). *Let $\mathcal{X}$ be a compact ultrametric space, let O_p be the ring of integers of $\mathbb{C}_p$ and let $\{f_i\}$ be a system of continuous functions from $\mathcal{C}(\mathcal{X}, O_p)$. If the $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$ and $\{\lambda_i\}$ is an arbitrary system of elements of O_p, then the following assertions are equivalent:*

a) *There is $\mu \in \mathcal{M}(X, O_p)$ with the property $\int_{\mathcal{X}} f_i \, \mathrm{d}\mu = \lambda_i$.*

b) *For an arbitrary choice of elements $\gamma_i \in \mathbb{C}_p$ almost all of which vanish,*

$$\sum_i \gamma_i f_i(x) \in p^n O_p \ \text{ for all } \ x \in \mathcal{X} \ \text{ implies } \ \sum_i \gamma_i \lambda_i \in p^n O_p.$$

Now, let $\{a_n\}_{n\geq 0}$ be a bounded sequence of $\mathbb{Q}_p$. Let us define $\varphi : C_G(O(x), \mathbb{C}_p) \to \mathbb{Q}_p$ such that $\varphi(M_n(x)) = a_n$. As we know from Proposition 1 any $f \in C_G(O(x), \mathbb{C}_p)$ can be written as $f = \sum_{n\geq 0} \alpha_n M_n(x)$, with $\alpha_n \to 0$. We define $\varphi(f) := \sum_{n\geq 0} \alpha_n a_n$. Using Theorem 1 we have a similar result of abstract Kummer congruences. More precisely we have:

Proposition 7. *Let $\{a_n\}_{n\geq 0}$ be a bounded sequence of $\mathbb{Q}_p$. There exists a unique functional $\varphi : C(O(x), \mathbb{C}_p) \to \mathbb{C}_p$ such that $\varphi(M_n(x)) = a_n$, for any $n \geq 0$.*

Theorem 2. *If μ is an equivariant measure on $O(x)$ with values in $\overline{\mathbb{Q}}_p$ then the mapping $f \rightsquigarrow \int_{O(x)} f(t)d\mu(t)$ is a functional on $C_G(O(x), \mathbb{C}_p)$ with values in $\mathbb{Q}_p$.*

Proof of Theorem 1. We use the notations from Proposition 1. For any $s \geq 0$ let us denote $A_s = \varphi(M_s(x))$. Then for any $u = \sum_s a_s M_s(x)$, one has $\varphi(u) = \sum_s a_s A_s$. Since the equality is true for any $u \in \widetilde{\mathbb{Q}_p[x]}$, there results that the sequence $\{A_s\}_s$ of p-adic numbers is bounded and so the set of all $\mathbb{Q}_p$-functionals on $\widetilde{\mathbb{Q}_p[x]} = C_G(O(x), \mathbb{C}_p)$ is in one to one correspondence with the set of bounded sequences $\{A_s\}_s$ of p-adic numbers.

Now let U be an open ball on $O(x)$, $x \in U$, and denote $H(U) = \{\sigma \in G : \sigma U = U\}$. Then $H(U)$ is a subgroup of G, and denote by K_U the subfield of $\overline{\mathbb{Q}}_p$ fixed by $H(U)$. Since $H(x) \subseteq H(U)$, then $K_U \subseteq K_x = \widetilde{\mathbb{Q}_p[x]} \cap \overline{\mathbb{Q}}_p$. It is clear that $[G : H(U)] < \infty$ and let $S_n = \{\sigma_1 = e, \sigma_2, \ldots, \sigma_n\}$ be a system of representatives for the right cosets of G with respect to $H(U)$. Also choose $\alpha \in \overline{\mathbb{Q}}_p$ such that $K_u = \mathbb{Q}_p(\alpha)$. It is clear that the elements $\{\sigma_1(\alpha), \ldots, \sigma_n(\alpha)\}$ are distinct, and the balls $\{\sigma_i(U)\}_{1\leq i\leq n}$ are pair-wise disjoint and cover $O(x)$. Let us put

$$
(4) \qquad \begin{cases} f_0^{(U)} = \sum_{\sigma_i \in S_n} \sigma_i(1)\xi_{\sigma_i(U)} = \xi_{O(T)} \\ f_1^{(U)} = \sum_{\sigma_i \in S_n} \sigma_i(\alpha)\xi_{\sigma_i(U)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^{(U)} = \sum_{\sigma_i \in S_n} \sigma_i(\alpha^{n-1})\xi_{\sigma_i(U)}. \end{cases}
$$

We assert that the function $f_i^{(U)}$ is *equivariant* for all i, $0 \le i < n$, i.e. $f_i^{(U)}(\sigma(y)) = \sigma(f_i^{(U)}(y))$ for all $y \in O(x)$ and any $\sigma \in G$. For that it is enough to assume $i = 1$. Firstly, we remark that one has: $f_i^{(U)}(y) = \sigma_i(y)$ for $y \in \sigma_i(U)$, $1 \le i \le n$. Now we can write: $f_{\sigma,1}^{(U)} = \sigma f_1^{(U)}(\sigma^{-1}(y)) = \sigma\sigma_i(\alpha)$ if $\sigma^{-1}(y) \in \sigma_i U$ (or equivalently $x \in \sigma\sigma_i(U)$).

It is clear that the following permutations

$$(5) \qquad \begin{pmatrix} \sigma_1 U & \sigma_2 U & \dots & \sigma_n U \\ \sigma\sigma_1 U & \sigma\sigma_2 U & \dots & \sigma\sigma_n U \end{pmatrix}, \begin{pmatrix} \sigma_1(\alpha) & \sigma_2(\alpha) & \dots & \sigma_n(\alpha) \\ \sigma\sigma_1(\alpha) & \sigma\sigma_2(\alpha) & \dots & \sigma\sigma_n(\alpha) \end{pmatrix}$$

coincide, so the applications

$$\begin{pmatrix} \sigma_1 U & \sigma_2 U & \dots & \sigma_n U \\ \sigma_1(\alpha) & \sigma_2(\alpha) & \dots & \sigma_n(\alpha) \end{pmatrix}, \begin{pmatrix} \sigma\sigma_1 U & \sigma\sigma_2 U & \dots & \sigma\sigma_n U \\ \sigma\sigma_1(\alpha) & \sigma\sigma_2(\alpha) & \dots & \sigma\sigma_n(\alpha) \end{pmatrix}$$

also coincide. This shows that $f_1^{(U)} = f_{\sigma,1}^{(U)}$, i.e. f_1 is equivariant.

Furthermore (4) is a Cramer system whose determinant is different from zero. It follows that

$$(6) \qquad \sum_{i=0}^{n-1} \mathbb{Q}_p(\alpha) f_i^{(U)} = \sum_{i=1}^{n} \mathbb{Q}_p(\alpha)\xi_{\sigma_i U},$$

and this sum is direct. This shows that the functional φ can be extended uniquely to a functional

$$\widetilde{\varphi} : \mathcal{C}_G(O(x), \mathbb{C}_p) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_p.$$

According to ([Sch], page 273), the $\mathbb{C}_p$-algebra $\mathcal{C}(O(x), \mathbb{C}_p)$ has an orthonormal basis consisting of characteristic functions of balls. Then the $\overline{\mathbb{Q}}_p$-subalgebra $\mathcal{C}_G(O(T), \mathbb{C}_p) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ is dense in the $\mathbb{C}_p$-algebra $\mathcal{C}(O(T), \mathbb{C}_p)$. But then $\widetilde{\varphi}$ can be extended uniquely to a $\mathbb{C}_p$-functional:

$$\widetilde{\varphi} : \mathcal{C}(O(T), \mathbb{C}_p) \to \mathbb{C}_p.$$

Then by Riesz's Theorem (Proposition 5) there exists a unique measure μ_φ on $O(x)$ associated to $\widetilde{\varphi}$. It is clear that for any ball U in $O(x)$, $\mu_\varphi(U) = \widetilde{\varphi}(\xi_U)$, can be obtained from (4) by applying the functional $\widetilde{\varphi}$. We must show that the measure μ_φ is equivariant. For that let as above U be an open ball of $O(x)$ which contains x. Denote $A = (a_{ij})$ the $n \times n$ matrix, where $a_{ij} = \sigma_i(\alpha^j)$, $1 \le i \le n$, $0 \le j < n-1$. Then by (4) there results:

$$\begin{pmatrix} \xi_{\sigma_1(U)} \\ \vdots \\ \xi_{\sigma_n(U)} \end{pmatrix} = A^{-1} \begin{pmatrix} f_0^{(U)} \\ \vdots \\ f_{n-1}^{(U)} \end{pmatrix}$$

and so by applying $\widetilde{\varphi}$ one has:

$$\begin{pmatrix} \mu_\varphi(\sigma_1(U)) \\ \vdots \\ \mu_\varphi(\sigma_n(U)) \end{pmatrix} = A^{-1} \begin{pmatrix} \varphi(f_0^{(U)}) \\ \vdots \\ \varphi(f_{n-1}^{(U)}) \end{pmatrix}.$$

Now for $\sigma \in G$ denote by A_σ the matrix obtained from A by applying σ to all the entries of A. Then $(A_\sigma)^{-1} = A_\sigma^{-1}$.

Since $f_i^{(U)}$ are equivariant functions, one has: $f_i^{(\sigma(U))} = f_i^{(U)}$ for all $\sigma \in G$. Then one obtains $\mu_\varphi(\sigma U) = \sigma\mu_\varphi(U)$, for all $\sigma \in G$, as claimed. □

PROOF OF THEOREM 2. Let μ be an equivariant measure on $O(x)$ with values in $\overline{\mathbb{Q}}_p$. Denote by $\varphi : \mathcal{C}(O(x), \mathbb{C}_p) \to \mathbb{C}_p$ the functional associated to μ. We must show that for any $f \in \mathcal{C}_G(O(x), \mathbb{C}_p)$ one has $\varphi(f) \in \mathbb{Q}_p$. Since any element $f \in \mathcal{C}_G(O(x), \mathbb{C}_p) = \widetilde{\mathbb{Q}_p[x]}$ is a limit of a sequence $\{a_n\}_n$ of elements of K_x (see [APZ1]) it is enough to show that for $a \in K_x$, one has $\varphi(a) \in \mathbb{Q}_p$. But this follows by the proof of Proposition 5 (see [Sch]) since μ is equivariant. Some details are left to the reader (see also the proof of Theorem 3). □

3. By the above considerations there results that if $\varphi : \mathcal{C}_G(O(x), \mathbb{C}_p) \to \mathbb{Q}_p$ is a functional then the associated measure μ_φ (Proposition 5) takes values in $K_x = \widetilde{\mathbb{Q}_p[x]} \cap \overline{\mathbb{Q}}_p$. At this point we describe the equivariant measures on $O(x)$ with values in $\mathbb{Q}_p$.

THEOREM 3. *For the element $x \in \mathbb{C}_p$ the following assertions are equivalent:*

1) *There exists a functional $\varphi : \mathcal{C}_G(O(x), \mathbb{C}_p) \to \mathbb{Q}_p$ such that for any open ball U of $O(x)$ one has: $\mu_\varphi(U) \in \mathbb{Q}_p$.*

2) *The function $Tr : K_x \to \mathbb{Q}_p$ defined by $Tr(\alpha) = \dfrac{1}{\deg \alpha} \cdot tr_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}(\alpha)$ is continuous and one has: $\varphi(\alpha) = Tr(\alpha)$ for all $\alpha \in K_x$.*

PROOF. Let $\alpha \in K_x$, and $\alpha = \sigma_1(\alpha), \cdots, \sigma_n(\alpha)$ be all the conjugates of α over $\mathbb{Q}_p$, $n = \deg(\alpha)$. Since $\alpha \in \widetilde{\mathbb{Q}_p[x]}$, denote $\overline{\alpha} : O(x) \to \mathbb{C}_p$ the local constant function defined by $\overline{\alpha}(\sigma(x)) = \sigma(\alpha)$. One has: $\varphi(\alpha) = \varphi(\overline{\alpha})$. The elements $\sigma_i(x)$, $1 \leq i \leq n$ are all distinct and let $\varepsilon > 0$ be a real number such that all the balls $B(\sigma_i(x), \varepsilon)$ are pairwise disjoint. Denote $H(x, \varepsilon) = \{\sigma \in G : |\sigma(x) - x| < \varepsilon\}$. Then $H(x, \varepsilon)$ is a subgroup of G of finite index. If $N = [G : H(x, \varepsilon)]$ and $\{\tau_i\}_{1 \leq i \leq N}$ is a set of right representatives for the cosets of G with respect to $H(x, \varepsilon)$, then the balls $\{B(\tau_i(x), \varepsilon)\}_{1 \leq i \leq N}$ cover $O(x)$

and are any two disjoint. One has $B(\tau_i(x),\varepsilon)=\tau_i(B(x,\varepsilon))$, and so by hypothesis there results that $\mu_\varphi(B(\tau_i(x),\varepsilon))=\mu_\varphi(B(x,\varepsilon))$. Since $\varphi(1)=1$, one has:

$$\mu_\varphi(\sigma(x))=1=\sum_i \mu_\varphi(B(\tau_i(x),\varepsilon))=N\mu_\varphi(B(x,\varepsilon)).$$

According to our choice of ε, there results that on any ball $B(\tau_i(x),\varepsilon)$ the function $\overline{\alpha}$ is constant and $N=k\deg(\alpha)$, where k is an integer. Then

$$\varphi(\overline{\alpha})=\sum_i \overline{\alpha}(\tau_i(x))\mu_\varphi(B(\tau_i(x),\varepsilon)).$$

Since for exactly k balls $B(\tau_i(x),\varepsilon)$ the function $\overline{\alpha}$ takes the same values $\sigma_j(\alpha)$, by the above considerations one further obtains

$$\varphi(\alpha)=\varphi(\overline{\alpha})=\sum_{j=1}^{n} k\sigma_j(\alpha)\mu\varphi(B(x,\varepsilon))=nk\mu_\varphi(B(x,\varepsilon)\cdot\frac{1}{n}\left(\sum_{j=1}^{n}\sigma_j(\alpha)\right)=Tr(\alpha).$$

The implication from 2) to 1) is left to the reader. □

REMARK 3. Let φ be as in Theorem 3. By the above considerations there results that for any $\varepsilon>0$, one has: $\mu_\varphi(B(x,\varepsilon))=\dfrac{1}{N}$, where $N=[G:H(x,\varepsilon)]$. This shows that the p-adic measure μ_φ coincides with the p-adic Haar measure π_x defined in [APZ2].

Under the hypothesis of Theorem 3 there results that if $\mathbb{Q}_p\subset K_1\subset\cdots\subset K_n\subset\cdots\subset K_x$ is a tower of finite extensions of $\mathbb{Q}_p$ such that $\cup_n K_n=K_x$, then by considerations from Section 3, one has $\varphi=\varphi^A$, when $A=\left\{\dfrac{1}{[K_n:\mathbb{Q}_p]}\right\}_{n\geq 1}$.

5. Cauchy Transforms on $O(x)$.

1. Let $x\in\mathbb{C}_p$. For any real number $\varepsilon>0$ denote $B(x,\varepsilon)=$ $=\{y\in\mathbb{C}_p:|x-y|<\varepsilon\}$ and $B[x,\varepsilon]=\{y\in\mathbb{C}_p:|x-y|\leq\varepsilon\}$. Also denote $E(x,\varepsilon)=\{y\in\mathbb{C}_p:|y-t|\geq\varepsilon,\ \text{for all } t\in O(x)\}$. The complement of $E(x,\varepsilon)$ in $\mathbb{C}_p\cup\{\infty\}$ is denoted by $V(x,\varepsilon)$. Both sets $E(x,\varepsilon)$ and $V(x,\varepsilon)$ are open and closed, and one has: $\cap_\varepsilon V(x,\varepsilon)=O(x)$. Denote $E(x)=\cup_\varepsilon E(x,\varepsilon)=$ $=\mathbb{C}_p\cup\{\infty\}\setminus O(x)$.

For any $x\in\mathbb{C}_p$ and $\varepsilon>0$ denote $H(x,\varepsilon)=\{\sigma\in G:|\sigma(x)-x|<\varepsilon\}$ and $H[x,\varepsilon]=\{\sigma\in G:|\sigma(x)-x|\leq\varepsilon\}$. Let S_ε (respectively $\overline{S}_\varepsilon$) be a complete system of representatives for the right cosets of G with respect to $H(x,\varepsilon)$ (respectively $H[x,\varepsilon]$). Then $E(x,\varepsilon)=\cup_{\sigma\in S_\varepsilon}B(\sigma(x),\varepsilon)$.

2. Let $\varphi : C(O(x),\mathbb{C}_p)\to K$ be a functional, where K is a suitable closed subfield of $\mathbb{C}_p$. Also denote by μ the measure associated to φ according to Proposition 5. Then for any $f\in C(O(x),\mathbb{C}_p)$ one has:

$$\varphi(f)=\int_{O(x)} f(t)d\mu(t).$$

For any $z\in E(x)$ and any $f\in C(O(x),\mathbb{C}_p)$, the function $T(f,z):O(x)\to\mathbb{C}_p$ defined by

$$T(f,z)(t)=\frac{f(t)}{z-t},\quad t\in O(x)$$

belongs to $C(O(x),\mathbb{C}_p)$. Hence for any $z\in E(x)$ one can define the element

$$F_K(\mu,f,z)=\int_{O(x)}\frac{f(t)}{z-t}d\mu(t),$$

called the *Cauchy transform of f with respect to μ (or φ)*.

Now let $z_0\in E(x)$. Then for any $z\in E(x)$ and $t\in O(x)$ one can write:

$$\frac{f(t)}{z_0-t}=\frac{f(t)}{z-t}+\frac{f(t)(z-z_0)}{(z-t)^2}+\cdots+\frac{f(t)(z-z_0)^n}{(z-t)^{n+1}}+\cdots. \tag{7}$$

Generally the series (7) does not converge, but it converges for a suitable choice of z. Indeed, let $|z_0-t|\geq\varepsilon$ for all $t\in O(x)$, and let a be a real number such that $0<\frac{\varepsilon}{a}<1$. Then for any $z\in B(z_0,\varepsilon^2/a)$, the series (7) converges since one has: $\left|\frac{z-z_0}{z-t}\right|^n\leq\left(\frac{\varepsilon}{a}\right)^n\cdot\frac{1}{\varepsilon}<1$, for any $t\in O(x)$. Hence for any $z\in B(z_0,\varepsilon^2/a)$ one can write:

$$\int_{O(x)}\frac{f(t)}{z_0-t}d\mu(t)=\int_{O(x)}\frac{f(t)}{z-t}d\mu(t)+\cdots+\int_{O(x)}\frac{f(t)(z-z_0)^n}{(z-t)^{n+1}}d\mu(t)+\cdots \tag{8}$$

and this series is also convergent (since φ is continuous) for all $z\in B(z_0,\varepsilon^2/a)$.

3. Let $\{\varepsilon_n\}_n$ be a strictly decreasing sequence of positive real numbers with limit zero. One can assume that $S_n\subseteq S_{n+1}$ for all $n\geq 0$, where $S_n=S_{\varepsilon_n}$

For any $n \geq 0$, choose an element $a_n \in K_x$ such that $|x - a_n| < \varepsilon_n$. Then by ([Sch], page 276) it follows that

$$F_K(\mu, f, z) = \lim_n \sum_{\sigma \in S_n} \mu(B(\sigma(x), \varepsilon_n)) \frac{f(\sigma(x))}{z - \sigma(x)}.$$

By this equality it also follows that

$$F_K(\mu, f, z) = \lim_n \sum_{\sigma \in S_n} \mu(B(\sigma(x), \varepsilon_n)) \frac{f(\sigma(a_n))}{z - \sigma(a_n)}. \tag{9}$$

By the above considerations one obtains the following result.

THEOREM 4. *Let $x \in \mathbb{C}_p$, and let $\varphi : C(O(x), \mathbb{C}_p) \to K$ be a functional, where K is a closed subfield of $\mathbb{C}_p$. Then for any $f \in C(O(x), \mathbb{C}_p)$ the function $F_K(\mu, f, z) : E(x) \to \mathbb{C}_p$ defined by* (8) *is a rigid analytic function on $E(x)$, and $\lim_{z \to \infty} F_K(\mu, f, z) = 0$.*

4. If the field K and the functional φ are fixed we shall write simply $F(f, z)$ instead of $F_K(\mu, f, z)$.

6. Equivariant Cauchy Transforms.

1. For $x \in \mathbb{C}_p$ consider an equivariant functional $\varphi : C_G(O(x), \mathbb{C}_p) \to \mathbb{Q}_p$. By Theorem 1 the associate measure $\mu = \mu_\varphi$ is equivariant. On has the following result:

THEOREM 5. *For any nonzero element $f \in C_G(O(x), \mathbb{C}_p)$, the function $F(f, z)$, the Cauchy transform of f with respect to φ, is an equivariant rigid analytic function defined on $E(x)$. Any element of $O(x)$ is a singular point for $F(f, z)$, if $F(f, z) \neq 0$.*

PROOF. We observe that for any $z \in E(x)$ and any $\sigma \in G$, one has $\sigma(z) \in E(x)$. Recall that $F(f, z)$ is *equivariant* if for any $\sigma \in G$ one has: $F(f, \sigma(z)) = \sigma(F(f, z))$. Now this equality is true since φ is equivariant (see (8)). Furthermore if $F(f, z) \neq 0$, it has a singular point (a pole) which must belong to $O(x)$. Then by equivariance all elements of $O(x)$ are poles for $F(f, z)$. $\square$

COROLLARY 1. *For an element $x \in \mathbb{C}_p$ the following assertions are equivalent:*

a) *$O(x)$ is a finite set.*

b) $x \in \overline{\mathbb{Q}}_p$.

c) *There exists an element $f \in C_G(O(x), \mathbb{C}_p)$, $f \neq 0$ such that $F(f,z) \in \mathbb{Q}_p(z)$.*

2. For $f \in C_G(O(x), \mathbb{C}_p)$ the element $f(x)$ belongs to $\widetilde{\mathbb{Q}_p[x]}$, and so with notations as in Proposition 1 one has: $f(x) = \sum_n a_n M_n(x)$. Then for any $t \in O(x)$ it follows $f(t) = \sum_n a_n M_n(t)$. Then by (8) one can write:

$$F(f,z) = \sum_{n \geq 0} \int_{O(x)} \frac{f(t)(z-z_0)^n}{(z-t)^{n+1}} d\mu(t) = \sum_{n \geq 0} \sum_{m \geq 0} \int_{O(x)} \frac{a_m M_m(t)(z-z_0)^n}{(z-t)^{n+1}} d\mu(t).$$

7. Cauchy Transforms of measures.

1. In what follows we shall prove that Barsky's Theorem (see [B]) is valid for $E(x)$, $x \in \mathbb{C}_p$. Namely, we shall prove that there exists a bijective mapping between the measures on $O(x)$ and rigid analytic functions on $E(x)$ with residue zero at infinity and which verify some boundary conditions.

Let $F : E(x) \to \mathbb{C}_p$ be a rigid analytic function such that $F(\infty) = 0$ and that the set of real numbers $\{\varepsilon \|F\|_{E(x,\varepsilon)}\}_{\varepsilon > 0}$ is bounded. (Here $\|F\|_{E(x,\varepsilon)} = \sup_{z \in E(x,\varepsilon)} |F(z)|$.)

One knows that $E(x,\varepsilon) = \mathbb{C}_p \cup \{\infty\} \setminus V(x,\varepsilon)$, and $V(x,\varepsilon) = \cup_{\sigma \in S_\varepsilon} B(\sigma(x), \varepsilon)$. By the Mittag-Leffler Theorem (see [FV]) one can write:

$$F(z) = \sum_{\sigma \in S_\varepsilon} \sum_{n \geq 1} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z - \sigma(x))^n}, \ z \in E(x,\varepsilon), \ a_{n,\sigma}^{(\varepsilon)} \in \mathbb{C}_p.$$

One also has

$$F(z) = \sum_{\sigma \in S_\varepsilon} F_\sigma^{(\varepsilon)}(z),$$

where $F_\sigma^{(\varepsilon)}(z) = \sum_{n \geq 1} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z-\sigma(x))^n}$, $\frac{|a_{n,\sigma}^{(\varepsilon)}|}{\varepsilon^n} \to 0$. Then by Cauchy's inequalities, one obtains

$$|a_{n,\sigma}^{(\varepsilon)}| \leq \varepsilon^n \|F\|_{E(x,\varepsilon)}, \ n \geq 1. \tag{10}$$

By the unicity of Mittag-Leffler's conditions for any $0 < \varepsilon' < \varepsilon$ one has

$$F_\sigma^{(\varepsilon)}(z) = \sum_{\substack{\tau \in S_{\varepsilon'} \\ \hat{\tau} = \hat{\sigma}}} F_\tau^{(\varepsilon')}(z), \quad z \in E(x, \varepsilon)$$

(here $\hat{\tau} = \hat{\sigma}$ means $\tau \in \sigma H(x, \varepsilon)$) and so

$$\sum_{n \geq 1} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z - \sigma(x))^n} = \sum_{\substack{\tau \in S_{\varepsilon'} \\ \hat{\tau} = \hat{\sigma}}} \sum_{m \geq 1} \frac{a_{m,\tau}^{(\varepsilon')}}{(z - \tau(x))^m}. \tag{11}$$

Since $|z - \sigma(x)| \geq \varepsilon$, and $|\sigma(x) - \tau(x)| < \varepsilon$, it follows that $|z - \tau(x)| = |z - \sigma(x)|$ and so:

$$\begin{aligned} \frac{a_{m,\tau}^{(\varepsilon')}}{(z - \tau(x))^m} &= \frac{a_{m,\sigma}^{(\varepsilon')}}{(z - \sigma(x))^m \left(1 - \frac{\tau(x) - \sigma(x)}{z - \sigma(x)}\right)^m} \\ &= \frac{a_{m,\tau}^{(\varepsilon')}}{(z - \sigma(x))^m} \sum_{k \geq 0} \binom{m + k - 1}{k} \left[\frac{\tau(x) - \sigma(x)}{z - \sigma(x)}\right]^k. \end{aligned} \tag{12}$$

If we denote $m + k = n$, then by identifying the coefficients of the terms of degree n in (11) and (12) one obtains:

$$a_{n,\sigma}^{(\varepsilon)} = \sum_{\substack{\tau \in S_{\varepsilon'} \\ \hat{\tau} = \hat{\sigma}}} \sum_{k=0}^{n-1} \binom{n-1}{k} a_{n-k,\tau}^{(\varepsilon')} (\tau(x) - \sigma(x))^k, \tag{13}$$

where $n \geq 1$.

Now for any $n \geq 1$ one defines a sequence $\{\mu_{n,\varepsilon}\}_{n,\varepsilon}$ of measures on $O(x)$ by the equality

$$\mu_{n,\varepsilon} = \sum_{\sigma \in S_\varepsilon} a_{n,\sigma}^{(\varepsilon)} \cdot \delta_{\sigma(x)}, \tag{14}$$

where δ_y denotes the Dirac measure concentrated at $y \in \mathbb{C}_p$.

By (13) one obtains, for $n = 1$:

$$a_{1,\sigma}^{(\varepsilon)} = \sum_{\substack{\tau \in S_{\varepsilon'} \\ \hat{\tau} = \hat{\sigma}}} a_{1,\tau}^{(\varepsilon')}, \quad 0 < \varepsilon' \leq \varepsilon.$$

This equality further implies that for any ball B of radius δ, $\varepsilon \leq \delta$, one has $\mu_{1,\varepsilon}(B) = \mu_{1,\varepsilon'}(B)$ whereas $\varepsilon' \leq \varepsilon$. Then by (10) and the Banach-Steinhaus Theorem (see [R]) there results that the mapping

$$B \rightsquigarrow \mu_1(B) = \lim_{\varepsilon} \mu_{1,\varepsilon}(B)$$

(where B runs over all the open balls of $O(x)$) defines a p-adic measure on $O(x)$. One says that $\mu = \mu_1$ is the *measure associated* to the rigid analytic function F.

Furthermore, for $n = 2$, by (13) one obtains

$$a_{2,\sigma}^{(\varepsilon)} = \sum_{\substack{\tau \in S_{\varepsilon'} \\ \hat{\tau} = \hat{\sigma}}} [a_{2,\tau}^{(\varepsilon')} + a_{1,\tau}^{(\varepsilon')}(\tau(x) - \sigma(x))].$$

If B is an open ball of $O(x)$ of radius δ, by the previous equality and (14) there results that

$$\mu_{2,\varepsilon} - \mu_{2,\varepsilon'} = \sum_{\substack{\sigma \in S_\varepsilon \\ \sigma(x) \in B}} \left[\sum_{\substack{\tau \in S_{\varepsilon'} \\ \hat{\tau} = \hat{\sigma}}} a_{1,\tau}^{(\varepsilon')}(\tau(x) - \sigma(x)) \right]$$

and so by (10) one has:

$$|\mu_{2,\varepsilon} - \mu_{2,\varepsilon'}| \leq \varepsilon\varepsilon' \|F\|_{E(x,\varepsilon')} \leq \varepsilon M$$

where $M = \sup_{\varepsilon > 0} \varepsilon \|F\|_{E(x,\varepsilon)} < \infty$, by hypothesis. Then by (10) and the Banach-Steinhaus Theorem, there exists a measure μ_2 on $O(x)$ defined by

$$\mu_2(B) = \lim_{\varepsilon} \mu_{2,\varepsilon}(B),$$

for all the open balls B of $O(x)$. In the same manner for all $n \geq 3$ one can define a measure μ_n on $O(x)$ by:

$$\mu_n(B) = \lim_{\varepsilon} \mu_{n,\varepsilon}(B).$$

Next, by an easy computation it follows that for all $n \geq 2$, one has $\|\mu_{n,\varepsilon}\| \leq M\varepsilon^{n-1}$, and so $\mu_n = 0$ for all $n \geq 2$. In what follows we shall prove that

$$F(z) = \int_{O(x)} \frac{1}{z-t} d\mu(t), \quad z \in E(x). \tag{15}$$

Indeed, with the above notations and using (10) one has:

$$\left| F(z) - \int_{O(x)} \frac{1}{z-t} d\mu_{1,\varepsilon}(t) \right| = \left| F(z) - \sum_{\sigma \in S_\varepsilon} \frac{a_{1,\sigma}^{(\varepsilon)}}{z - \sigma(x)} \right|$$
$$\leq \varepsilon^2 \|F\|_{E(x,\varepsilon)} \leq M\varepsilon,$$

for $|z|$ sufficiently large. It is clear, by the definition of $\mu = \mu_1$, that (see (14)) one has:

$$\int_{O(x)} \frac{1}{z-t} d\mu(t) = \lim_{\varepsilon \to 0} \int_{O(x)} \frac{1}{z-t} d\mu_{1,\varepsilon}(t)$$

and so by the last inequality one obtains (15) for $|z|$ sufficiently large. Because $E(x)$ is infra-connected, by analytic continuation one obtains (15) for all $z \in E(x)$. Finally one has the following result:

THEOREM 6. *For any $x \in \mathbb{C}_p$, there exists a bijective mapping between the p-adic measures on $O(x)$ and the functions $F : E(x) \to \mathbb{C}_p$ which verify the following conditions:*

i) *F is rigid analytic on $E(x)$, and $F(\infty) = 0$.*
ii) *The set of real numbers $\{\varepsilon \|F\|_{E(x,\varepsilon)}\}_{\varepsilon>0}$ is bounded.*

Moreover, by this bijective mapping the rigid analytic and equivariant functions are in one to one correspondence with equivariant measures on $O(x)$.

8. Cauchy Transforms of Lipschitzian Distributions.

Let X be an arbitrary compact subset of $\mathbb{C}_p$ without isolated points. A map $f : X \to \mathbb{C}_p$ is said to be λ-Lipschitzian provided there exists a positive real number λ such that $|f(x) - f(y)| \le \lambda |x-y|$, for any $x, y \in X$. A map $f : X \to \mathbb{C}_p$ is said to be Lipschitzian provided there exists a real number λ for which f is λ-Lipschitzian. Let us remark that $Lip(X, \mathbb{C}_p)$, the set of Lipschitzian functions, is a $\mathbb{C}_p$-vector space. Moreover, it is a Banach space with the norm

$$\|f\| = \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|} + \sup_{x \in X} |f(x)|.$$

Let us denote by $\Omega(X)$ the set of open compact subsets of X. It is clear that any element of $\Omega(X)$ is a disjoint finite union of open balls of X.

A distribution μ on X with values in $\mathbb{C}_p$ is a map $\mu : \Omega(X) \to \mathbb{C}_p$ which is finitely additive, thus if $D = \bigcup_{i=1}^{n} D_i$ with $D_i \in \Omega(X)$ for $1 \le i \le n$ and $D_i \cap D_j = \emptyset$ for $1 \le i \ne j \le n$, then $\mu(D) = \sum_{i=1}^{n} \mu(D_i)$.

We call a distribution $\mu : \Omega(X) \to \mathbb{C}_p$ Lipschitzian provided that for any $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for any $0 < \delta \le \delta_\varepsilon$ and any $x \in X$, $\delta |\mu(B(x,\delta))| \le \varepsilon$.

One knows (see [VZ, Theorem 1]) that any Lipschitzian function $f : X \to \mathbb{C}_p$ is integrable with respect to any Lipschitzian distribution $\mu : \Omega(X) \to \mathbb{C}_p$. Moreover, by the proof of this result, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if

$$S(\mu, f, B_1, \dots, B_n, x_1, \dots, x_n) = \sum_{i=1}^{n} f(x_i)\mu(B_i)$$

is an arbitrary Riemann sum with $x_i \in B_i$ and B_i open ball of radius $\delta_i \le \delta_\varepsilon$ for $1 \le i \le n$, the following inequality holds:

$$\left| S(\mu, f, B_1, \dots, B_n, x_1, \dots, x_n) - \int_X f(t)d\mu(t) \right| \le \varepsilon \max\{1, 2\lambda\}, \tag{16}$$

where λ is the Lipschitzianity constant with respect to f.

Let us consider now an element $z \in \mathbb{P}^1(\mathbb{C}_p) \setminus X$, and $f \in Lip(X, \mathbb{C}_p)$. Define $T(f, z) : X \to \mathbb{C}_p$ by

$$T(f, z)(t) = \frac{f(t)}{z - t}, \ t \in X. \tag{17}$$

It is easy to see that $T(f, z)$ is well defined and Lipschitzian. In fact, if $d = d(z, X)$ is the distance from z to X, and $t_1, t_2 \in X$ we have

$$|T(f, z)(t_1) - T(f, z)(t_2)| = \left| \frac{f(t_1)}{z - t_1} - \frac{f(t_2)}{z - t_2} \right| \le A|t_1 - t_2|, \tag{18}$$

where $A = A(z, f, X) = \dfrac{1}{d^2}\max\{\lambda|z|, \ \lambda \sup_{t\in X}|t|, \ \sup_{t\in X}|f(t)|\}$.

We can integrate this function with respect to any Lipschitzian distribution, so the map

$$F_X(\mu, f, z) = \int_X T(f, z)(t)d\mu(t)$$

is well defined. It is the Cauchy transform of the Lipschitzian function f with respect to the Lipschitzian distribution μ. Using a similar argument as in the proof of Theorem 6.1 from [APZ2] one obtains the following result.

THEOREM 7. *Let X be a compact subset of $\mathbb{C}_p$ without isolated points, μ a Lipschitzian distribution on X, and $f \in Lip(X, \mathbb{C}_p)$. Then*

$$F_X(\mu, f, z) := \int_X \frac{f(t)}{z - t} d\mu(t) \tag{19}$$

is well defined, rigid analytic on $\mathbb{P}^1(\mathbb{C}_p)\setminus X$, and vanishes at infinity. Moreover, any element of X is a singular point of $F_X(\mu,f,z)$.

Next, let $f\in Lip(X,\mathbb{C}_p)$ and $z\in\mathbb{P}^1(\mathbb{C}_p)\setminus X$. From (16), with $T(f,z)$ instead of f and Σ for the corresponding Riemann sum, one has

$$|\Sigma - F_X(\mu,f,z)|\le \varepsilon\max\{1,2A\}, \tag{20}$$

where A is defined above. If $(B_i)_{1\le i\le n}$ is a covering of X with disjoint open balls of radius δ_ε, we have from the definition of μ that

$$|\Sigma|\le\frac{\varepsilon M}{d\delta_\varepsilon}, \tag{21}$$

where $M=\sup_{x\in X}|f(x)|$. From (20) and (21) one has

$$|F_X(\mu,f,z)|\le\varepsilon\max\left\{1,2A,\frac{M}{d\delta_\varepsilon}\right\}, \tag{22}$$

and so

$$d^2|F_X(\mu,f,z)|\le\varepsilon\max\left\{d^2,2d^2A,\frac{Md}{\delta_\varepsilon}\right\}. \tag{23}$$

Denoting by $E_d(X)$ the complement in $\mathbb{P}^1(\mathbb{C}_p)$ of a d-neighborhood of X, and using the definition of A, from (23) we obtain

$$\lim_{d\to 0} d^2\|F_X(\mu,f,z)\|_{E_d(X)}\le B\varepsilon, \tag{24}$$

where B is an absolute constant that does not depend on d. Letting $\varepsilon\to 0$, we obtain the following result.

THEOREM 8. *Let X be a compact subset of $\mathbb{C}_p$ without isolated points. Let $f\in Lip(X,\mathbb{C}_p)$ and let μ be a Lipschitzian distribution on X. To any pair (μ,f) as above, we can associate a rigid analytic function $F_X(\mu,f,z)$ in such a way that it vanishes at infinity, and satisfies the boundary condition:*

$$\lim_{d\to 0} d^2\|F_X(\mu,f,z)\|_{E_d(X)}=0. \tag{25}$$

If X, μ and f are equivariant then $F_X(\mu,f,z)$ is also equivariant.

A natural question that arises is to provide a converse to the statement of this theorem.

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