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Groups in which the Derived Groups of all 2-Generator Subgroups are Cyclic.

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Dedicated to Guido Zappa on his 90th birthday

1. Introduction.

Let us denote by $\mathcal{C}$ the class of groups G for which $\langle x, y\rangle'$ is cyclic for all $x, y \in G$. It is easy to check that a group G belongs to $\mathcal{C}$ if and only if, for all $x, y \in G$, there is an integer $n = n(x, y)$ such that $[x, y]^y = [x, y]^n$. Groups G with this property appear to have been first studied in [1]. Theorem 2 of that paper states that a finite nilpotent group G of odd order in which every two-generator subgroup has cyclic derived group is metabelian, and it is remarked that it is not known whether the stated restriction on the order of G is necessary. We show that it is indeed necessary, that is, we exhibit a finite 2-group in $\mathcal{C}$ that is not metabelian (see Theorem 4 below). We also generalise the above result of Alperin by proving the following.

THEOREM 1. *If G is a finite group of odd order that belongs to the class $\mathcal{C}$ then G is metabelian.*

Another result from [1] is that a torsion-free nilpotent group with our property is metabelian. It turns out that there is considerably more that may be said about torsion-free groups in $\mathcal{C}$.

THEOREM 2. *If G is a torsion-free group in the class $\mathcal{C}$ then G has a normal nilpotent subgroup N of class at most 2 and index at most 2 in G. In*

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particular, a torsion-free locally nilpotent group in $\mathcal{C}$ is nilpotent of class at most 2.

In fact, a satisfactory classification is available for torsion-free groups in $\mathcal{C}$, as follows.

THEOREM 3. *Let G be a torsion-free group. Then $G \in \mathcal{C}$ if and only if either G is nilpotent of class at most 2 or $G/Z(G) = F/Z(G).\langle g\rangle Z(G)/Z(G)$, where $F/Z(G)$ is abelian, $g^2 \in Z(G)$ and $a^g \cong a^{-1}$ mod $Z(G)$ for all $a \in F$.*

Our next result shows that Theorem 2 cannot be improved upon in at least one direction; it also provides us with a finite 2-group that answers the question from [1] referred to above. We make the obvious remark that subgroups and homomorphic images of a group in $\mathcal{C}$ are again in $\mathcal{C}$.

THEOREM 4. *There exists a torsion-free group G that lies in the class $\mathcal{C}$ but is not metabelian. Furthermore, G has a homomorphic image of order 2^{10} that is not metabelian.*

For p a fixed prime, it is easy to exhibit finite p-groups of arbitrarily large nilpotency class that belong to $\mathcal{C}$, for the group $G_n :=$ $:= \langle a, b : a^{p^n} = 1 = b^{p^{n-1}}, b^{-1}ab = a^{p+1}\rangle$ is metacyclic and has class exactly n for each positive integer n. One might try to use these easy examples to construct a locally finite p-group in $\mathcal{C}$ that is not nilpotent but, at least for odd p, such a construction is not possible, as the following result indicates.

THEOREM 5. *Let p be an odd prime and let G be a p-group in the class $\mathcal{C}$. Then G is nilpotent.*

It is obvious that every two-generator subgroup of a group in $\mathcal{C}$ is supersoluble, and by a result from [3] we have immediately that a finite group in $\mathcal{C}$ is supersoluble. This was pointed out in [2], where several structural properties were established for finite groups in $\mathcal{C}$. Our next result shows that local supersolubility is a characteristic of many groups in $\mathcal{C}$. It was established in [7] that a finitely generated hyper(abelian-by-finite) group in which every two generator subgroup is supersoluble is itself supersoluble, while our stronger hypothesis allows us to establish local supersolubility for an arbitrary *locally graded* group, where each nontrivial finitely generated subgroup is assumed to have a nontrivial finite image.

Indeed, we do not know whether even this extra hypothesis is necessary, but if there is a counterexample at all then it is easily shown that there is one that is finitely generated, infinite and simple.

THEOREM 6. *Let G be a locally graded group in the class $\mathcal{C}$. Then G is locally supersoluble and has a finite series of characteristic subgroups $1 \leq L \leq M \leq N \leq G$ where L is metabelian and consists of all the elements of G that have odd order, M/L is a (locally finite) 2-group, N/M is torsion-free and nilpotent of class at most 2, and G/N has order at most 2.*

Theorem 6 presents by no means a satisfactory description of locally supersoluble groups in $\mathcal{C}$. We would like to know more about finite 2-groups in $\mathcal{C}$ - perhaps such a group always has its derived subgroup, or even its square, of bounded nilpotency class, though we have insufficient evidence to present this as a conjecture. We are, however, able to assert the following, our final main result.

THEOREM 7. *If G is a 2-group in $\mathcal{C}$ then G^2 is hypercentral, of length at most ω.*

It will be seen that this last result is an easy consequence of a lemma that is required in order to establish Theorem 6. We mention (without providing a proof) that, if G is a 2-group in $\mathcal{C}$ then G^2 is also soluble, and we conjecture that it is nilpotent. Possibly G itself is nilpotent, but again we are reluctant to offer this as conjecture.

2. Finite groups in the class $\mathcal{C}$.

In this section we prove Theorem 1. It is convenient to divide the proof into a sequence of lemmas as follows

LEMMA 2.1. *Let $G \in \mathcal{C}$ and suppose that $G = P \rtimes \langle x \rangle$, with P a finite p-subgroup and x a p'-element of odd order, where p is an odd prime. If $a, b \in P \setminus C_P(x)$ and $[a,x]^x = [a,x]^r, [b,x]^x = [b,x]^s$, then $r \equiv s$ mod p and $1 \not\equiv r$ mod p.*

PROOF. Let i be maximal such that neither $[a,x]$ nor $[b,x]$ is contained in $Z_i(P)$, the *ith* term of the upper central series of P. Working modulo $Z_i(P)$, we may assume that $1 \neq [a,x] \in Z(P), [b,x] \neq 1$. Then for some integer t we have $[ab,x]=[a,x][b,x], [ab,x]^x=[ab,x]^t=[a,x]^t[b,x]^t$, and $[ab,x]^x=[a,x]^x[b,x]^x=[a,x]^r[b,x]^s$. If $\langle[a,x]\rangle\cap\langle[b,x]\rangle$ contains a nontrivial element c then $c^x=c^r=c^s$ and $r\equiv s \bmod |c|$, while if the intersection is trivial then $[a,x]^{t-r}=1=[b,x]^{s-t}$ and $t\equiv r \bmod |[a,x]|, s\equiv t \bmod |[b,x]|$, and in particular $r\equiv s \bmod p$. Also $r\not\equiv 1$, for otherwise $[a,_j x]=1$ for some j and hence $[a,x]=1$, a contradiction.

LEMMA 2.2. *Let $G\in\mathcal{C}$ and suppose that $G=P\rtimes\langle x\rangle$, with P a finite p-subgroup and x a p'-element of odd order, where p is an odd prime. Then $[P,\langle x\rangle]$ is abelian.*

PROOF. Supposing the result false, there exist $a,b\in P$ such that $c:=[[a,x],[b,x]]\neq 1$. As G is supersoluble there is a normal subgroup N of G that has order p, and by induction on $|G|$ we may assume that $[P,\langle x\rangle]$ is abelian modulo N. By Lemma 2.1 we have $[a,x]^x=[a,x]^r, [b,x]^x=[b,x]^s$ with $r\equiv s \bmod p$. If $N\leq C_G(x)$ then $1\neq c=c^x=c^{rs}$ and so $r^2\equiv rs\equiv 1 \bmod p$, a contradiction since $|x|$ is odd and $r\not\equiv 1 \bmod p$, by Lemma 2.1. Thus $N\not\leq C_G(x)$ and $[n,x]^x=[n,x]^t$ for every nontrivial element n of N, where $t\equiv r \bmod p$, again by Lemma 2.1. Now $1\neq c^t=c^x$, as $c=[n,x]^v$ for some $n\in N, v\in\mathbb{Z}$, and since $c^x=c^{rs}$ we have $r^2\equiv r \bmod p$. Lemma 2.1 gives the required contradiction.

LEMMA 2.3. *Let $G\in\mathcal{C}$ and suppose that $G=P\rtimes\langle x\rangle$, with P a finite p-subgroup and x a p'-element, where p is an odd prime. Then G is metabelian.*

PROOF. By Theorem 3.5 of [4] we have $P=C_P(x)[P,\langle x\rangle]$. Moreover, $[P,\langle x\rangle]\lhd G$, and so $G'=(C_P(x))'[P,\langle x\rangle]$. By Theorem 2 of [1] and Lemma 2.2 above, each of $(C_P(x))'$ and $[P,\langle x\rangle]$ is abelian, and it suffices to show that $(C_P(x))'\leq C_G([P,\langle x\rangle])$. But if $a\in P, g\in C_P(x)$ then we have $[a,x]=[ga,x]$ and, since $G\in\mathcal{C}$, each of a, ga and x is in the normalizer K of $\langle[a,x]\rangle$ in G. Thus $C_P(x)\leq K$ and $(C_P(x))'\leq C_G(\langle[a,x]\rangle)$, and the result follows.

LEMMA 2.4. *Let $G\in\mathcal{C}$ and suppose that $G=P\rtimes\langle x\rangle$, with P a finite p-subgroup and x a p'-element of odd order, where p is an odd prime. If $Z(P)\leq C_G(x)$ then G is nilpotent.*

PROOF. Assume the result false and let $i\geq 1$ be maximal such that $Z_i(P)\leq C_G(x)$. Then x does not act nilpotently on $Z_{i+1}(P)$ and we may assume that $i=1$. Let $a\in Z_2(P)$, $b\in P$. By Lemma 2.2, $[[a,x],[b^{-1},x]]=1$, and since $G \in \mathcal{C}$ it follows that $[x, b^{-1}]$ centralizes $[a, x]^x$ and hence that $[a, x]^{bx} = [a, x]^{xb}$. Also, $\langle [a, x] \rangle^P$ is abelian, and we deduce easily that $1 = [a, x, b, x] = [a, x, x, b]$, and since b was arbitrary it follows that $[a, x, x] \in Z(P)$. Thus $[a, x, x] \in C_G(x)$ and so $a \in C_P(x)$, and we have a contradiction to the fact that x does not centralize $Z_2(P)$.

We are now in a position to complete the proof of the theorem.

PROOF OF THEOREM 1. Let G be a counterexample of minimal order. Since G is supersoluble we have G' nilpotent, and it follows by minimality that G' is a p-group for some prime p. Let P be a Sylow p-subgroup of G; thus $G' \leq P$ and $G = P \rtimes X$ for some abelian p'-subgroup X, and $G' = P'[P, X]$. Now P' is abelian, by [1], and by Lemma 2.3 $P'[P, \langle x \rangle]$ is abelian for every $x \in X$ and X is not cyclic. Thus $[P, X]$ is not abelian and we may choose elements $a, b \in P$ and $x, y \in X$ such that $[[a, x], [b, y]] \neq 1$. By minimality we have $G = \langle a, b, x, y \rangle = P\langle x, y \rangle$; also $Z(P)$ is cyclic since G has just one normal subgroup N, say, of order p. Write $N = \langle n \rangle$.

If $n^x = n^\alpha$ and $n^y = n^\beta$, with each of $\alpha, \beta \not\equiv 1 \bmod p$, then we have $n^{xy^\gamma} = n$ for some γ, and so if $z = xy^\gamma$ then $N \leq C_G(z)$, so that $Z(P) \leq C_G(z)$ (see Theorem 3.10 of [4]) and $P \leq C_G(z)$ by Lemma 2.4. But then $G = P\langle y, z \rangle$ and $[P\langle y \rangle, \langle z \rangle] = 1$, and so G' is abelian. This contradiction concludes the proof of the theorem.

3. Torsion-free groups.

Our main objective in this section is the classification of torsion-free groups in $\mathcal{C}$, as presented in Theorem 3. For much of the following discussion we concern ourselves with $\mathcal{C}$-groups G whose derived subgroups are torsion-free. We begin with an easy lemma, but one which has a very useful consequence. All we need for the proof is that, for elements a, x of a group G, $\langle [a, x] \rangle^{\langle a \rangle} = [\langle a \rangle, x]$.

LEMMA 3.1. *Let G be a group, $a, x \in G$, and suppose that $[a, x]^a = [a, x]$ or $[a, x]^a = [a, x]^{-1}$. Then a^2 centralizes $[\langle a \rangle, x]$.*

The next two results are also very straightforward.

LEMMA 3.2. *Let $G \in \mathcal{C}$ and suppose that $\langle x, y \rangle'$ is either infinite or trivial for all $x, y \in G$. Let $g \in G$. Then, for each $x \in G$, either $[x, g^2] = 1$ or $[x, g, g] = 1$ (with both holding if and only if $[x, g] = 1$).*

PROOF. If $g, x \in G$ then $\langle [x,g] \rangle \lhd \langle x, g \rangle$ and so $[x,g]^g = [x,g]$ (and $[x,g,g] = 1$) or $[x,g]^g = [x,g]^{-1}$. In the latter case we have $[x,g^2] = $ $= [x,g]^2[x,g,g] = [x,g]^2[x,g]^{-2} = 1$.

LEMMA 3.3. *Let G be a torsion-free locally nilpotent group in $\mathcal{C}$. Then G is nilpotent of class at most* 2*.*

PROOF. If $x, y \in G$ then each of x and y normalizes $\langle [x,y] \rangle$ and hence centralizes it, by local nilpotency. Thus G is 2-Engel and hence nilpotent of class at most 2 (see, for example, [9; Theorem 7.14]).

COROLLARY 3.4. *Let $G \in \mathcal{C}$ and suppose that G has no nontrivial normal torsion subgroups. Suppose further that $\langle x, y \rangle'$ is either infinite or trivial for all $x, y \in G$. Then G^2 is nilpotent of class at most* 2 *(and so G' is nil-*2*).*

PROOF. Let $a \in G$. If $x \in G$ then, using Lemma 3.1, we see that a^2 centralizes $[\langle a \rangle, x]$ and, since x was arbitrary, a^2 centralizes $[\langle a \rangle, G]$. Thus $\langle a^2 \rangle^G$ is abelian. Since this is true for all $a \in G$ it follows that G^2 is a product of normal abelian subgroups and hence locally nilpotent, and the result follows by Lemma 3.3.

We are almost in a position to establish a result that says much about the structure of certain groups in $\mathcal{C}$. But first we must prove the following "global version" of Lemma 3.2.

LEMMA 3.5. *Let $G \in \mathcal{C}$ and suppose that $\langle x, y \rangle'$ is either infinite or trivial for all $x, y \in G$. Then, for each $g \in G$, either $g^2 \in Z(G)$ or $[x,g,g] = 1$ for all $x \in G$.*

PROOF. Fix $g \in G$ and let $T_g = \{x \in G : [x,g,g] = 1\}$.

Claim. T_g is a subgroup of G.

Let $x, y \in T_g$, so $[x,g,g] = 1 = [y,g,g]$, and suppose for a contradiction that $[xy,g,g] \neq 1$. Then $[xy,g^2] = 1$, by Lemma 3.2, and we deduce that $[x,g^2] = ([y,g^2]^{-1})^{y^{-1}}$ and hence that $[x,g^2] \in \langle x, g \rangle' \cap \langle y, g \rangle'$ and so $N := \langle [x,g^2] \rangle$ is normal in $H := \langle x, y, g \rangle$. In the following, all congruences are modulo N: since $[x,g,g] = 1$ we have $[x,g]^2 \equiv [x,g^2] \equiv 1$ and similarly $[y,g]^2 \equiv 1$. Also $[x,g,x] = 1$ or $[x,g]^{-2}$ and so $[x,g,x] \equiv 1$, and similarly $[y,g,y] \equiv 1$. Next, $[x,g,gy] \equiv [x,g,y] \equiv [x,g,xy]$ and, since y

normalizes $\langle [[x, g], y] \rangle$ and xy normalizes $\langle [[x, g], gy] \rangle$ it follows that $\langle y, xy, gy \rangle$ normalizes $K := \langle [x, g, y] \rangle$ mod N, so H normalizes K mod N, and hence H' centralizes K mod N. In particular, $[[x, g, y], [x, g]] \in N$, and so $1 \equiv [[x, g]^2, y] \equiv [x, g, y]^2$, and it follows that K has order at most 2 mod N and hence that $[K, H] \leq N$. We deduce that $[x, g]N \in Z_2(H/N)$ and similarly that $[y, g]N \in Z_2(H/N)$. Thus $[xy, g]N \in Z_2(H/N)$ and we have $[xy, g, g, g] \in N$. But $[x, g, g] = 1$ and $N = \langle [x, g^2] \rangle$ together imply $[N, g] = 1$ and hence $[xy, g, g, g, g] = 1$. But g^2 centralizes xy and hence $\langle xy \rangle^{\langle g \rangle}$, and we get $1 = [xy, g, g, g^2] = [xy, g, g, g]^2$ and so $[xy, g, g, g] = 1$. Similarly $[xy, g, g] = 1$ and this contradiction establishes the claim.

Again by Lemma 3.2 we have $G = C_G(g^2) \cup T_g$ and so either $G = C_G(g^2)$ or $G = T_g$, as required.

PROPOSITION 3.6. *Let $G \in \mathcal{C}$ and suppose that G has no nontrivial normal torsion subgroups. Suppose further that $\langle x, y \rangle'$ is either infinite or trivial for all $x, y \in G$, and let $F = Fitt(G)$, the Fitting radical of G. Then the following hold.*

(i) *If $g \in G \setminus F$ then $a^g \equiv a^{-1}$ mod $Z(G)$ for all $a \in F$.*
(ii) *G/F has order at most 2.*
(iii) *$F' \leq Z(G)$.*

PROOF. By Lemma 3.3 and its corollary, F is (torsion-free) nilpotent of class at most 2 and contains G^2, and so G/F has exponent at most 2. Suppose that there exists $g \in G \setminus F$ and let $a \in F$. If $[u, ag, ag] = 1$ for all $u \in F$ then $[u, g, g] \in F'$ for all $u \in F$ and so $[F, \langle g \rangle, \langle g \rangle] \leq F'$ and $F\langle g \rangle / F'$ is nilpotent. But then $F\langle g \rangle$ is nilpotent (see [5]) and so $g \in F$ (since $F\langle g \rangle \lhd G$), a contradiction. It follows from Lemma 3.5 that $(ag)^2 \in Z(G)$ for all $a \in F$ (and in particular $g^2 \in F$). Thus, again for all $a \in F$, $a^g \equiv a^{-1}$ mod $Z(G)$, and (i) is proved. If h and gh are also in $G \setminus F$ then we have $a^h \equiv a^{-1}$ and $a^{gh} \equiv a^{-1}$ mod $Z(G)$ for all $a \in F$, but $a^{gh} \equiv a$ and so $a \equiv a^{-1}$ for all a, and hence $[F, \langle g \rangle] \leq Z(G)$ and again we have the contradiction that $F\langle g \rangle$ is nilpotent. Thus G/F has order at most 2. Finally, suppose $F < G$, choose $g \in G \setminus F$ and let $a, b \in F$. Then, modulo $Z(G)$, $a^{-1} \equiv a^{bg} = (a[a, b])^g \equiv a^{-1}[a, b]^{-1}$ and so $[a, b]^{-1} \in Z(G)$ and statement (iii) follows.

Theorem 2 follows immediately from Proposition 3.6 and Lemma 3.3. Proposition 3.6 also establishes that a torsion-free group in $\mathcal{C}$ has the structure described in Theorem 3. Suppose now that G is a torsion-free group; if G is nil-2 then of course $G \in \mathcal{C}$, so assume that $G = F\langle g \rangle$ where F

and g are as described in Theorem 3. Let H be an arbitrary two-generator subgroup of G. If $H \leq F$ then H' is cyclic. Otherwise, $H = \langle a, bg \rangle$ for some $a, b \in F$; then $a^g = a^{-1}z$ for some $z \in Z(G)$ and one checks easily that $[a, bg] = z[a, b]a^{-2}$ and $[a, bg]^{bg} = [a, bg]^{-1}$, so H' is again cyclic. Thus Theorem 3 is also proved.

PROOF OF THEOREM 4. Let $A = \langle a \rangle \times \langle z \rangle$, a free abelian group of rank 2, and let H be the group with presentation $\langle x, y : [x, y]^x = [y, x] = [x, y]^y \rangle$. It is routine to check that H is torsion-free; indeed, H is an extension $\langle x, [x, y] \rangle \rtimes \langle y \rangle$, where x acts by inversion on $\langle [x, y] \rangle$ and the action of y on $\langle x, [x, y] \rangle$ is via $x^y = x[x, y], [x, y]^y = [x, y]^{-1}$. We define an action of H on A by setting $a^x = a^{-1}z, a^y = a^{-1}, z^x = z = z^y$. It is easily verified that $a^{x^{-1}} = a^{-1}z, a^{y^{-1}} = a^{-1}, a^{[x,y]} = az^2, a^{[y,x]} = az^{-2}, a^{xy^{-1}} = az = a^{xy}$, and that the relations for H are thereby respected. With the above action, set $G = A \rtimes H$, which is also torsion-free.

First let us observe that G' is not abelian, for it contains both $[a, x] = a^{-2}z$ and $[x, y]$, but $(a^{-2}z)^{[x,y]} = a^{-2}z^{-3}$. Next, it is clear that x^2 and y^2 are central in G, as of course is z. Since $[a, [x, y]] = z^2$, the centralizer of a in H is easily seen to be $\langle x^2, y^2 \rangle$, and so $Z(G) = \langle z, x^2, y^2 \rangle$. Let $F = \langle A, x^2, y^2, [x, y], xy^{-1} \rangle$. From the above calculations and the fact that $[[x, y], xy^{-1}] = 1$ it is immediate that $F/Z(G)$ is abelian. Also $G = F\langle y \rangle$, and since y inverts every element of F modulo $Z(G)$ we may apply Theorem 3 to deduce that $G \in \mathcal{C}$.

Finally, let N be the subgroup of G generated by a^8, z^8, x^2, y^2 and $[x, y]^4$. Since $[a, [x, y]] = z^2$ we have $[a, [x, y]^4] = z^8$ and it follows that N is normal in G. Certainly $|HN/N| = 16$ and $|AN/N|$=64 and so G/N has order 2^{10}. Furthermore, $[a, x, [x, y]] = z^{-4} \notin N$ and so $(G/N)'$ is not abelian, and the proof of Theorem 4 is complete.

4. p-groups in the class $\mathcal{C}$, where p is odd.

Our first result in this section is very easy to prove but is nonetheless of some interest. In any case, the special case where G is a p-group will turn out to be an important ingredient in the proof of Theorem 5.

PROPOSITION 4.1. *Let G be a group, n an integer, and suppose that $[x, y]^y = [x, y]^n$ for all $x, y \in G$. Then G is 2-Engel and therefore nilpotent of class at most 3, and of class at most 2 if G contains no elements of order 3.*

PROOF. Let $x, y \in G$. Then $[y,x]^y = ([x,y]^{-1})^y = [x,y]^{-n} = [y,x]^n = [y,x]^x$ and so xy^{-1} centralizes $[y,x]$ and hence centralizes $\langle [y,x] \rangle = \langle x,y \rangle'$. In particular, $[x, xy^{-1}, xy^{-1}] = 1$. If $a, b \in G$ then we set $x = a, y = b^{-1}a$ in the above to get $[a,b,b] = 1$, as required. The consequences for the nilpotency class of G follow from Theorem 7.14 of [9].

COROLLARY 4.2. *Suppose $G = H \times K$ and that $G \in \mathcal{C}$. Suppose further that H is finite but not nilpotent of class at most 3, and let $x, y \in K$. Then the order of $[x,y]$ is less than the exponent of H'. In particular, if K is finite and of odd order then $exp(K') < exp(H')$.*

PROOF. First note that if K has odd order then K' is abelian, by Theorem 1, and so all we need establish is the statement concerning each commutator of K. Suppose for a contradiction that $x, y \in K$ and $|[x,y]| \geq exp(H')$. We have $[x,y]^y = [x,y]^m$ for some integer m, and by Proposition 4.1 there are elements a, b of H such that $[a,b]^b \neq [a,b]^m$, but $[a,b]^b = [a,b]^n$ for some integer n. Now $[ax, by]^{by} = [a,b]^n[x,y]^m = ([a,b][x,y])^r$ for some r, and we have $r \equiv n \bmod |[a,b]|$ and $r \equiv m \bmod |[x,y]|$, and by our choice of x, y this gives $m \equiv n \bmod |[a,b]|$ and so $[a,b]^b = [a,b]^m$, a contradiction.

Suppose now that p is an odd prime and that G is a p-group in $\mathcal{C}$. Every two-generator subgroup of G has cyclic derived subgroup and is therefore regular, and in particular $[x^{p^n}, y] = [x,y]^{p^n} = [x,y]^{p^n}$ for all $x, y \in G$ and for all positive integers n. (For these properties of regular p-groups we refer the reader to Chapter 4 of [8].) Now if $x \in G$ and $|x| = p$ then we deduce that $[x,y]^p = 1$ and so y centralizes $[x,y]$, so x is a 2-Engel element and, since p is odd, it follows that x lies in $Z_3(G)$ - see Corollary 2 to Theorem 7.13 of [9]. By induction, an element of order at most p^n lies in $Z_{3n}(G)$, for each positive integer n, and so G is hypercentral, with hypercentral length at most ω. In particular, G is locally finite and therefore, by Theorem 2 of [1], metabelian. These facts about G will be used in our subsequent discussion.

LEMMA 4.3. *Let G be a p-group in $\mathcal{C}$, where p is an odd prime, and let A be a normal nilpotent subgroup of G, with G/A abelian.*

(i) *If G/A has finite exponent then G is nilpotent.*
(ii) *If G/A is divisible then G is nilpotent.*

PROOF. (i) By induction we may suppose that G/A has exponent p. If G/A' is nilpotent then so is G [5], and so we may factor by A' and assume that A is abelian. Let $a \in A, g \in G$. Then $1 = [a, g^p] = [a, g]^p$ and so $[A, G]^p = 1$ and $[A, G] \leq Z_3(G)$, which gives $A \leq Z_4(G)$ and G nilpotent.

(ii) As in part (i) we may suppose that A is abelian. We shall show that $A \leq Z(G)$, and for this we may assume that $G/A \cong C_{p^\infty}$, and so there are elements $g_1, g_2, \ldots$ such that $G = A\langle g_1, g_2, \ldots\rangle, g_1^p \in A$ and $g_{i+1}^p \equiv g_i \bmod A$ for each $i \geq 1$. Let $a \in A$ and suppose that a has order p^n. Then, for $i \geq n+1$, we have $1 = [a^{p^n}, g_i] = [a, g_i^{p^n}] = [a, g_{i-n}]$. Thus $[a, g_j] = 1$ for all $j \geq 1$ and the result follows.

PROOF OF THEOREM 5. We have seen that G is metabelian; let A be a normal abelian subgroup of G such that G/A is also abelian, and let B/A be a basic subgroup of G/A - so B/A is a direct product of cyclic subgroups and G/B is divisible (see [10; 4.3.4]). Assuming for a contradiction that G is not nilpotent, we have from Lemma 4.3 that B is not nilpotent, so we may assume that G/A is a direct product of cycles. Choose a finite subgroup F_1 of G such that F_1 is not nil-4 and F_1A/A is a direct factor of G/A, and write $G/A = F_1A/A \times M/A$.

M is normal in G, as therefore is M^{p^n}, where p^n is the exponent of F_1. Since G/M^{p^n} has finite exponent we have from Lemma 4.3 that $N := M^{p^n}$ is not nilpotent. For $x \in F_1$ and $y \in M$ we have $1 = [x^{p^n}, y] = [x, y^{p^n}]$, and it follows that $[F_1, N] = 1$. Note that $N \cap F_1 \leq Z(F_1)$, therefore.

Let $D = \{a \in A : a^{p^n} = 1\}$. If every finite subgroup F_2 of N has derived subgroup of exponent at most p^n mod D then $exp(F_2') \leq p^{2n}$ and hence $F_2' \leq Z_{6n}$ for all such F_2, and so $N' \leq Z_{6n}(G)$ and we obtain the contradiction that N is nilpotent. Thus we may choose a finite subgroup F_2 of N such that $exp(DF_2'/D) > p^n$.

Let $E = F_1 \cap F_2$. Then $E \leq Z(F_1), [E, F_2] = 1$ and $E \leq D$ (recall that $F_1 \cap N \leq A$.) Let $H = \langle F_1, F_2\rangle$. Then $E \lhd H$ and $H/E = F_1/E \times F_2/E$, and since $E \leq Z(F_1)$ we see that F_1/E is not nil-3. By means of Corollary 4.3 we deduce that $exp(EF_2'/E) < exp(EF_1'/E) \leq exp(F_1') \leq p^n$. But $E \leq D$ and so $exp(DF_2')/D) < p^n$, a contradiction that concludes the proof of Theorem 5.

5. Locally graded groups.

First let G be a locally supersoluble group in $\mathcal{C}$. The set L of elements of odd order in G forms a subgroup, which of course is characteristic in G, and if M is the maximal normal torsion subgroup of G then M/L is a locally finite 2-group. Now G' is locally nilpotent, and we may apply Proposition 3.6 to the group G/M to obtain a normal torsion-free nil-2 subgroup N/M of index at most 2 in G/M. Thus, in order to establish Theorem 6, it suffices to prove the following.

PROPOSITION 5.1. *Let G be a locally graded group in the class $\mathcal{C}$. Then G is locally supersoluble.*

This in turn requires a preliminary lemma.

LEMMA 5.2. *Let G be a group in the class $\mathcal{C}$ and suppose that every torsion element of G has 2-power order.*

(i) *If $a \in G$ and $|a| = 4$ then $[a^2, x^2] = 1$ for all $x \in G$.*
(ii) *c^2 is in the ω-hypercentre of G^2 for every 2-element c of G.*

PROOF. For (i), let a be as stated and let $x \in G$. If $[a, x]$ has infinite order then $[a, x]^a = [a, x]^{-1}$, for if $[a, x]^a = [a, x]$ then we obtain the contradiction that $[a, x]^4 = 1$. Thus $[a^2, x] = 1$ and so $[a^2, x^2] = 1$ in this case. Now suppose that $[a, x]$ is a 2-element; then $[a, x]^a = [a, x]^{1+2s}$ for some integer s and so $[a, x]^{a^2} = [a, x]^{1+4h}$ for some h, and it follows that $1 = [a^4, x] = [a^2, x]^{2+4h} = [a^2, x]^{2t}$, where t is odd. Thus $[a^2, x]^2 = 1$, and we have $[a^2, x]^x = [a^2, x]$ and hence $[a^2, x^2] = 1$, as required. Now suppose that c has order 2^n. Statement (ii) follows by an easy induction on n, provided that $G/Z(G^2)$ inherits from G the property that there are no nontrivial elements of odd order. But if $g \in G$ and $g^k \in Z(G^2)$ for some odd k then it is clear that $g \in Z(G^2)$, and the lemma is proved.

PROOF OF PROPOSITION 5.1. We may assume that G is finitely generated. Let R denote the finite residual of G and suppose that G/R is supersoluble. Now $\langle x\rangle^{\langle y\rangle}$ is finitely generated for all x, y in G, and we may apply Lemma 3 of [6] to deduce that R is finitely generated. If R is nontrivial then it has a proper G-invariant subgroup S of finite index, and then G/S is polycyclic and hence residually finite, giving the contradiction $R = S$. Thus we may assume that G is residually finite. Every finite image is supersoluble and hence nilpotent-by-abelian, and so G' is residually a finite nilpotent group; it is also finitely generated, again by Lemma 3 of [6], and if G' is supersoluble then G is soluble and the result follows from [7]. Thus we may assume that G is residually (finite nilpotent), and since the $2'$-radical of every finite nilpotent image is metabelian, we may suppose that G is residually finite-2.

Let H be the hypercentre of G^2; by Lemma 5.1, $c^2 \in H$ for all 2-elements c of G. Let $x, y \in G$; if $[x, y]$ has infinite order then $[x, y]^y = [x, y]$ or $[x, y]^{-1}$, but the same holds mod H if $[x, y]$ has 2-power order, for in that case we have $[x, y]^2 \in H$. Arguing as in the proof of Corollary 3.4, we deduce that G^2/H is locally nilpotent. But G^2 has finite index in G and is therefore finitely generated, so G^2/H is nilpotent, and another application of Lemma 3 of [6] gives H finitely generated and hence nilpotent. So G is soluble, and the result follows from [7].

Finally, it is clear that Theorem 7 follows immediately from Lemma 5.2.

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