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On the Occurrence of the Complete Graph K_5 in the Hasse Graph of a Finite Group.

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Dedicated to Professor Guido Zappa
on the occasion of his 90th birthday

1. Introduction.

In [7] we determined all finite groups with planar subgroup lattices and also those with planar Hasse graphs. Here a finite lattice L is called planar (see [4]) if it is possible to draw its Hasse diagram in the plane in such a way that no two line segments intersect; the Hasse graph L^* of L is its Hasse diagram considered as an undirected graph in the usual way (see §2) and for a finite group G, the Hasse graph $L(G)^*$ of its subgroup lattice $L(G)$ is also called the Hasse graph of G.

By Kuratowski's theorem, a finite graph is planar if and only if it does not contain a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph (see [2, p. 24]). Figure 1 shows the graphs K_5 and $K_{3,3}$; a subdivision is obtained from a graph by subdividing some of the edges,

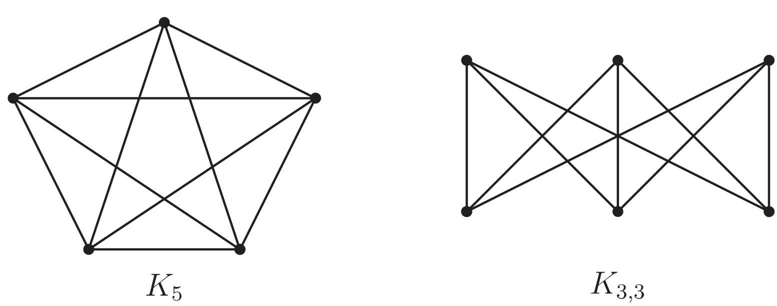

Figure 1

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that is, by replacing the edges by paths having at most their endvertices in common. We were able to show in [7] that the Hasse graph of a finite group is planar if and only if it contains no subdivision of $K_{3,3}$ as a subgraph.

The aim of the present paper is to determine all finite groups whose Hasse graph contains no subdivision of K_5 as a subgraph; we shall call such a graph K_5-*free*, for short. For p-groups, we obtain the following results.

THEOREM A. *Let* $|G| = p^n$ *where* $2 < p \in \mathbb{P}$ *and* $5 \leq n \in \mathbb{N}$. *Then* $L(G)^*$ *is* K_5-*free if and only if* G *is metacyclic of exponent at least* p^{n-2}.

THEOREM B. *Let* $|G| = 2^n$ *where* $6 \leq n \in \mathbb{N}$. *Then* $L(G)^*$ *is* K_5-*free if and only if* G *is lattice-isomorphic to an abelian group of type* (2^n), $(2^{n-1}, 2)$, *or* $(2^{n-2}, 4)$, *or* G *is one of the following two groups*

$$G = \langle a, b \mid a^4 = b^{2^{n-2}} = 1, a^b = a^{-1} \rangle,$$
$$G = \langle a, b \mid a^8 = 1, b^{2^{n-3}} = a^4, a^b = a^{-1} \rangle.$$

The assumption $n \geq 5$ is needed in Theorem A since for every odd prime p, there are two groups of order p^4 with K_5-free Hasse graph which are not metacyclic. Theorem B holds for $n \leq 5$ if one includes in the list the quaternion group Q_{32} of order 2^5 and all metacyclic groups of order at most 2^4 (see Remarks 3.8 and 3.11).

It might be interesting to note that only rather few of these K_5-free p-groups have planar (or $K_{3,3}$-free) Hasse graphs. In fact, we showed in [7] that for $p > 2$, a group of order p^n has this property if and only if it is either cyclic or lattice-isomorphic to an abelian group of type (p^{n-1}, p). For $p = 2$, the same groups and, in addition, only the dihedral group of order 8, the quaternion groups Q_8 and Q_{16}, and the semidihedral group of order 16 have planar Hasse graphs. This latter group is the unique p-group with planar Hasse graph and non-planar subgroup lattice.

In [7, Lemma 3.2] we showed that the Hasse graph of a cyclic group C_k of order k is K_5-free if and only if it is planar, and that C_k has this property if and only if k is of the form $p^n, p^n q^m$, or $p^n qr$ where $n, m \in \mathbb{N}$ and p, q, r are primes; here the groups C_{p^n} and $C_{p^n q^m}$ even have planar subgroup lattices whereas $L(C_{pqr})$ is non-planar. Therefore the following two theorems complete the determination of all finite groups with K_5-free Hasse graph.

THEOREM C. *Let* G *be a finite nilpotent group which is not cyclic and not of prime power order. Then* $L(G)^*$ *is* K_5-*free if and only if* $G \simeq C_p \times Q_8$ *where* $2 < p \in \mathbb{P}$ *or* $G \simeq C_p \times C_q \times C_q$ *with different primes* p *and* q.

THEOREM D. *Let G be a finite group which is not nilpotent. Then $L(G)^*$ is K_5-free if and only if $G = PK$ where $P \trianglelefteq G$ and K is cyclic, and one of the following holds (where p, q, r are pairwise different primes and $n \in \mathbb{N}$).*

(a) $|P| = p, |K| = q^n$ *and* $|K : C_K(P)| \leq q^3$.
(b) $|P| = p$ *and* $|K| = qr$.
(c) $|P| = p^2$, *P is cyclic,* $|K| = q^n$ *and* $|K : C_K(P)| = q$.
(d) $|P| = p^2$, *P is elementary abelian,* $|K| = q^n$, $|K : C_K(P)| \leq q^3$, $|C_K(P)| \leq q$, *and* $\Omega(K/C_K(P))$ *operates irreducibly on P.*
(e) $|P| = p^2$, *P is elementary abelian,* $|K| = qr$ *and both minimal subgroups of K operate irreducibly on P.*
(f) $|P| = p^3$, *P is nonabelian of exponent p,* $|K| = q$ *and K operates irreducibly on $P/\Phi(P)$.*
(g) $P \simeq Q_8$, $|K| = 3$ *or* 9, $G/C_K(P) \simeq SL(2, 3)$.
(h) $|P| = p^4$, *P is abelian of type* (p^2, p^2), $|K| = q$ *and K operates irreducibly on $P/\Phi(P)$.*

Again, by [7], only few of these groups have planar Hasse graphs, namely those in (a) of Theorem D for which $|K : C_K(P)| = q$ and those in (d) for which $|K| = q$. These groups, in fact, also have planar subgroup lattices whereas for all other groups in Theorems C and D, the Hasse graph is non-planar.

2. Notation and Preliminary Remarks.

In the whole paper, G is a finite group and p, q are primes. We denote by $L(G)$ the subgroup lattice of G; so $L(G) = \{X \mid X \leq G\}$ with the set-theoretical inclusion as relation and we write $X \cap Y$ for the intersection and $X \cup Y$ for the join of the subgroups X, Y of G. Further we write $X \lessdot Y$ if X is a maximal subgroup of Y.

For $n \in \mathbb{N}$ and a p-group G, we let

$$\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1 \rangle$$
$$\mho_n(G) = \langle x^{p^n} \mid x \in G \rangle$$

and we denote by

C_n the cyclic group of order n
D_{2^n} the dihedral group of order 2^n $(n \geq 2)$
Q_{2^n} the quaternion group of order 2^n $(n \geq 3)$
S_{2^n} the semidihedral group of order 2^n $(n \geq 4)$.

Further notation is standard (see [3] and [5]).

All graphs considered are undirected graphs. In particular, we define the *Hasse graph* L^* of a finite lattice L to be the (undirected) graph with vertex set L in which the unordered pair $\{x, y\}$ of elements of L is an edge if and only if one of x, y is a lower neighbour of the other. The degree of a vertex is the number of edges in which it is contained. For a group G, the Hasse graph $L(G)^*$ of its subgroup lattice $L(G)$ is also called the Hasse graph of G.

A path γ (of length n) from u to v in L^* is a sequence of elements $x_0, x_1, \ldots, x_n$ of L such that $x_0 = u,\ x_n = v$ and for all i there is an edge between x_i and x_{i+1} in L^*, that is, x_i is a lower or upper neighbour of x_{i+1}. We usually write $\gamma = (x_0, \ldots, x_n)$ for such a path and call the vertices different from x_0 and x_n the *internal* vertices of γ. Finally, a collection $\{\gamma_1, \ldots, \gamma_m\}$ of paths is called *internally disjoint* if each internal vertex of γ_i $(i = 1, \ldots, m)$ lies on no γ_j $(j \neq i)$.

If Ω is a subgraph of L^* which is a subdivision of K_5, then there exist 5 points $x_1, \ldots, x_5 \in \Omega$ and an internally disjoint set of 10 paths γ_{ij} from x_i to x_j $(1 \leq i < j \leq 5)$ in Ω, and hence in L^*. Conversely, every subset $\Gamma = \{x_1, \ldots, x_5\}$ of L with $|\Gamma| = 5$ together with an internally disjoint set of paths γ_{ij} from x_i to x_j $(1 \leq i < j \leq 5)$ in L^* yields a subgraph Ω of L^* which is a subdivision of K_5. We shall call such a set Γ a K_5-*set* in L^* and the members of Γ are called the K_5-*points* of Ω. They are uniquely determined by Ω since all the other vertices of Ω have degree 2 in Ω.

If we want to show, for a certain group G, that $L(G)^*$ has such a subgraph Ω, we usually only give a K_5-set Γ and describe the nontrivial paths γ_{ij} between the members of Γ; the trivial ones being edges or, for example, one of the $p + 1$ possible paths of length 2 between the bottom and the top of an elementary abelian section of order p^2 in G.

Finally, as mentioned in the introduction, we call a graph K_5-*free* if it contains no subdivision of K_5 as a subgraph.

3. p-Groups and Subdivisions of K_5.

In this section we shall determine all finite p-groups whose Hasse graphs are K_5-free. We start with the following trivial, but fundamental, result.

LEMMA 3.1. *If G is elementary abelian of order p^3, then $L(G)^*$ contains a subdivision of K_5.*

PROOF. Let $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ where $o(a) = o(b) = o(c) = p$ and let $\Gamma := \{1, \langle b\rangle, \langle a, b\rangle, \langle b, c\rangle, G\}$. There are five edges between these points in $L(G)^*$ and the other five paths needed are $(1, \langle ab\rangle, \langle a, b\rangle)$, $(1, \langle bc\rangle, \langle b, c\rangle)$, $(1, \langle abc\rangle, \langle ab, c\rangle, G)$, $(\langle b\rangle, \langle b, ac\rangle, G)$, and $(\langle a, b\rangle, \langle a\rangle, \langle a, c\rangle, \langle c\rangle, \langle b, c\rangle)$. So Γ together with these paths yields a subdivision of K_5 in $L(G)^*$. □

REMARK 3.2. By [7, Theorem A], all other groups of order p^3 have planar subgroup lattices except the nonabelian group H of exponent p $(p > 2)$. Since all subgroups of order p different from $Z(H)$ have degree 2 in the graph $L(H)^*$, it is easy to see that $L(H)^*$ is K_5-free. So the elementary abelian group is the unique group of order p^3 whose Hasse graph contains a subdivision of K_5. □

Lemma 3.1 shows that if G is a p-group with K_5-free Hasse graph, then every subgroup of G is generated by two elements. Groups with this property were studied by Blackburn [1]; most of them are metacyclic.

LEMMA 3.3. *Let $|G| = p^n$, $n \geq 5$. If $L(G)^*$ is K_5-free, then G is metacyclic and has exponent at least p^{n-2}.*

PROOF. We show first that G is metacyclic. By Lemma 3.1 and [1, Theorems 4.1 and 5.1], this is true except possibly when G is a 3-group of maximal class. (Blackburn has a weaker assumption in his Theorem 4.1, so he gets a further possibility for $p > 2$, namely a group G of order p^5 in which $\Omega(G)$ is nonabelian of order p^3 and exponent p and $G/\Omega(G)$ is cyclic. However, if we take $x \in G$ such that $G = \langle x\rangle\Omega(G)$ and a maximal subgroup S of G containing x, then $\Phi(S)$ is a cyclic normal subgroup of order p^2 in G and so $\Phi(S)\Omega(G)/\Phi(S) \cap \Omega(G)$ is elementary abelian of order p^3.) So suppose, for a contradiction, that G is a 3-group of maximal class. Then G has a factor group of order 3^5 which is also of maximal class. We show that the Hasse graph of such a group contains a subdivision of K_5; this will be the desired contradiction.

So let G_0 be of order 3^5 and class 4. Then it is well-known [3, pp. 370-371] that G_0 has normal subgroups G_i with $G_0 > G_1 > G_2 > G_3 > G_4 > 1$ such that G_1 is metacyclic, $G_3 = \Omega(G_1) = \Phi(G_1)$ and $G_2 = (G_0)'$ is abelian. Let M be a maximal subgroup of G_0 such that $M \neq G_1$. Then M has maximal class [3, p. 374] and so $\Phi(M) = G_3$ since G_0 has only one normal subgroup of order 3^2. Similarly, $\Phi(G_2) = G_4 = Z(G_0)$. If M/G_4 would contain a cyclic subgroup H/G_4 of order 3^2, then H would be abelian and hence $H \cap G_2 \leq Z(M)$, a contradiction since M has maximal class. Thus M/G_4 has exponent 3. Now we claim that $\Gamma := \{M, G_1, G_2, G_3, G_4\}$ is a K_5-set in

$L(G_0)^*$. Here we use 3 nontrivial paths, namely $(M, G_0, G_1), (M, M_1, M_2, G_4)$ with maximal subgroups $M_1 \neq G_2$ of M and $M_2 \neq G_3$ of M_1 (which exist since M/G_4 has exponent 3), and $(G_1, S\Phi(G_1), S, \Omega(S), 1, G_4)$ where S is a cyclic subgroup of order 3^2 of G_1 not contained in G_2 and not containing G_4 (which exists since $\Omega(G_1) = \Phi(G_1)$). In addition, 4 paths are edges and in the elementary abelian factors $M/G_3, G_1/G_3$, and G_2/G_4 there are intermediate subgroups not yet used. So we get a subdivision of K_5 in $L(G_0)^*$ and this completes the proof that G is metacyclic.

It remains to be shown that $\exp G \geq p^{n-2}$. So suppose, for a contradiction, that $\exp G \leq p^{n-3}$. Since G is metacyclic, $G = NX$ where $N \trianglelefteq G$ and N and X are cyclic. Then $|N : N \cap X| = |G : X| \geq p^3$ and $|X : N \cap X| = |G : N| \geq p^3$. So $G/N \cap X$ contains a subgroup isomorphic to a semidirect product of a cyclic group N_1 of order p^3 by a cyclic group X_1 of order p^3. For $p > 2$, such a group has modular subgroup lattice and is therefore lattice-isomorphic to an abelian group of type (p^3, p^3) [5, 2.3.1 and 2.5.9]. For $p = 2$, $\operatorname{Aut} C_8$ is elementary abelian of order 4 and hence X_1 induces an automorphism of order at most 2 in N_1. So to get a contradiction, we show that the Hasse graph of a semidirect product $H = \langle a\rangle\langle b\rangle$, where $\langle a\rangle \trianglelefteq H$, $o(a) = o(b) = p^3$ and $b^p \in C_H(a)$, contains a subdivision of K_5. For this let $\Gamma := \{H_1, \dots, H_5\}$ where $H_1 = \langle a^{p^2}\rangle$, $H_2 = \langle a^{p^2}, b^{p^2}\rangle$, $H_3 = \langle a^p, b^{p^2}\rangle$, $H_4 = \langle a^p, b^p\rangle$ and $H_5 = \langle a, b^p\rangle$. Clearly, H_5 is abelian of type (p^3, p^2) and we have 4 edges $\{H_i, H_{i+1}\}$, 2 nondiagonal paths $(H_1, \langle a^p\rangle, \langle a\rangle, \langle a, b^{p^2}\rangle, H_5)$ and $(H_2, \langle a^{p^2}, b^p\rangle, H_4)$, 3 diagonal paths $(H_1, \langle a^p b^{p^2}\rangle, H_3)$, $(H_3, \langle ab^p\rangle, H_5)$ and $(H_1, 1, \langle a^{p^2} b^{p^2}\rangle, \langle a^p b^p\rangle, \langle a^{p^2}, a^p b^p\rangle, H_4)$, and finally the path $(H_2, \langle b^{p^2}\rangle, \langle b^p\rangle, \langle b\rangle, \langle a^{p^2}, b\rangle, \langle a^p, b\rangle, H, H_5)$. So $L(H)^*$ contains a subdivision of K_5. But $L(G)^*$ has a subgraph isomorphic to such an $L(H)^*$, a contradiction. Thus $\exp G \geq p^{n-2}$. □

For $p > 2$, the converse of Lemma 3.3 also holds. To prove this, we need some preliminaries which will also be used in the case $p = 2$.

DEFINITION 3.4. Let G be a group.

(a) We denote by $L_5(G)$ the set of all subgroups H of G for which there exists a K_5-set Γ in $L(G)^*$ such that $H \in \Gamma$.

(b) For every subset $\mathcal{L}$ of $L(G)$ containing $L_5(G)$ we define the graph $\hat{\mathcal{L}}$ in the following way: the set of vertices of $\hat{\mathcal{L}}$ is $\mathcal{L}$ and a two element subset $\{H, K\}$ of $\mathcal{L}$ is an edge if and only if there exists a path $(X_0, \dots, X_r)$ in $L(G)^*$ such that $X_0 = H$, $X_r = K$ and $X_i \in L(G) \setminus \mathcal{L}$ for all $i = 1, \dots, r-1$.

Then we have the following trivial result.

LEMMA 3.5. *Let $L_5(G) \subseteq \mathcal{L} \subseteq L(G)$. If Γ is a K_5-set in $L(G)^*$, then it is also a K_5-set in $\hat{\mathcal{L}}$.*

PROOF. Let $\Gamma = \{H_1, \ldots, H_5\}$. Since $L_5(G) \subseteq \mathcal{L}$, clearly $\Gamma \subseteq \mathcal{L}$. Furthermore there exist internally disjoint paths γ_{ij} from H_i to H_j in $L(G)^*$ $(1 \leq i < j \leq 5)$. If we remove in these paths all the elements $X \in L(G)^* \setminus \mathcal{L}$, we get internally disjoint paths $\hat{\gamma}_{ij}$ from H_i to H_j in the graph $\hat{\mathcal{L}}$. Thus Γ is a K_5-set in $\hat{\mathcal{L}}$. □

LEMMA 3.6. *Let $p \in \mathbb{P}$, $2 \leq m \in \mathbb{N}$ and $G = \langle a \rangle \times \langle b \rangle$ where $o(a) = p^m$ and $o(b) = p^2$. Then $L(G)^*$ is K_5-free.*

PROOF. Let $A := \langle a \rangle, B := \langle b \rangle$ and $\mathcal{L} := \{A_i \times B_j \,|\, 0 \leq i \leq m,\ 0 \leq j \leq 2\}$ where $A_i := \langle a^{p^{m-i}} \rangle$ is the subgroup of order p^i of A and $B_j := \langle b^{p^{2-j}} \rangle$ is the subgroup of order p^j of B. We want to apply Lemma 3.5 with this lattice $\mathcal{L}$ and therefore have to show that $L_5(G) \subseteq \mathcal{L}$.

For this let $H \in L_5(G)$ and suppose first that G/H is cyclic. Since every element of $L_5(G)$ clearly has degree at least 4 in the graph $L(G)^*$, it follows that H is not cyclic. Suppose, for a contradiction, that H has type (p^k, p) for some $k \geq 1$. If $k = 1$, then $H = \Omega(G)$ and G/H would not be cyclic. Thus $k \geq 2$ and H has exactly one noncyclic and p cyclic maximal subgroups. There are 4 internally disjoint paths from H to the other K_5-points of a K_5-set in $L(G)^*$. Since G/H is cyclic, at least two of these paths have to use cyclic maximal subgroups of H, at most one of these can go on to $\Phi(H)$. So at least one path has to stop at a maximal cyclic subgroup X of H or to go on to a subgroup $Y \neq H$ covering X. In the first case, $X \in L_5(G)$ and again there exists $Y \neq H$ covering X. Clearly, Y is cyclic; otherwise $\Omega(G) \leq$ $\leq Y \cap H = X$, a contradiction. Hence $X = \Phi(Y) \leq \Phi(G)$ (see [3, p. 269]) and so $H = X\Omega(G) \leq \Phi(G)$; but this contradicts the fact that G/H is cyclic. Thus H cannot have type (p^k, p) and since G is abelian of type (p^m, p^2), the type of H must be (p^k, p^2) for some $k \geq 2$. Therefore H contains the unique subgroup $\Omega_2(G)$ of type (p^2, p^2) of G. We have shown:

(1) If $H \in L_5(G)$ and G/H is cyclic, then $\Omega_2(G) \leq H$.

It follows that $H = A_i \times B$ for some $i \geq 2$ and so $H \in \mathcal{L}$. Since $L(G)$ is self-dual [5, p. 454], (1) also implies that a cyclic subgroup K in $L_5(G)$ is contained in $\mho_2(G)$ and hence $K = A_i \in \mathcal{L}$ (where $i \leq m - 2$). Finally, if neither H nor G/H is cyclic, then $\Omega(G) \leq H \leq \Phi(G)$ and hence $H = A_i \times B_1 \in \mathcal{L}$ (where $1 \leq i \leq m - 1$). So we have shown that

(2) $L_5(G) \subseteq \mathcal{L}$.

We now determine $\hat{\mathcal{L}}$. By Definition 3.4, every edge in $L(G)^*$ between two elements of $\mathcal{L}$ is also an edge in $\hat{\mathcal{L}}$. Furthermore, every elementary abelian section $A_{i+1} \times B_{j+1}/A_i \times B_j$ $(0 \le i < m,\ 0 \le j < 2)$ yields an edge between $A_i \times B_j$ and $A_{i+1} \times B_{j+1}$. Finally, there is an edge between A_i and $A_{i+2} \times B$ $(0 \le i \le m-2)$ since there is a diagonal path in the factor group $A_{i+2} \times B/A_i \simeq C_{p^2} \times C_{p^2}$ given by

$$\big(\langle a^{p^{m-i}}\rangle,\ \langle a^{p^{m-i-1}} b^p\rangle,\ \langle a^{p^{m-i-2}} b\rangle,\ \langle a^{p^{m-i-2}} b, b^p\rangle,\ \langle a^{p^{m-i-2}}\rangle \times \langle b\rangle\big).$$

We claim that there are no more edges in $\hat{\mathcal{L}}$:

(3) The edges $\{A_i \times B_j, A_{i+1} \times B_j\}$ $(i < m)$, $\{A_i \times B_j, A_i \times B_{j+1}\}$ $(j < 2)$, $\{A_i \times B_j, A_{i+1} \times B_{j+1}\}$ $(i < m,\ j < 2)$, and $\{A_i, A_{i+2} \times B\}$ $(i \le m-2)$ are all the edges of the graph $\hat{\mathcal{L}}$.

For this, let H, K be in $\mathcal{L}$ with $H \neq K$ and let $(H, X_1, \ldots, X_r, K)$ be a path in $L(G)^*$ such that $X_i \in L(G) \setminus \mathcal{L}$ for all i (so that $\{H, K\}$ is an edge in $\hat{\mathcal{L}}$). Clearly, we may assume that $r \ge 1$ and we suppose first that $H = A_k$ for some k. Then $k \le m-1$; and if $k = m-1$, it follows that $r = 1$ and $K = A_m \times B_1$, so that $\{H, K\}$ is one of the edges in (3). Therefore let $k \le m-2$. Then we prove by induction on i that for $i = 1, \ldots, r$,

(α) $H < X_i < A_{k+2} \times B =: L$,
(β) $A \cap X_i = H = HB \cap X_i$ if X_i is cyclic, and
(γ) $X_i \lessdot L$ if X_i is not cyclic.

This is clear for $i = 1$. So suppose it holds for some $i < r$ and consider first the case that X_i is cyclic. If $X_{i+1} \lessdot X_i$, then by (α), $H \le X_{i+1}$ and since $X_{i+1} \notin \mathcal{L}$, we have that $H < X_{i+1}$. Since $L(X_i)$ is a chain, (β) holds for X_{i+1}. If $X_i \lessdot X_{i+1}$ and X_{i+1} is cyclic, then again X_{i+1} satisfies (β) since X_i does; furthermore, $X_{i+1} \le A_{k+2} \times B$ since $o(b) = p^2$. Finally, if $X_i \lessdot X_{i+1}$ and X_{i+1} is not cyclic, then $X_{i+1} = X_i \Omega(G) \le L$. Furthermore, since $X_{i+1} \neq A_{k+1} \times B_1$, we have $|X_i : H| = p^2$ and so $X_{i+1} \lessdot L$. Now suppose that X_i is not cyclic. Then $\Omega(G) \le X_i$ and since $X_i \notin \mathcal{L}$, we have $\Omega_2(G) \not\le X_i$ and $X_i \not\le \Phi(G)$; in particular, G/X_i is cyclic. So if $X_i \lessdot X_{i+1}$, it follows that $X_{i+1} \le X_i \Omega_2(G) \le L$. By ($\gamma$), $X_i \lessdot L$ and so $X_{i+1} = L$, a contradiction. Thus $X_{i+1} \lessdot X_i \lessdot L$. Since $A_{k+2} \times B_1 \neq X_i \neq A_{k+1} \times B$, every cyclic maximal subgroup of X_i satisfies (β) and the only noncyclic maximal subgroup of X_i is $\Phi(L) = A_{k+1} \times B_1 \in \mathcal{L}$. So we have shown that (α)–(γ) hold. If X_r is cyclic and $|X_r : H| = p^2$, then (β) shows that neither $\Phi(X_r)$ nor $X_r \Omega(G)$ belong to $\mathcal{L}$. Thus $|X_r : H| = p$ and $K = X_r \Omega(G) = A_{k+1} \times B_1$. If X_r is not cyclic, then by (γ), X_r is one of the $p-1$ maximal subgroups different from $A_{k+2} \times B_1$ and $A_{k+1} \times B$ of L. Then G/X_r is cyclic and it follows that $K = L$ or $K = \Phi(L) = A_{k+1} \times B_1$. In all cases, $\{H, K\}$ is one of the edges in (3).

Since $L(G)$ is self-dual, the above argument also covers the case that $H = A_k \times B$ for some k. It remains to consider the case that $H = A_k \times B_1$ for some k. Here, if $k = 0$ or $k = m$, then $r = 1$ and $K = A_1 \times B$ or $K = A_{m-1}$, respectively. So suppose that $1 \leq k \leq m - 1$. Again using the duality of $L(G)$, we may assume that X_1 is a cyclic maximal subgroup of $A_k \times B_1$. Then $\Phi(X_1) = A_{k-1}$ and so either $K = A_{k-1}$ or $(A_{k-1}, X_1, \ldots, X_r, K)$ is one of the paths considered above. In the latter case, K is one of the groups A_k, $A_{k-1} \times B_1, A_{k+1} \times B$, as we have shown; so $\{H, K\}$ is one of the edges in (3).

We finally show that the graph

(4) $\hat{\mathcal{L}}$ is planar.

For this we just map the element $A_i \times B_j$ of $\mathcal{L}$ to the point (j, i) in $\mathbb{R}^2$. Then we connect pairs of points of the form $\{(r, s), (r, s+1)\}$, $\{(r, s), (r+1, s)\}$ and $\{(r, s), (r+1, s+1)\}$ in the obvious way by straight line segments. Finally we have to connect the points $(0, s)$ with $(2, s+2)$ and we can do this using the straight line segment from $(0, s)$ to $(-2s-2, -s-2)$ together with the lower half of the circle with center $(-s, 0)$ and radius $(s+2)\sqrt{2}$. Since these circles do not intersect, this is a planar representation of the graph $\hat{\mathcal{L}}$; a similar one is shown in Figure 2 for $m = 4$.

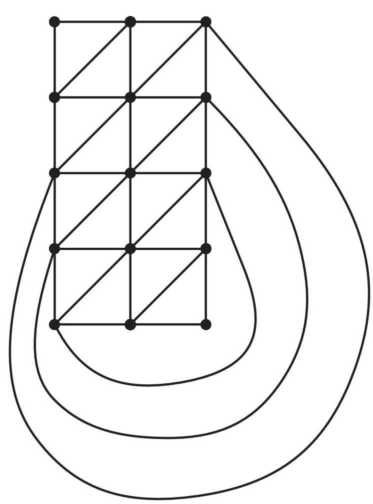

Figure 2

It is obvious that Lemma 3.6 follows from (4) and Lemma 3.5. For, if $L(G)^*$ would contain a subdivision of K_5, then so would $\hat{\mathcal{L}}$; but a planar graph contains no subdivision of K_5. □

Now we prove Theorem A; we also determine the occurring groups.

THEOREM 3.7. *Let $|G| = p^n$ where $2 < p \in \mathbb{P}$ and $5 \leq n \in \mathbb{N}$. Then the following properties are equivalent.*

(*a*) *$L(G)^*$ is K_5-free.*
(*b*) *G is metacyclic and* $\exp G \geq p^{n-2}$.
(*c*) *G is either abelian of type (p^n), (p^{n-1}, p), or (p^{n-2}, p^2), or G is one of the following nonabelian groups:*

(*c4*) $G = \langle a, b \mid a^{p^{n-1}} = b^p = 1,\ a^b = a^{1+p^{n-2}} \rangle$,
(*c5*) $G = \langle a, b \mid a^{p^{n-2}} = b^{p^2} = 1,\ a^b = a^{1+p^{n-3}} \rangle$,
(*c6*) $G = \langle a, b \mid a^{p^{n-2}} = b^{p^2} = 1,\ a^b = a^{1+p^{n-4}} \rangle$,
(*c7*) $G = \langle a, b \mid a^{p^2} = b^{p^{n-2}} = 1,\ a^b = a^{1+p} \rangle$,
(*c8*) $G = \langle a, b \mid a^{p^3} = 1,\ b^{p^{n-3}} = a^{p^2},\ a^b = a^{1+p} \rangle$.

PROOF. By Lemma 3.3, (a) implies (b), and that (c) implies (b) is obvious. So suppose that (b) holds. Then again by Iwasawa's and Baer's theorems [5, 2.3.1 and 2.5.9], since $p > 2$, $L(G)$ is modular and therefore is isomorphic to the subgroup lattice of an abelian p-group which clearly also has exponent at least p^{n-2}. Thus (a) follows from Lemma 3.6 and [7, Theorem A]. Furthermore, G is one of the groups in (c) if $\exp G \geq p^{n-1}$. If $\exp G = p^{n-2}$ and G has a cyclic normal subgroup $\langle a \rangle$ of order p^{n-2}, then by [5, 2.3.11], $\langle a \rangle$ has a complement $\langle b \rangle$ in G and G is abelian or the group in (c5) or (c6). Finally, suppose that $\exp G = p^{n-2}$ and G has no cyclic normal subgroup of order p^{n-2}. By [5, 2.3.18] there are an abelian normal subgroup A, an element b of order p^{n-2} in G and $s \in \mathbb{N}$ such that $G = A\langle b \rangle$ and $a^b = a^{1+p^s}$ for all $a \in A$. Then $|A : A \cap \langle b \rangle| = |G : \langle b \rangle| = p^2$ and since $G' \not\leq \langle b \rangle$, it follows that $s = 1$ and $A/A \cap \langle b \rangle$ is cyclic. So $x = x^b = x^{1+p}$ for $\langle x \rangle = A \cap \langle b \rangle$ and hence $|A \cap \langle b \rangle| \leq p$. It follows easily that G is the group in (c7) or (c8). □

For $p > 2$, Theorem 3.7 gives the groups of order p^n with K_5-free Hasse graphs if $n \geq 5$. Of course, it is not difficult to determine also the groups of order p^4 with this property. We give the result without proof.

REMARK 3.8. Let $|G| = p^4$, $p > 2$. Then $L(G)^*$ is K_5-free if and only if G is metacyclic or one of the following groups:

(a) $G = \langle x, y, z \mid x^p = y^p = z^{p^2} = 1 = [y, z], y^x = yz^p, z^x = zy\rangle$,
(b) $G = \langle x, y, z \mid x^p = y^p = z^{p^2} = 1 = [y, z], y^x = yz^{sp}, z^x = zy\rangle$
where $p \geq 5$ and s is a quadratic nonresidue modulo p,
(c) $G = \langle x, y, z \mid y^3 = z^9 = 1 = [y, z], x^3 = z^3, y^x = yz^{-3}, z^x = zy\rangle$.

We turn to the case $|G| = 2^n$. Here, for simplicity, we shall prove our main result only for $n \geq 6$. But for this, we first have to study certain groups of order 2^5.

LEMMA 3.9. *If G is a dihedral or semidihedral group of order 2^5 or if $G = \langle a, b \mid a^8 = b^4 = 1, a^b = a^{-1}\rangle$, then $L(G)^*$ contains a subdivision of K_5.*

PROOF. Suppose first that $G \simeq D_{32}$ or $G \simeq S_{32}$. Then in both cases, G has exactly three maximal subgroups A, D, M such that $A = \langle a\rangle$ is cyclic, $D \simeq D_{16}$ and M is quaternion or dihedral; furthermore $Z(G) = \langle a^8\rangle$ and $G/Z(G) \simeq D_{16}$. Let $\Gamma := \{Z(G), \langle a^4\rangle, U, V, D\}$ where U and V are the two dihedral subgroups of order 8 of D. Since $U \cap V = \langle a^4\rangle$, we see that there are 5 edges in $L(G)^*$ between members of Γ. Further obvious paths are $(\langle a^4\rangle, \langle a^2\rangle, D)$ and $(D, G, M, M_1, M_2, Z(G))$ with noncyclic $M_1 \lessdot M$ and $\langle a^4\rangle \neq M_2 \lessdot M_1$. Since $U/Z(G)$ and $V/Z(G)$ are elementary abelian of order 4, there are trivial paths $(Z(G), U_1, U)$ and $(Z(G), V_1, V)$ with $U_1 \neq \langle a^4\rangle \neq V_1$ and there are further noncyclic maximal subgroups U_2 of U and V_2 of V. So we finally get a path $(U, U_2, X, 1, Y, V_2, V)$ with $|X| = |Y| = 2$ and $X \neq Z(G) \neq Y$. All these paths are internally disjoint and so Γ is a K_5-set in $L(G)^*$.

Now let $G = \langle a, b \mid a^8 = b^4 = 1, a^b = a^{-1}\rangle$. Then $Z(G) = \langle a^4, b^2\rangle$ and $G/\langle b^2\rangle \simeq D_{16}$. Furthermore G has three maximal subgroups $M = \langle a\rangle \times \langle b^2\rangle$ and M_1, M_2 for which $M_i/\langle b^2\rangle \simeq D_8$ $(i = 1, 2)$. Clearly $\Phi(G) = M_1 \cap M_2 = \langle a^2, b^2\rangle$ and we let $\Gamma := \{M_1, M_2, \Phi(G), Z(G), \langle b^2\rangle\}$. This time there are 4 edges between members of Γ, two trivial paths from M_i to $Z(G)$ and a third noncyclic maximal subgroup H_i in M_i which yields a path from M_i to $\langle b^2\rangle$ for $i = 1, 2$. The final two paths are (M_1, G, M_2) and $(\langle b^2\rangle, 1, \langle a^4\rangle, \langle a^2\rangle, \Phi(G))$. So Γ is a K_5-set in $L(G)^*$. □

LEMMA 3.10. *Let $6 \leq n \in \mathbb{N}$ and suppose that G is a nonabelian group of order 2^n such that $L(G)^*$ is K_5-free. Then G is one of the following groups.*

(a) $G = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{1+2^{n-2}}\rangle$
(b) $G = \langle a, b \mid a^{2^{n-2}} = b^4 = 1, a^b = a^{1+2^{n-3}}\rangle$
(c) $G = \langle a, b \mid a^{2^{n-2}} = b^4 = 1, a^b = a^{1+2^{n-4}}\rangle$
(d) $G = \langle a, b \mid a^4 = b^{2^{n-2}} = 1, a^b = a^{-1}\rangle$
(e) $G = \langle a, b \mid a^8 = 1, b^{2^{n-3}} = a^4, a^b = a^{-1}\rangle$

Proof. By Lemma 3.3, G is metacyclic and has exponent at least 2^{n-2}. If $\exp G = 2^{n-1}$, then either G is the group with modular subgroup lattice in (a) or G is dihedral, quaternion, or semidihedral [3, p. 90]. In the latter three cases, $G/Z(G) \simeq D_{2^{n-1}}$ which is impossible by Lemma 3.9 since $n \geq 6$. Thus G satisfies (a).

So let $\exp G = 2^{n-2}$. Since G is metacyclic, there exists $N \trianglelefteq G$ such that $N = \langle a \rangle$ is cyclic and $G = N\langle x \rangle$ for some $x \in G$. Consider first the case that no such N has order 2^{n-2} and let $H = N\langle x^2 \rangle$. Since $a^{x^2} = a^r$ with $r \equiv 1 (\mathrm{mod}\, 4), L(H)$ is modular [5, 2.3.4] and since H is generated by two elements of order at most 2^{n-3}, it follows from [5, 2.3.5] that $\exp H \leq 2^{n-3}$. Hence there exists $b \in G \setminus H$ such that $o(b) = 2^{n-2}$. Then $G = N\langle b \rangle$ and $|N : N \cap \langle b \rangle| = |G : \langle b \rangle| = 4$. Clearly, $a^b = a^t$ for some $t \in \mathbb{N}$ and if $t \equiv 1 (\mathrm{mod}\, 4)$, then $[a, b] \in \langle b \rangle$ and $\langle b \rangle \trianglelefteq G$ which would contradict our assumption in this case. So $t \not\equiv 1 (\mathrm{mod}\, 4)$ and it follows that $|N \cap \langle b \rangle| \leq 2$. Thus either $|N| = 4$ and G is the group in (d) or $|N| = 8$, $b^{2^{n-3}} = a^4$ and $a^b = a^{-1}$ or $a^b = a^3$. In the latter case, $(ab^{2^{n-4}})^b = a^3 b^{2^{n-4}} = (ab^{2^{n-4}})^{-1}$ and we may replace a by $ab^{2^{n-4}}$ to obtain the group in (e).

It is left to consider the case that $G = N\langle x \rangle$ where $N = \langle a \rangle \trianglelefteq G$ and $o(a) = 2^{n-2}$. Again $a^x = a^t$ for some $t \in \mathbb{N}$ and if $t \equiv 1 (\mathrm{mod}\, 4)$, then by [5, 2.3.4], $L(G)$ is modular. Since $\exp G \geq 2^4$, G is not hamiltonian; then [5, 2.3.11] implies that N has a complement $\langle b \rangle$ in G. Thus G is one of the groups in (b) or (c).

So let $t \not\equiv 1 (\mathrm{mod}\, 4)$. Then again $|N \cap \langle x \rangle| \leq 2$ and we show that this case cannot occur. If x induces an automorphism of order 2 in N, then $a^x = a^{-1}$ or $a^x = a^{-1+2^{n-3}}$ and so $G/\langle x^2 \rangle$ is a dihedral or semidihedral group of order 2^{n-1} or 2^{n-2}. This contradicts Lemma 3.9 except when $n = 6$ and $N \cap \langle x \rangle \neq 1$. But in this case, $G/\langle a^8 \rangle$ is the third group of order 2^5 in Lemma 3.9, again a contradiction. Thus x induces an automorphism of order 4 in N. Since $\mathrm{Aut}\, N$ has only two cyclic subgroups of order 4 [3, p. 84] and $t \not\equiv 1 (\mathrm{mod}\, 4)$, it follows that $a^b = a^{-1+2^{n-4}}$ for $b = x$ or $b = x^{-1}$. Then $(ab)^4 = a^{2^{n-3}} b^4$; so if $N \cap \langle b \rangle \neq 1$, we have $(ab)^4 = 1$ and can replace b by ab. Thus we may assume that $N \cap \langle b \rangle = 1$. Then $G/\langle a^{2^{n-3}}, b^2 \rangle$ is semidihedral of order 2^{n-2} and Lemma 3.9 implies that $n = 6$. So $G = \langle a, b \mid a^{16} = b^4 = 1, a^b = a^3 \rangle$ and G has the following maximal subgroups: $H := \langle a, b^2 \rangle$ has modular subgroup lattice, $S := \langle a^2, b \rangle$ and $T := \langle a^2, ab \rangle$. Put $W := \Omega_2(H) = \langle a^4, b^2 \rangle$ and $Z := Z(G) = \langle a^8 \rangle$ and let $\Gamma := \{Z, W, \Phi(G), S, T\}$. There are 3 edges (connecting $\Phi(G)$ to W, S, T) and 4 obvious paths $(S, G, T), (Z, Z\langle b^2 \rangle, W), (Z, \langle a^4 \rangle, \langle a^2 \rangle, \Phi(G))$ and $(Z, 1, \langle b^2 \rangle, \langle b \rangle, \langle a^8, b \rangle, \langle a^4, b \rangle, S)$. Since $G/W \simeq D_8$, there are trivial paths (S, S_1, W) and (T, T_1, W) with $T_1 \neq \Phi(G) \neq S_1 \neq \langle a^4, b \rangle$. Finally, $(ab)^2 =$

$= a^{12}b^2$ and so if $X = \langle (ab)^2 \rangle$, then $T/X \simeq D_8$ and $Z \lessdot X$. Thus there exists a noncyclic subgroup T_2/X of order 4 with $T_2 \neq T_1$ and a path (T, T_2, T_3, X, Z) so that all the paths considered are internally disjoint. So $L(G)^*$ contains a subdivision of K_5, a contradiction. □

PROOF OF THEOREM B. Let $|G| = 2^n$ where $6 \le n \in \mathbb{N}$. Again by [5, 2,3.1 and 2.5.9], the groups in (a) – (c) of Lemma 3.10 have modular subgroup lattice and are therefore lattice-isomorphic to the abelian groups mentioned in Theorem B. So if $L(G)^*$ is K_5-free, then by Lemmas 3.3 and 3.10, G is one of the groups given in this theorem. Conversely, if G is lattice-isomorphic to an abelian group of type $(2^n), (2^{n-1}, 2)$, or $(2^{n-2}, 4)$, then by Lemma 3.6 and [7, Theorem A], G is K_5-free.

So, finally, let G be one of the other two groups given in Theorem B. Then in both cases, $b^2 \in Z(G)$; let $\langle b^2 \rangle =: Z$ and $H := \langle a, b^2 \rangle$. Then H is abelian of exponent 2^{n-3} and hence there exists a complement $\langle c \rangle$ to Z in H; so $H = Z \times \langle c \rangle$ with $o(c) = 4$.

Clearly, $G/Z \simeq D_8$. So every $x \in G \setminus H$ satisfies $x^2 \in Z$ and $\langle a \rangle \langle x \rangle = G$ since H is the unique maximal subgroup of G containing $\langle a \rangle$. Then $H = \langle a \rangle (H \cap \langle x \rangle) = \langle a \rangle \langle x^2 \rangle$ and since $x^2 \in Z$, it follows that $\langle x^2 \rangle = Z$. Since every subgroup of G not contained in H contains an element $x \in G \setminus H$, it follows that $L(G)$ consists of the subgroups of H, four cyclic subgroups $X_1, \dots, X_4$ containing Z as a maximal subgroup, two maximal subgroups M_1 and M_2 different from H, and G.

Now it is easy to see that $L_5(G) \subseteq L(H)$. For, these cyclic subgroups X_i have degree 2 in $L(G)^*$, G has degree 3, and both M_i have degree 4; however, also M_i cannot be a member of a K_5-set since two of the four paths from M_i to the other members would have to start with a cyclic maximal subgroup of M_i and then use $\Phi(M_i)$, which is impossible.

Now suppose, for a contradiction, that Γ is a K_5-set in $L(G)^*$. Then by Lemma 3.5, it is also a K_5-set in $\hat{\mathcal{L}}$ where $\mathcal{L} = L(H)$. Clearly, $\hat{\mathcal{L}}$ contains $L(H)^*$ and a further edge can only connect two of the three groups $Z, Z \times \langle c^2 \rangle, H$ since no other subgroup of H is covered by an element of $L(G) \setminus L(H)$. Since there are already edges from $Z \times \langle c^2 \rangle$ to Z and to H in $L(H)^*$, the only additional edge is $\{Z, H\}$. Let $\overline{Z}$ be a cyclic 2-group containing Z as a maximal subgroup and consider $H = Z \times \langle c \rangle$ as a subgroup of index 2 in $\overline{G} := \overline{Z} \times \langle c \rangle$. In the subdivision of K_5 given by Γ in $\hat{\mathcal{L}}$, we replace the edge $\{Z, H\}$ – if it appears in one of the paths – by the path $(Z, \overline{Z}, \overline{Z} \times \langle c^2 \rangle, \overline{G}, H)$. Then we get a subdivision of K_5 in $L(\overline{G})^*$; but this contradicts Lemma 3.6. □

As in the case $p > 2$ it is not difficult to determine the 2-groups of order less than 2^6 with K_5-free Hasse graph. Again we give the result without proof.

REMARK 3.11. Let $|G| = 2^n, n \leq 5$. Then $L(G)^*$ is K_5-free if and only if one of the following holds:

(a) $n \leq 4$ and G is metacyclic,

(b) G is one of the groups in Theorem B for $n = 5$,

(c) $G \simeq Q_{32}$.

4. Proof of Theorem C.

It remains to determine the nonprimary finite groups with K_5-free Hasse graphs. For this, we first look at direct products of lattices.

LEMMA 4.1. *Let $L = L_1 \times L_2$ with finite lattices L_i. Then L^* contains a subdivision of K_5 if one of the following holds.*

(a) *L_1 contains $L(C_2 \times C_2)$ as a sublattice and $|L_2| \geq 3$.*

(b) *L_1 contains $L(C_4 \times C_2)$ as a sublattice and $|L_2| \geq 2$.*

PROOF. (a) We may assume that $L_1 = L(H) = \{1, M_1, M_2, M_3, H\}$ where $|H| = 4$ and $|M_i| = 2$ for $i = 1, 2, 3$ and that L_2 is a chain $A < B < C$ of length 2. Then

$$\Gamma := \{H \times A, 1 \times B, M_1 \times B, M_2 \times B, H \times B\}$$

is a K_5-set in L^* since there are 5 trivial paths between members of Γ in the interval $[H \times B/1 \times B]$, furthermore 4 obvious paths from $H \times A$ to $K \times B$ via $K \times A$, and finally the path $(M_1 \times B, M_1 \times C, H \times C, M_2 \times C, M_2 \times B)$.

(b) Here we may assume that $L_1 = L(H)$ where $H = \langle a \rangle \times \langle b \rangle$ with $o(a) = 4$, $o(b) = 2$ and that $L_2 = \{0, I\}$ is a chain of length 1. Then

$$\Gamma := \{1 \times 0, \langle a^2 \rangle \times 0, \langle a^2, b \rangle \times 0, \langle a^2 \rangle \times I, \langle a^2, b \rangle \times I\}$$

is a K_5-set in L^* since there are 5 edges between members of Γ and we have the further paths $(1 \times 0, \langle a^2 b \rangle \times 0, \langle a^2, b \rangle \times 0), (1 \times 0, 1 \times I, \langle a^2 \rangle \times I)$, $(1 \times 0, \langle b \rangle \times 0, \langle b \rangle \times I, \langle a^2, b \rangle \times I)$, $(\langle a^2 \rangle \times 0, \langle ab \rangle \times 0, \langle ab \rangle \times I, H \times I, \langle a^2, b \rangle \times I)$, and $(\langle a^2, b \rangle \times 0, H \times 0, \langle a \rangle \times 0, \langle a \rangle \times I, \langle a^2 \rangle \times I)$. □

There is a similar result for the Hasse graph of a semidirect product of two groups. This graph contains a subdivision of K_5 if a certain smaller configuration occurs in the Hasse graph of the complement. We first give this configuration a name.

DEFINITION 4.2. A (3,6)-*gon* in a graph is a set of 3 distinct points R, S, T together with an internally disjoint set of 6 paths between them, one between R and S, two between S and T, and three between R and T.

Since at most one path between two points can be an edge, there are intermediate points in one of the paths between S and T and in two of the paths between R and T (see Figure 3).

LEMMA 4.3. *Let $G = NK$ where $1 \neq N \trianglelefteq G$ and $N \cap K = 1$. If $L(K)^*$ contains a (3,6)-gon, then $L(G)^*$ contains a subdivision of K_5.*

PROOF. Let R, S, T be the points of a (3,6)-gon in $L(K)^*$, let $\varepsilon, \delta_1, \delta_2, \rho_1, \rho_2, \rho_3$ be the appropriate paths from R to S, S to T, and R to T with interior points U, V, W of δ_2, ρ_2, ρ_3, respectively. For every path $\gamma = (X_0, \ldots, X_r)$, we let $\gamma^{-1} := (X_r, \ldots, X_0)$ be the path with the same endpoints in opposite direction. The isomorphism theorem for groups implies that if all the X_i are subgroups of K, then $\hat{\gamma} := (NX_0, \ldots, NX_r)$ is a path in $L(G)^*$ and that the set $\{\hat{\varepsilon}, \hat{\delta}_1, \hat{\delta}_2, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\}$ is internally disjoint. Finally, for every subgroup X of K, we let $\gamma(X) = (X_0, X_1, \ldots, X_r)$ be a fixed path in $L(G)^*$ from X to NX such that $X_i \lessdot X_{i+1}$ for all i. By Dedekind's law, $X_i = (N \cap X_i)X$ and hence $X_i \cap K = X$ for all i. This shows that for $X \neq Y \leq K$, the paths $\gamma(X)$ and $\gamma(Y)$ are disjoint.

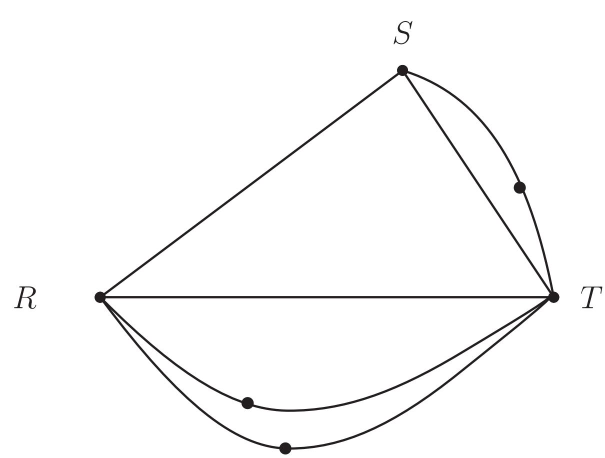

Figure 3

We now let $\Gamma := \{R, S, T, NR, NT\}$. There are obvious paths $\varepsilon, \delta_1, \rho_1$ between R, S, T, $\gamma(R)$ between R and NR, $\gamma(T)$ between T and NT, and $\hat{\rho}_1$ between NR and NT. Furthermore, $\gamma(S)$ and $\hat{\varepsilon}^{-1}$ yield a path from S to NR (via NS), and we get a path from S to NT using δ_2 from S to U, then $\gamma(U)$, and finally $\hat{\delta}_2$ from NU to NT. Similarly we get paths from R to NT via $\rho_2, \gamma(V), \hat{\rho}_2$ and from T to NR via $\rho_3^{-1}, \gamma(W), \hat{\rho}_3^{-1}$. Thus Γ is a K_5-set in $L(G)^*$. □

We shall need the following result on groups without (3,6)-gons.

PROPOSITION 4.4. *Let K be a group such that $L(K)^*$ contains no (3,6)-gon. Then K has one of the following properties (where p and q are primes and $2 \leq n \in \mathbb{N}$).*

(*a*) *K is cyclic.*
(*b*) *K is abelian of type $(2^n, 2)$ or lattice-isomorphic to this group.*
(*c*) *K is elementary abelian of order p^2 or nonabelian of order pq.*
(*d*) *K is isomorphic to Q_8, $C_4 \times C_4$, or $C_p \times C_2 \times C_2$ where $p > 2$.*

PROOF. Let K be a minimal counterexample and suppose first that K is a 2-group. The trivial subgroup T and the two noncyclic subgroups R, S of order 4 clearly yield a (3,6)-gon in $L(D_8)^*$. So D_8 cannot be involved in K and [5, 2.3.3] shows that $L(K)$ is modular. Since $L(D_8)^*$ is a subgraph of $L(C_2 \times C_2 \times C_2)^*$, K is generated by two elements and so, again by [5, 2.3.1 and 2.5.9], either $K \simeq Q_8$ or $L(K) \simeq L(H)$ where H is abelian of type (2^n) or $(2^n, 2^m)$. However, if $C_8 \times C_4 \simeq H_0 \leq H$, then $R = \Omega_2(H_0), T = \Omega(H_0), S = \mho_2(H_0)$ would yield a (3,6)-gon in $L(H)^*$, a contradiction. It follows that $K \simeq H \simeq C_4 \times C_4$ or that $m = 1$. In every case, K is one of the groups in (a) – (d), a contradiction. So K is not a 2-group.

Now suppose, for a contradiction, that there are subgroups $A \trianglelefteq P \leq K$ such that P/A is elementary abelian of order $p^2, p > 2$, or nonabelian of order $pq, p > q, p$ and q primes. Suppose first that $P = K$. Then since K is a counterexample, $A \neq 1$. Let $A < S < K$. If S would have a maximal subgroup $B \neq A$, then $R = K$, $T = A$ and S would yield a (3,6)-gon in $L(K)^*$ with 5 paths inside $[K/A]$ and the sixth connecting S to T via B and $T \cap B$. This contradiction shows that all subgroups strictly between A and K are cyclic of prime power order; since $A \neq 1$, it follows that K is a p-group all of whose maximal subgroups are cyclic. Hence $K \simeq Q_8$ (see [3, p. 311]), a contradiction since $p > 2$.

Thus $P < K$ and the minimality of K implies that $A = 1$ and that P is a maximal subgroup of K. Let $x \in K \setminus P$. If $P^x = P$ and x normalizes a minimal subgroup M of P, then $R=1$, $T=P$ together with $S=P\cap\langle x\rangle$ if $P\cap\langle x\rangle\neq 1$ and $S=M$ if $P\cap\langle x\rangle=1$ yield a (3,6)-gon in $L(K)^*$ with 5 paths inside $[P/1]$, the sixth path being $(S,\langle x\rangle,K,T)$ or $(S,S\langle x\rangle,K,T)$, respectively. This is a contradiction; so if $P^x=P$, then x operates irreducibly on P. In particular, $\langle x\rangle\cap P=1$; now $R=P,\ T=1$ and $S=K$ yield a (3,6)-gon, again a contradiction. It follows that $P^x\neq P$. If $P\cap P^x\neq 1$, again $R=1,\ T=P$ and $S=P\cap P^x$ would yield a (3,6)-gon with sixth path (S,P^x,K,T). So, finally, $P\cap P^x=1$ for all $x\in K\setminus P$; but then again $R=P,\ T=1$ and $S=K$ yield a (3,6)-gon. This is the desired contradiction which shows that for $p>2$, neither $C_p\times C_p$ nor a nonabelian group of order pq ($p>q\in\mathbb{P}$) are involved in K.

It follows that Sylow p-subgroups are cyclic; moreover, Burnside's theorem [3, p. 419] implies that K has a normal p-complement. The intersection of all these normal p-complements ($2<p\in\mathbb{P}$) is a normal Sylow 2-subgroup H of K with cyclic factor group. The minimality of K implies that H is one of the groups in (a) – (d) and $K=HC$ with cyclic group C of odd order. If C would operate nontrivially on H, then it would also operate nontrivially on $H/\Phi(H)$ (see [3, p. 275]). So $|H/\Phi(H)|=4$ and it would follow that the alternating group A_4 would be involved in K. But $L(A_4)^*$ has a (3,6)-gon given by $R=A_4,\ T=1$ and the subgroup S of order 4. This contradiction shows that $K=H\times C$.

Since K is not a 2-group, $C\neq 1$; since K is a counterexample, H is not cyclic. So $|H:\Phi(H)|=4$ and if $\Phi(H)\neq 1$, then $R=H,\ T=\Phi(H)$ and $S=T\times C$ would yield a (3,6)-gon in $L(K)^*$, a contradiction. Thus $|H|=4$. Since K is a counterexample, C contains a subgroup $D\simeq C_p\times C_q$ or $E\simeq C_{p^2}$, p and q primes. However, in $L(H\times D)^*$ we get a (3,6)-gon $R=H,\ S=D$ and $T=1$ with obvious paths. In $L(H\times E)^*$ we may take $R=E,\ S=\Omega(E)$ and $T=H\times S$; here, if A,B,C are the three subgroups of order 2 of H, we have the paths $(S,1,A,H,T),(S,A\times S,T)$ and $(R,A\times R,H\times R,T),(R,B\times R,B\times S,T),(R,C\times R,C\times S,T)$ to get a (3,6)-gon. This is a final contradiction. □

Since one of the three points of a (3,6)-gon has degree at least 5 in the graph, it is easy to see that the Hasse graphs of the groups in (b) – (d) of Proposition 4.4 contain no (3,6)-gon and that the cyclic groups with this property are those with planar subgroup lattice and those whose order is the product of 4 pairwise different primes. We shall not need this.

An immediate consequence of 4.1 – 4.4 is the following result on coprime direct products which we shall use quite often. First of all, it completes the determination of nilpotent groups with K_5-free Hasse graphs, that is, it implies Theorem C.

LEMMA 4.5. *Let $G = A \times B$ where $A \neq 1 \neq B$ and $(|A|, |B|) = 1$. If $L(G)^*$ is K_5-free, then either G is cyclic or $G = P \times K$ where $|P| = p$ and K is isomorphic to Q_8 or elementary abelian of order q^2 or nonabelian of order qr with pairwise different primes p, q, r.*

PROOF. Suppose that G is not cyclic. Then one of the two factors A and B, say B, is not cyclic. By [6, Theorem 1.1], $L(B)$ contains $L(C_2 \times C_2)$ as a sublattice. By [5, 1.6.4], $L(G) \simeq L(A) \times L(B)$ and so Lemma 4.1, (a) shows that $|L(A)| \leq 2$. Thus $|A| = p$ for some prime p and now Lemma 4.1, (b) yields that $L(B)$ does not contain $L(C_4 \times C_2)$ as a sublattice. By Lemma 4.3, $L(B)^*$ contains no (3,6)-gon and so Proposition 4.4 implies that G is one of the groups in Lemma 4.5 or $B \simeq C_q \times C_2 \times C_2$ with $q \neq 2 \neq p$. But in this case, $G \simeq (C_p \times C_q) \times (C_2 \times C_2)$ which would contradict Lemma 4.1,(a). □

PROOF OF THEOREM C. If $L(G)^*$ is K_5-free, then by Lemma 4.5, G is one of the groups given in the theorem. Conversely, if $G = P \times Q$ with $|P| = p$ is one of these groups, then $L(G)^*$ contains only four points of degree at least 4, namely $Z(Q), Q, P \times Z(Q), G$ if $Q \simeq Q_8$ and $1, Q, P, G$ if $Q \simeq C_q \times C_q$. Hence there is no K_5-set in $L(G)^*$. □

5. Proof of Theorem D.

LEMMA 5.1. *All the groups in (a) – (h) of Theorem D have K_5-free Hasse graphs.*

PROOF. In most cases this is rather obvious; in some cases it even follows already from the fact that there do not exist 5 subgroups of G with degree at least 4 in $L(G)^*$. This, for example, holds if G satisfies (a). For, then $C_K(P)$ and the subgroups properly containing $P \times C_K(P)$ have degree $p+1$ or $p+2$, whereas all other subgroups have degree at most 3; since $|K : C_K(P)| \leq q^3$, there are at most 4 subgroups of degree at least 4. And if G satisfies (b), then G has precisely 4 subgroups of degree at least 4. If G satisfies (d), then in addition to $C_K(P)$ and the subgroups properly containing $P \times C_K(P)$, also $1, P$, and $P \times C_K(P)$ have degree at least 4. But every path between one of the groups $1, P$, or $C_K(P)$ and one of the groups containing $P \times C_K(P)$ has to use $C_K(P)$ or $P \times C_K(P)$. Since we would need at least 4 paths of this type for a K_5-set, it follows that $L(G)^*$ is K_5-free. Similarly, if G satisfies (e), then the only subgroups of degree at least 4 are $1, P, PQ, PR, G$ where $K = QR$ with $|Q| = q$ and $|R| = r$; however, there is no path from P to G which does not use PQ, PR, or 1. Finally, if G satisfies (f), then $q \nmid p-1$ since K is irreducible on $P/\Phi(P)$. Hence K centralizes $\Phi(P)$ and it follows that $1, \Phi(P), P, G$ are the only subgroups of degree at least 4 in $L(G)^*$.

Now suppose that G satisfies (c). Then subgroups of order q^n have degree 2 and subgroups of order pq^n of G have degree $p+2$ in $L(G)^*$; however, all but 2 of the paths from such a subgroup to the other members of a K_5-set would have to start with a cyclic subgroup of order q^n and then use $C_K(P)$, which is impossible. So if we let $H := P \times C_K(P)$, it follows that $L_5(G) \subseteq \subseteq L(H) \cup \{G\} =: \mathcal{L}$. Since $C_K(P) \leq \langle x \rangle$ for every $x \in G \setminus H$, the edges of $\hat{\mathcal{L}}$ are the edges of $L(H)^*$ together with $\{H, C_K(P)\}, \{H, G\}, \{C_K(P), G\}$ and $\{\Omega(P) \times C_K(P), G\}$. Thus $\hat{\mathcal{L}}$ is planar and by Lemma 3.5, $L(G)^*$ is K_5-free.

Now let G be the group in (g) with $|K| = 9$. Then there are only 6 subgroups of degree at least 4, namely $\Phi(P), P, C_K(P), \Phi(P) \times C_K(P)$, $P \times C_K(P)$, and G. So $L_5(G) \subseteq \mathcal{L}$ where $\mathcal{L}$ is the set of these 6 subgroups. It is easy to see that $P, C_K(P)$ and G have degree 3 in $\hat{\mathcal{L}}$. By Lemma 3.5, $L(G)^*$ is K_5-free. The other group in (g) is isomorphic to $G/C_K(P)$.

Finally, suppose that G satisfies (h). Then every maximal subgroup M of P has degree $p+2$; however, all but 2 of the paths from M to a member of a K_5-set would have to start with a cyclic maximal subgroup of M and then use $\Phi(M)$. This is impossible. Since K is also irreducible on $\Phi(P) = = \Omega(P)$, a dual argument shows that also $\Phi(M)$ cannot be a member of a K_5-set. Furthermore, $\Phi(P)K$ has degree p^2+2, but again all but 2 paths would start with cyclic subgroups of order q and contain 1, which is impossible. This leaves only $1, \Phi(P), P, G$ as possible members of $L_5(G)$. So there is no K_5-set in $L(G)^*$. □

We now prove in a couple of steps that every nonnilpotent finite group with K_5-free Hasse graph is one of the groups in (a) – (h) of Theorem D. First we study groups with a normal Sylow p-subgroup. For this we need two simple results on small groups.

LEMMA 5.2. *Let $p, q \in \mathbb{P}$ such that $p \neq q$ and suppose that $G = PQ$ where P is an elementary abelian normal subgroup of order p^2, $|Q| = q$ and $[P, Q] \neq 1$. If $L(G)^*$ is K_5-free, then Q operates irreducibly on P.*

Proof. If not, then by Maschke's theorem, $P = P_1 \times P_2$ where $P_i \trianglelefteq G$ $(i = 1, 2)$ and P_1Q, say, is nonabelian of order pq. Then $\Gamma := \{1, P_2, P, P_1Q, G\}$ is a K_5-set in $L(G)^*$; here 7 paths are trivial in the sense of §2 and since $p > q$, there are further paths (P, P_1, P_1Q), (P_2, P_2Q, Q, P_1Q) and $(1, Q^x, P_2Q^x, G)$ for some $x \in P$. This contradicts our assumption. □

Lemma 5.3. *Let $G = PK$ where $P \trianglelefteq G$, $|P| = p$ or p^2, p does not divide $|K|$ and $C_K(P) = 1$. If $L(G)^*$ is K_5-free, then $|L(K)| \leq 4$.*

Proof. Suppose, for a contradiction, that $|L(K)| \geq 5$. If P is cyclic, we may assume that $|P| = p$. Then K is cyclic, so $|K| > 8$ and hence $|P| > 10$. If P is not cyclic, then by Lemma 5.2, every nontrivial subgroup of K operates irreducibly on P. Hence $|K|$ is odd and $|L(K)| \geq 5$ implies that $|P| > 10$. In both cases, $N_G(K) = K$; so there are at least 10 complements to P in G and any two of them intersect trivially (see [5, 4.1.1]).

By assumption, there are pairwise different subgroups H_i satisfying $P < H_i \leq G$ $(i = 1, \dots, 4)$; let $H_5 = 1$. We choose 10 pairwise different complements K_{ij} $(1 \leq i < j \leq 5)$ to P in G and paths δ_{ij} in $L(K_{ij})^*$ from $H_i \cap K_{ij}$ to $H_j \cap K_{ij}$ via the join of the two groups. Then the set of paths (H_i, δ_{ij}, H_j) for $1 \leq i < j \leq 4$ and (H_i, δ_{i5}) for $1 \leq i \leq 4$ is internally disjoint. So $\{H_1, \dots, H_5\}$ is a K_5-set in $L(G)^*$, a contradiction. □

Lemma 5.4. *Let $G = PK$ where P is a normal p-subgroup, p a prime, K a p'-group and $[P, K] \neq 1$. If $L(G)^*$ is K_5-free, then there are primes q, r such that either*

(a) *K is a cyclic q-group and $|K : C_K(P)| \leq q^3$, or*
(b) *K is cyclic of order qr.*

Proof. Let G be a minimal counterexample. Since then also $G/\Phi(P)$ is a counterexample [3, p. 275], $\Phi(P) = 1$ and Lemma 3.1 implies that

(1) $|P| = p$ or P is elementary abelian of order p^2.

Let $K_0 := C_K(P)$. By Lemma 5.3, $|L(K/K_0)| \leq 4$. Since the subgroup lattice of every noncyclic group contains $L(C_2 \times C_2)$ as a sublattice [6, Theorem 1.1], it follows that K/K_0 is cyclic and that there are primes q, r such that

(2) K/K_0 is cyclic of order qr or q^k where $1 \leq k \leq 3$.

Finally, by Lemma 4.3,

(3) $L(K)^*$ contains no (3,6)-gon.

Now suppose first that K/K_0 is cyclic of order qr, $q \neq r$. Since G is a counterexample, $K_0 \neq 1$. By Proposition 4.4, K is cyclic; otherwise $K \simeq C_q \times C_2 \times C_2$ and PH with $C_2 \times C_2 \simeq H \leq K$ would be a counterexample of order less than $|G|$. Let Q and R be the subgroups between K_0 and K such that $|Q : K_0| = q$ and $|R : K_0| = r$. The minimality of G implies that Q and R satisfy (a) or (b) of the lemma; since $K_0 \neq 1$, it follows that $|K_0|$ is a prime and $|K_0| \neq r$, say. Lemma 4.3, applied with $N = K_0$, yields that $L(PR)^*$ contains no (3,6)-gon; by Proposition 4.4, $|P| = p$. We claim that $\Gamma := \{1, K_0, PQ, PR, G\}$ is a K_5-set in $L(G)^*$. Here only 3 of the required paths are trivial, but, in addition, there are 3 obvious paths $(1, P, P \times K_0, PQ)$, (K_0, Q, PQ) and (K_0, R, PR). Since $p \geq 5$, there exist $a, b, c, d \in P$ such that K, K^a, K^b, K^c, K^d are pairwise different. So if we let S be the subgroup of order r in K, we have 4 further paths $(1, S^a, PS, PR)$, $(1, S^b, K_0S^b, K^b, G)$, (K_0, Q^c, K^c, G) and $(PQ, PQ \cap K^d, K^d, PR \cap K^d, PR)$. Since any two different conjugates of K intersect in K_0, the set of these paths is internally disjoint. So Γ is a K_5-set, a contradiction.

It follows that K/K_0 is cyclic of order q^k where $1 \leq k \leq 3$. We claim that $k = 1$ and

(4) K is elementary abelian of order q^2 or nonabelian of order qt with $q < t \in \mathbb{P}$.

For this, suppose first that K is a q-group. If $k \geq 2$, then every maximal subgroup of K would operate nontrivially on P and hence would be cyclic by the minimality of G. So K would be isomorphic to $C_q \times C_q$ or Q_8 [3, p. 311], which would be impossible since $k \geq 2$. Thus $k = 1$ and hence $\Phi(K) \leq K_0$. So $\Phi(K) \trianglelefteq G$ and the minimality of G implies that $\Phi(K) = 1$. By Lemma 3.1, (4) holds. – Now suppose that K is not a q-group. If $K = S \times T$ with $S \simeq C_2 \times C_2$ and $|T| \in \mathbb{P}$, then the minimality of G would imply that $t = q$; so $S = K_0$ and $G = PT \times S$, contradicting Lemma 4.5. Thus Proposition 4.4 implies that (4) holds or that K is cyclic. In the latter case, let Q be the Sylow q-subgroup of K. Then Lemma 4.3, applied with a q-complement N in K_0, yields that $L(PQ)^*$ contains no (3,6)-gon; by Proposition 4.4, $|PQ| = pq$. It follows that $k = 1$ and $G = PQ \times K_0$. Now Lemma 4.5 implies that $|K_0|$ is a prime; but this is impossible since G is a counterexample. Thus (4) holds in all cases.

Let $Q \leq K$ such that $|Q| = q$ and $Q \neq K_0$. By Lemma 4.3, applied with $N = K_0$, $L(PQ)^*$ contains no (3,6)-gon and Proposition 4.4 implies that $|P| = p$ and PQ is nonabelian of order pq. We claim that $\Gamma := \{1, P, K_0, K, G\}$ is a K_5-set in $L(G)^*$. Since G/K_0, G/P and K are nonabelian of order pq or qt or elementary abelian of order q^2, 7 of the required 10 paths are trivial. Furthermore we have paths $(P, P \times K_0, K_0)$, (P, PQ, Q, K) and (even if $p = 3$ and $q = 2$, so that $G \simeq D_{12}$) we may choose a conjugate $K^x \neq K$ to get the final path $(1, Q^x, K^x, G)$ internally disjoint to all the trivial paths chosen above. Thus Γ is a K_5-set, a final contradiction. □

Now we come to the main step in the proof of Theorem D. The following Lemma shows, for example, that every group with K_5-free Hasse graph has a Sylow tower and hence is soluble.

Lemma 5.5. *Let G be a finite group such that $L(G)^*$ is K_5-free and let $P \in \mathrm{Syl}_p(G)$. If P is not cyclic, then $P \trianglelefteq G$.*

Proof. Let G be a minimal counterexample and let p be the smallest prime dividing $|G|$ for which a Sylow p-subgroup P of G is not cyclic and not normal in G. Clearly, if $P \leq X < G$, then $P \trianglelefteq X$ and hence

(1) $H := N_G(P)$ is the unique maximal subgroup of G containing P.

We show next that

(2) G has no normal r-complement ($r \in \mathbb{P}, r \mid |G|$).

This is clear if $r \neq p$ because $P \leq N < G$ for such a normal r complement N and this would imply that $P \trianglelefteq G$. So suppose that N is a normal p-complement. Then for every prime q dividing $|N|$, there would exist a P-invariant Sylow q-subgroup Q of N [5, 4.1.3(d)]. Since P is not cyclic, Lemma 5.4 would imply that $[Q, P] = 1$. Thus $[N, P] = 1$ and $G = N \times P$, a contradiction.

(3) $p = 2$.

To see this, let q be the smallest prime dividing $|G|$ and let $Q \in \mathrm{Syl}_q(G)$. By (2) and Burnside's theorem [3, p. 420], Q is not cyclic. So if $q \neq p$, then the choice of p would imply that $Q \trianglelefteq G$. But then Lemma 5.4 would yield that $[Q, P] = 1$ and this would contradict Lemma 4.5. Thus $p = q$. Now (2) and Frobenius' theorem [3, p. 436] imply that there exist $P_0 \leq P$ and $x \in G$ of prime power order r^n where $r \neq p$ such that x induces a nontrivial automorphism on P_0 and hence also on $P_0/\Phi(P_0)$. By Lemma 3.1, $|P_0/\Phi(P_0)| = p^2$ and so $r \mid p^2 - 1$. Since $p < r$, it follows that $r = p + 1$. Thus $p = 2$.

(4) $O_2(G) = 1$.

Suppose, for a contradiction, that $O_2(G) \neq 1$ and let $N \leq O_2(G)$ be a minimal normal subgroup of G. By Lemma 3.1, $|N| \leq 4$. The minimality of G implies that P/N is cyclic; by Burnside's theorem, G/N has a normal 2-complement A/N. By Schur-Zassenhaus [3, p. 126], there exists a complement K to N in A. If $N \leq Z(A)$, then K would be a normal 2-complement of G, contradicting (2). So $N \not\leq Z(A)$, hence $|N| = 4$ and $|A : C_A(N)| = 3$. Clearly, $C_A(N) = N \times C_K(N)$ and P normalizes $C_K(N) = O_{2'}(A)$. So Lemmas 4.5 and 5.4 imply that $C_K(N) = 1$. Thus $A \simeq A_4$ and if $A < B \leq G$ such that $|B : A| = 2$, then $C_B(A) = 1$, again by Lemma 3.1. It follows that $B \simeq S_4$ and the minimality of G implies that $B = G$. We claim that $\Gamma := \{1, N, P, P^x, G\}$ with $o(x) = 3$ is a K_5-set in $L(G)^*$. For, we have paths $(1, Q, A, G)$ with $Q = \langle x \rangle$, $(1, Z(P), S, P)$ and $(1, Z(P^x), S^x, P^x)$ with $S \leq P$ cyclic of order 4, and connect 1 to N via the third subgroup of order 2 of N. Furthermore, $N_G(Q) \simeq S_3$ and therefore $P \cap N_G(Q) =: T$ has order 2 and is not contained in N. So we have the path

$$(P, TZ(P), T, N_G(Q), T^x, T^x Z(P^x), P^x)$$

and, in addition, can take 5 trivial paths inside $[G/N]$ so that all paths are internally disjoint. Thus Γ is a K_5-set, the desired contradiction which shows that (4) holds.

(5) If $S, T \in \mathrm{Syl}_2(G)$ such that $S \neq T$, then $S \cap T = 1$.

Choose S and T such that $S \cap T =: D$ is maximal. We show that $D \trianglelefteq G$; then (4) will imply that $D = 1$ and (5) holds. So suppose, for a contradiction, that $N_G(D) < G$. Since $\langle N_S(D), N_T(D) \rangle$ is not a 2-group, $N_G(D)$ contains more than one Sylow 2-subgroup and the minimality of G implies that the Sylow 2-subgroups of $N_G(D)$ are cyclic. In particular, $N_S(D)$ is cyclic; since S is not cyclic, it follows that $N_S(D) < S$. But then $D \trianglelefteq N_S(N_S(D)) > N_S(D)$, a contradiction. Thus (5) holds.

It follows from (1) and (5) that if $P = H$, then G would be a Frobenius group with Frobenius complement P. But then the Frobenius kernel would be a normal 2-complement, contradicting (2). Thus $P < H$ and so $H = PK$ with $K \neq 1$ and $|K|$ odd. If $[P, K] = 1$, Lemma 4.5 would imply that P is isomorphic to $C_2 \times C_2$ or Q_8. So proper subgroups of P would have no automorphisms of odd order and again G would have a normal 2-complement [3, p. 436]. So $[P, K] \neq 1$ and hence $|P/\Phi(P)| = 4$ and $|K/C_K(P)| = 3$. Thus

(6) $H = PK$ where $|K/C_K(P)| = 3$ and $|C_K(P)|$ is a prime or 1,

by Lemma 4.5. Finally, we claim that

(7) there exists $x \in N_G(K)$ such that $H \cap H^x = K$.

For this let $T = K$ if K is a 3-group and $T = C_K(P)$ if $1 \neq |C_K(P)| \neq 3$. If $N_G(T) \leq H$, then T would be a Sylow subgroup of G and $T \leq Z(N_G(T))$; by Burnside's theorem, T would have a normal complement and this would contradict (2). So $N_G(T) \not\leq H$; let $x \in N_G(T) \setminus H$. Then $T \leq H \cap H^x$; therefore, if $T = C_K(P)$, it would follow that $T \leq Z(H) \cap Z(H^x)$ and hence $T \trianglelefteq G$. But then the minimality of G would imply that $PT/T \trianglelefteq G/T$ and so $P \trianglelefteq G$, a contradiction. Thus $T = K$ and $x \in N_G(K)$; by (5), $|H \cap H^x|$ is odd, so $H \cap H^x = K$ and (7) holds.

Let $N := \Phi(P)C_K(P)$. Then $H/N \simeq A_4$ and we choose $y \in P$ such that $KN \neq K^yN$. We claim that $\Gamma := \{1, H, H^x, H^{xy}, G\}$ is a K_5-set in $L(G)^*$. For, we have edges $(H,G), (H^x,G), (H^{xy},G)$ and obvious paths from H to H^x via KN, K and KN^x, from H to H^{xy} via K^yN, $K^y = H \cap H^{xy}$ and K^yN^{xy} and from H to 1 via PN and P. Since $H^x \simeq H \simeq H^{xy}$, there are complements K_1 to P^x in H^x and K_2 to P^{xy} in H^{xy} such that $K_1N^x \neq KN^x$ and $K_2N^{xy} \neq K^yN^{xy}$; so we have paths from H^x to 1 via K_1N^x and K_1 and from H^{xy} to 1 via K_2N^{xy} and K_2. Since $|H \cap H^x|$ is odd and 4 divides $|G|$, $|G : H| > 3$. Therefore there is a conjugate H^z different from H, H^x and H^{xy} and a path from 1 to G via P^z and H^z. Finally, if a is an involution in P^x, then $\langle a, a^y \rangle$ is a dihedral group of order $2m$ where m is odd, by (5). We connect H^x to H^{xy} via $P^x, \langle a \rangle, \langle a, a^y \rangle, \langle a^y \rangle$ and P^{xy}; here all groups properly between $\langle a \rangle$ and $\langle a^y \rangle$ in this path have nonnormal Sylow 2-subgroups whereas the members of the other paths are G or are contained in normalizers of Sylow 2-subgroups. Therefore the set of all these paths is internally disjoint and Γ is a K_5-set. This is a final contradiction proving the lemma. □

We can now complete the

Proof of Theorem D. It remains to be shown that if G is a nonnilpotent finite group such that $L(G)^*$ is K_5-free, then G has one of properties (a) – (h) of Theorem D. For this we use induction on $|G|$.

We show first that $G = PK$ with $[P, K] \neq 1$ for some normal Sylow p-subgroup P and p-complement K of G. This follows from Zassenhaus' theorem [3, p. 420] if all Sylow subgroups of G are cyclic. And if P is a noncyclic Sylow p-subgroup of G, then by Lemma 5.5, $P \trianglelefteq G$. By Schur-Zassenhaus [3, p. 126], there exists $K \leq G$ such that $G = PK$ and $P \cap K = 1$. If $G = P \times K$, then Lemma 4.5 would imply that $|K|$ is a prime and hence G would be nilpotent, a contradiction. Thus $[P, K] \neq 1$.

By Lemma 5.4, K is a cyclic q-group and $|K : C_K(P)| \leq q^3$ for some prime q or K is cyclic of order qr with primes $q \neq r$. Therefore if $|P| = p$, then G satisfies (a) or (b) of Theorem D. So let $|P| > p$. If $|K| = qr$ with primes $q \neq r$ and $C_K(P) \neq 1$, then $K = C_K(P) \times R$ with $|R| = r$, say, and by Lemma 4.3, $L(PR)^*$ would contain no (3,6)-gon; but this would contradict Proposition 4.4. So, in the sequel, $|P| > p$ and either

(i) K is a cyclic q-group and $|K : C_K(P)| \leq q^3$, or
(ii) $K = Q \times R$ where $|Q| = q,\ |R| = r,\ q \neq r$, and $C_K(P) = 1$.

Now assume that P is cyclic. Then by induction, either $|P| = p^2$ or $G/\Omega(P)$ satisfies (c) of Theorem D. In the latter case, $|P| = p^3$ and, again by induction, $C_K(P) = 1$. Thus $|G| = p^3q$ and there is an obvious K_5-set $\Gamma := \{\Omega(P), \Omega_2(P), T_1, T_2, G\}$ in $L(G)^*$ where $|T_i| = p^2q$ for $i = 1, 2$ and $T_1 \neq T_2$; here, since $G/\Omega_2(P)$ and $T_i/\Omega(P)$ are nonabelian of order pq, 8 of the 10 required paths are trivial, there is a further path from $\Omega(P)$ to G via a third subgroup of order p^2q and a final path $(T_1, X_1, Y_1, 1, Y_2, X_2, T_2)$ where $|X_i| = pq$ and $|Y_i| = q$ for $i = 1, 2$. So this case cannot occur, that is, $|P| = p^2$.

If K would satisfy (ii), then $\Gamma := \{1, \Omega(P), PQ, PR, G\}$ would be a K_5-set in $L(G)^*$. For, there are 3 edges between members of Γ and the path (PQ, P, PR); and since $p \geq 7$, there are enough conjugates of K (pairwise intersecting trivially) and of $\Omega(P)Q$ and $\Omega(P)R$ to connect 1 and $\Omega(P)$ to PQ, PR, G. This contradiction shows that K is a cyclic q-group. The induction assumption then yields that $C_K(P) = 1$ and $|K| = q^2$ if $|K : C_K(P)| \neq q$. But then $\Gamma := \{1, \Omega(P), \Omega(P)\Omega(K), P\Omega(K), G\}$ is a K_5-set in $L(G)^*$. For, there are 4 edges, 2 further trivial paths and the path $(\Omega(P)\Omega(K), \Omega(P)K, G)$; and since $p \geq 5$, there are enough conjugates of K to connect $\Omega(P)$ to G and 1 to $P\Omega(K)$ and G. Thus $|K : C_K(P)| = q$ and G satisfies (c) of Theorem D.

Assume next that P is elementary abelian and not cyclic. Then by Lemma 3.1, $|P| = p^2$. If (i) holds for K, then Lemmas 4.5 and 5.2 show that G satisfies (d) of Theorem D. And if K satisfies (ii), then (e) holds.

Finally, suppose that P is neither cyclic nor elementary abelian. Then $\Phi(P) \neq 1$ and by induction, $G/\Phi(P)$ satisfies (d) or (e). In both cases, K is irreducible on $P/\Phi(P)$. But if $|P| = p^n$ and $n \geq 5$, then by Lemma 3.3, P is metacyclic and has exponent p^m where $m \geq n - 2$. So if $L(P)$ is modular, then $\Omega_{m-1}(P)$ is a characteristic maximal subgroup of P [5, 2.3.5]. And if $L(P)$ is not modular, then $p = 2$ and $P = \langle a \rangle \langle b \rangle$ with $P' = \langle a^2 \rangle$ [5, 2.3.4]. Then P/P' has a characteristic maximal subgroup or $|P : \langle a \rangle| = 2$. In the latter case, $\langle a \rangle$ is the unique cyclic maximal subgroup of P. In every case, P has a characteristic maximal subgroup, a contradiction. It follows that $|P| \leq p^4$. If $|P| = p^3$, then, clearly, $P \simeq Q_8$ or P is nonabelian of exponent p. Finally, if $|P| = p^4$, then $|\Phi(P)| = p^2$ and so $|P/C_P(\Phi(P))| \leq p$. Since P has no characteristic maximal subgroup, it follows that $\Phi(P) \leq Z(P)$. So if $P = \langle a, b\rangle$, then $P' = \langle [a, b]\rangle$ and $[a, b]^p = [a^p, b] = 1$. So $|P'| \leq p$ and since P/P' has no characteristic maximal subgroup, $P' = 1$. Thus P is abelian and of type (p^2, p^2).

If $P \simeq Q_8$, then $|K/C_K(P)| = 3$ so that $G/C_K(P) \simeq SL(2, 3)$ and K does not satisfy (ii). So K is a 3-group and Lemma 4.5 implies that $|C_K(P)| \leq 3$. Thus (g) holds.

In the other two cases, Lemma 4.5 yields that $C_K(P) = 1$. Suppose, for a contradiction, that $|K|$ is not a prime. Then the induction hypotheses implies that $|K| = q^2$ or $|K| = qr$. Let Q be the subgroup of order q of K and take $x \in P$ such that $\Phi(P)Q \neq \Phi(P)Q^x$. Then we claim that $\Gamma := \{1, \Phi(P), \Phi(P)Q, \Phi(P)Q^x, PQ\}$ is a K_5-set in $L(G)^*$. Since $PQ = PQ^x$ and Q is irreducible on $P/\Phi(P)$, we have 5 trivial paths and three further ones $(1, Q, \Phi(P)Q)$, $(1, Q^x, \Phi(P)Q^x)$, $(\Phi(P)Q, \Phi(P)K, G, \Phi(P)K^x, \Phi(P)Q^x)$. Finally, we connect $\Phi(P)$ to PQ via $\Phi(P)Q^y$ for a suitable $y \in P$ and 1 to PQ via P and suitable subgroups of order p^i different from $\Phi(P)$. Thus Γ is a K_5-set, the desired contradiction. It follows that $|K| = q$ and G satisfies (f) or (h) of Theorem D. $\square$

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