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Cyclic Quasinormal Subgroups of Arbitrary Groups (*).

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To Professor Guido Zappa on his 90th birthday

ABSTRACT - In recent years several papers have appeared showing how cyclic quasinormal subgroups are embedded in finite groups and many structure theorems have been proved. The purpose of the present work is twofold. First we show that, without exception, all of these theorems remain valid for finite cyclic quasinormal subgroups of *infinite* groups. Secondly we obtain analogous results for infinite cyclic quasinormal subgroups, where the statements turn out to be even stronger.

1. Introduction and statement of results.

Let A be a cyclic *quasinormal* subgroup of a group G. Thus for every subgroup X of G, $AX = XA = \langle A, X \rangle$. When G is finite, then the structure of the normal closure A^G of A in G is quite well understood. If A has odd order, then $[A, G]$ is abelian and A acts on it by conjugation as a group of power automorphisms ([2]). When A has even order, then $[A, G]$ is nilpotent of class at most 2. When dihedral actions are excluded in certain subgroups of products AX, where X is cyclic (see below), then $[A, G]'$ has order at most 2 and A acts on $[A, G]/[A, G]'$ as power automorphisms ([3]). But always $[A, G, A]$ is abelian and A acts on it again as power automorphisms ([4]).

Many of the arguments used in obtaining these results involved induction on group orders; and at the time, infinite groups did not appear to be the natural setting for the work. However, it has now transpired that *all*

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the major results in [2], [3] and [4] are true without the hypothesis that G is finite. A large part of the present work, therefore, is devoted to extending those earlier theorems to the case where G is infinite. Many of the original proofs can be repeated almost verbatim, modulo some initial lemmas, and we shall endeavour to avoid tedious repetition. But of course when G is infinite we have the new situation allowing the cyclic quasinormal subgroup A to be infinite also. The arguments in this case are new and we give them in full. As one might intuitively expect, the structure of A^G is simpler when A is infinite than when A is finite.

Our work therefore divides naturally into three parts, dealing with the two cases when A is finite, corresponding to $|A|$ odd and $|A|$ arbitrary (analogous to the contents of [2], [3] and [4]); followed by the situation when A is infinite. In general

all groups G are assumed to be arbitrary,

i.e. finite or infinite. For any set π of primes, A_π denotes the π-component of a cyclic group A. Recall that all subgroups of a cyclic quasinormal subgroup are also quasinormal (see Lemma 2.3 below). Our main results are as follows.

THEOREM A. *Let A be a finite cyclic quasinormal subgroup of odd order in a group G. Then $[A, G]$ is abelian and A acts on it as a group of power automorphisms.*

THEOREM B. *Let A be a finite cyclic quasinormal subgroup of a group G. Suppose that each cyclic subgroup X of G is quasinormal in A_2X, and let $N = [A, G]'$. Then*

(i) $|N| \leqslant 2$ *and* $N \leqslant A$;

(ii) *A acts on $[A, G]/N$ as a group of power automorphisms*; *and*

(iii) *$[A, G]$ is abelian if and only if A acts on $[A, G]$ as a group of power automorphisms.*

In Example 5.2 of [3], a finite group G of order 2^{17} is constructed, with a cyclic quasinormal subgroup A of order 2^7, in which $[A, G]'$ has order 2 and the hypotheses of Theorem B are satisfied.

THEOREM C. *Let A be a finite cyclic quasinormal subgroup of a group G. Then*

(i) $[A, G] = [A_2, G] \times [A_{2'}, G]$;

(ii) *$[A_{2'}, G]$ is an abelian $2'$-group on which A acts as a group of power automorphisms*;

(iii) $[A_2, G]$ *is a* 2*-group of class at most* 2;
(iv) $[A_2, G, A]$ *is normal in* G *and lies in* $Z([A_2, G])$; *and*
(v) A *acts on* $[A_2, G, A]$ *as a group of power automorphisms.*

In Theorems A, B and C, the power automorphisms are *universal*, i.e. each element maps to the *same* power. Also one can easily check that if we form the product of A with any of the abelian subgroups on which we claim that A acts as a group of power automorphisms, then that product is a modular group, i.e. its subgroups form a modular lattice.

THEOREM D. *Let* A *be an infinite cyclic quasinormal subgroup of a group* G. *Then*

(i) $[A, G]$ *is abelian and* A *acts on it as a group of power automorphisms;*
(ii) $[A, G]$ *is periodic if and only if* $A \cap [A, G] = 1$; *and*
(iii) *when* $A \cap [A, G] \neq 1$, *i.e. when* $[A, G]$ *is not periodic, then* A^G *is abelian.*

We shall see in Section 4 that the power automorphisms in Theorem D (i) are not universal in general. It follows from Lemma 2.5 below that in all the above theorems, the normal closure A^G of A in G is locally nilpotent and so the periodic elements of A^G form a subgroup T, say. In [1], Theorem 2.2, Busetto shows that with the hypotheses of Theorem D, $A^G = AT$, T is abelian and $T \leqslant Z_\omega(G)$. Busetto also shows that A acts on T as a group of power automorphisms and A^G is a modular group. Clearly $A^G = A[A, G]$ and when $A \cap [A, G] = 1$, then $[A, G] = T$. But in general $[A, G] \neq T$. Indeed Example 5 in Section 5 shows that $T \nleqslant [A, G]$ in general.

Sections 2, 3 and 4 of our paper will establish the above results sequentially. The final Section 5 contains examples relevant to the many questions that arise naturally throughout our discussion. The notation is standard. Thus A^G and A_G denote, respectively, the normal closure and the core of a subgroup A in a group G. When A is a quasinormal subgroup of G, then we shall write $A \; qn \; G$. Cyclic groups of order n and infinity will be denoted by C_n and C_∞, respectively, and Q_8 denotes the quaternion group of order 8. The centre of G is $Z(G)$ and, for any ordinal α, the α-th term of the upper central series is $Z_\alpha(G)$. If p is prime and A is a p-group, then $\Omega(A)$ denotes the subgroup generated by the elements of order p. When π is a set of primes, then π' is the complementary set. Also G_p is a maximal p-subgroup of G and $G_{p'}$ a maximal p'-subgroup. The intersection of the normalisers of *all* the subgroups of a group G is the *norm*, written *norm*(G). Finally the subgroup lattice of G is denoted by $\mathcal{L}(G)$ and the lattice of all subgroups of G containing a subgroup H by $\mathcal{L}(G:H)$.

2. Finite cyclic quasinormal subgroups of odd order.

When p is an odd prime, Lemma 2.3 of [2] considers a finite p-group $G = AX$, where A and X are cyclic subgroups. Here we need the following generalisation.

LEMMA 2.1. *Let $G = AX$ be a nilpotent group, where $A = \langle a \rangle$ is a cyclic p-subgroup, with p an odd prime, and $X = \langle x \rangle$ is also cyclic (possibly infinite). Then*

(i) *G is metacyclic and G' is a p-group;*
(ii) *every subgroup of G is quasinormal;*
(iii) *$G' = \langle [a, x] \rangle$;*
(iv) *for each integer i, $\langle [a^i, x] \rangle = \langle [a, x^i] \rangle = \langle [a, x]^i \rangle$; and*
(v) *each element of G' has the form $[a, g]$, for some $g \in X$.*

PROOF. (i) If X is finite, then $X = X_p \times X_{p'}$. So $G = (AX_p) \times X_{p'}$ and AX_p is metacyclic, by [2], Lemma 2.3 (i). Thus G is metacyclic. Clearly $G' \leqslant AX_p$ and so G' is a p-group. If X is infinite, then $A \lhd G$ and both statements are immediate.

(ii) By (i) and Iwasawa's Theorem (see [11], Theorems 2.4.4 and 2.4.11), G is a modular group. Since G is also nilpotent, all its subgroups are subnormal and therefore quasinormal ([14], Proposition 1.7).

(iii) The easy argument is the same as in [2], Lemma 2.3 (iii).

(iv) When X is finite, then we argue as in (i) above and [2], Lemma 2.3 (iv). When X is infinite, then $A \lhd G$ and $X_1 = X/C_X(A) \cong C_{p^n}$, for some n. Again the result follows from [2], Lemma 2.3 (iv) applied to the split extension $A \rtimes X_1$.

(v) The argument of [2], Lemma 2.3 (v) applies, making use of (i), (iii) and (iv) above. $\square$

We shall establish Theorem A in a somewhat circuitous way, as in [2]. It turns out that each element of $[A, G]$ has a special form and we deduce this by considering first the case $A \cap [A, G] = 1$. In fact with this hypothesis we can prove the strongest statement of all our results for a very general situation.

THEOREM 2.2. *Let A be any quasinormal subgroup of a group G and suppose that $A \cap [A, G] = 1$. Then $[A, G]$ is abelian and A acts on it as a group of power automorphisms.*

PROOF. Let X be a subgroup of $[A, G]$. Then $[A, X] \leqslant AX \cap [A, G] = X$. So A and hence also A^G normalise all subgroups of $[A, G]$. Therefore $[A, G]$ is a Dedekind group. If $[A, G]$ is not abelian, we factor G by the $2'$-component of $[A, G]$. Then

$$[A, G] \cong Q_8 \times C_2 \times C_2 \times \cdots$$

and there is a cyclic subgroup Y of order 4 in $[A, G]$. If A does not centralise Y, then $A/C_A(Y)$ has order 2 and $AY/C_A(Y)$ is the dihedral group of order 8, with $A/C_A(Y)$ a non-central subgroup of order 2, contradicting the quasinormality of A. Therefore A must centralise $[A, G]$ and thus so also does A^G. But then $[A, G]$ is abelian, a contradiction. □

Next we consider *a cyclic quasinormal subgroup A of odd prime-power order* in a group G satisfying the hypotheses of Theorem 2.2, i.e. $A \cap [A, G] = 1$. Let $A = \langle a \rangle$. We will show that

$$[A, G] = \{[a, g] | g \in G\}. \quad (1)$$

In Lemma 2.4 of [2], this is proved for the case in which G is a finite p-group, with p an odd prime, using the following two results (see [11], Lemma 5.2.11 and [1], respectively).

LEMMA 2.3. *If A is a cyclic quasinormal subgroup of a group G, then every subgroup of A is quasinormal in G.*

LEMMA 2.4. *Let A be a quasinormal subgroup of prime order p in a group G. Then A^G is an elementary abelian p-group. Moreover, if A is not normal in G, then $A^G \leqslant Z_2(G)$.*

It is convenient to recall two more well-known results (see [9], [3] and [13], Lemma 2.1).

LEMMA 2.5. *A quasinormal subgroup of a finite group is subnormal. More generally, a quasinormal subgroup of any group is ascendant.*

LEMMA 2.6. *Let A be a quasinormal subgroup of a group G and let X be an infinite cyclic subgroup such that $A \cap X = 1$. Then X normalises A.*

The argument used to establish (1) for finite p-groups G in Lemma 2.4 of [2] proceeds by induction on $|G|$. When G is infinite we can use the following (see [8], Lemma 7.1.9).

LEMMA 2.7. *Suppose that a group $G = \langle A, g_1, ..., g_n \rangle$, where A qn G. Then $|A^G : A|$ is finite.*

Thus in order to prove (1) when G is infinite, clearly we may assume that G is finitely generated. Then $|A^G : A|$ is finite and so A^G is finite. Therefore suppose that G is a counter-example with $|A^G|$ minimal. Let $A_1 = A_G (< A)$, $A_2/A_1 = \Omega(A/A_1)$ and $A_2 = \langle a_2 \rangle$. Then by Lemmas 2.3 and 2.4

$$[A_2, G]A_1/A_1 \leqslant Z(G/A_1).$$

Hence $[A_2, G, G] \leqslant A_1 \cap [A, G] = 1$ and so $[A_2, G] \leqslant Z(G)$. Therefore there is an element x in G such that $[a_2, x] \in Z(G)$ and

$$[a_2, x] \neq 1. \tag{2}$$

We would like to write $[a_2, x]$ in the form $[a, g]$, where $A = \langle a \rangle$. In fact this can be achieved by generalising a result due to Busetto (see [1]; also [11], Theorem 5.2.12).

LEMMA 2.8. *Let A be a cyclic quasinormal subgroup of order p^n (p a prime) in a group G and let $|A_G| = p^m$, where $0 \leqslant m < n$. Then $A \leqslant Z_{2n-m}(G)$.*

PROOF. Busetto's theorem is the case $m = 0$. Suppose that the result is false and let G be a counter-example with $|A|$ minimal. Then $m \geqslant 1$. So $B = \Omega(A) \lhd G$ and by choice of G

$$A/B \leqslant Z_{2(n-1)-(m-1)}(G). \tag{3}$$

Let $g \in G$. If $|g| = \infty$ or $|g| = q^\ell$, where q is a prime different from p, then $A^g = A$. The first case follows from Lemma 2.6 and the second from Lemma 2.5. Thus $[A, g] \leqslant A^p$, by (3), and therefore $[B, g] = 1$. On the other hand, if $|g| = p^\ell$, then $B \lhd \langle B, g \rangle$ implies $[B, g] = 1$. It follows that $B \leqslant Z(G)$. Then $A \leqslant Z_{2n-m}(G)$, again by (3). □

Returning to (2) above, since $A \cap [A, G] = 1$, x does not normalise A. Then Lemma 2.8 shows that $\langle A, x \rangle$ is nilpotent. Therefore, by Lemma 2.1 (v), there is an element $y \in \langle x \rangle$ such that $[a_2, x] = [a, y]$. The argument of Lemma 2.4 in [2] now suffices to complete the proof of the following.

LEMMA 2.9. *Let $A = \langle a \rangle$ be a cyclic quasinormal subgroup of odd prime-power order in a group G. Suppose that $A \cap [A, G] = 1$. Then each element of $[A, G]$ has the form $[a, g]$, for some element g in G.*

We should point out that the hypothesis here that A has *odd* order is necessary, even when G is finite. This is illustrated in Example 2 in Section 5.

The next step is to remove the hypothesis $A \cap [A, G] = 1$ in Lemma 2.9.

LEMMA 2.10. *Let $A = \langle a \rangle$ be a cyclic quasinormal subgroup of odd prime-power order in a group G and suppose that A is not normal in G. Then each element of $[A, G]$ has the form $[a, g]$, for some $g \in G$.*

We have to exclude $A \triangleleft G$ here, as can be seen from the non-abelian group of order 6. In [2], Theorem 2.5, Lemma 2.10 is proved in the case when G is a p-group. We wish to use the same argument here. Since A is not normal in G, it follows from Lemma 2.8 that $A \leqslant Z_m(G)$ for some m. Then it is convenient for us to prove the following result.

Let $A = \langle a \rangle$ be a cyclic quasinormal subgroup of odd prime-power order in a group G and suppose that $A \leqslant Z_m(G)$, for some finite m. Then each element of $[A, G]$ has the form $[a, g]$, for some element g in G.

Now we can prove this stronger form of Lemma 2.10 by means of the argument in Theorem 2.5 of [2]. There a counter-example was chosen with $|A|$ minimal. Here we assume the above statement is false and so there is a finitely generated counter-example G. Then $|A^G|$ is finite and we can choose G with $|A^G|$ minimal. Let $|A| = p^n$. Then we need to know that the statement holds with A replaced by A^p. But this must be the case, because $|(A^p)^G| < |A^G|$ is an immediate consequence of the following elementary result.

LEMMA 2.11. *Let A be a cyclic quasinormal subgroup of finite order k in a group G. Then A^G has exponent k.*

PROOF. If $k = 1$, the Lemma is trivial. Thus we proceed by induction on k and assume the usual induction hypothesis. Let p be a prime divisor of k and let B be the subgroup of order p in A. Then B *qn* G and B^G is elementary abelian, by Lemma 2.4. By induction A^G/B^G has exponent k/p and hence A^G has exponent k. □

Thus our argument above can continue as in Theorem 2.5 of [2] and so Lemma 2.10 follows. As a consequence we obtain

THEOREM 2.12. *Let A be a cyclic quasinormal subgroup of odd prime-power order p^n in a group G. Then $[A, G]$ is an abelian p-group of exponent dividing $|A|$ and A acts on $[A, G]$ as a group of power automorphisms.*

The proof is exactly the same as that of Theorem 2.1 in [2], using Lemmas 2.10, 2.1 (iii) and 2.11 above. The power automorphisms here are universal, by [10], 13.4.3 (ii).

We can now extend Theorem 2.12 to the case where A has finite odd order, i.e. Theorem A. If $A = P_1 \times P_2 \times \cdots \times P_s$ is the primary decomposition of A, then $[A, G] = [P_1, G] \times [P_2, G] \times \cdots \times [P_s, G]$ and each P_i is quasinormal in G. Theorem A follows from Theorem 2.12 by the easy division algorithm argument which was used to establish Theorem 1.1 from Theorem 2.1 in [2]. The only difference here is that G may be infinite, but each step in the proof remains valid using the lemmas above.

3. Arbitrary finite cyclic quasinormal subgroups.

It suffices to prove Theorem B for the case where A is a 2-group. For, if $A = P \times Q$, with P the 2-component and Q the $2'$-component of A, then P and Q are quasinormal in G. Also $[A, G] = [P, G] \times [Q, G]$. By Theorem A, $[Q, G]$ is abelian and Q acts on it as a group of power automorphisms. By Lemma 2.5, P centralises $[Q, G]$ and Q centralises $[P, G]$. Thus if Theorem B holds for quasinormal 2-subgroups (P here), then it holds generally. The statements referring to power automorphisms follow as in Theorem A.

Thus we may assume that A in Theorem B is a 2-group and we have the hypothesis

$(*)$ *X is quasinormal in AX for all cyclic subgroups X of G.*

The key to making progress is again the understanding of the structure of the products AX, where X is a cyclic subgroup of G. We need the analogue of Lemma 2.1 for the case $p = 2$.

LEMMA 3.1. *Let $G = AX$, where $A = \langle a \rangle$ is a cyclic quasinormal 2-subgroup of G and $X = \langle x \rangle$ is a cyclic subgroup. Then*

(i) *G is metacyclic and G' is a 2-group;*
(ii) *$G' = \langle [a, x] \rangle$;*

(iii) *for each integer i,* $\langle [a^i, x] \rangle = \langle [a, x]^i \rangle$*;*
(iv) *a conjugates* $[a, x]$ *to a power congruent to* 1 *modulo* 4*; and*
(v) *each element of* G' *has the form* $[a^i, x]$*, for some integer i.*

If in addition $(*)$ *holds, then*

(iii)′ *for each integer i,* $\langle [a, x^i] \rangle = \langle [a, x]^i \rangle$*;*
(iv)′ *x conjugates* $[a, x]$ *to a power congruent to* 1 *modulo* 4*; and*
(v)′ *each element of* G' *has the form* $[a, x^i]$*, for some integer i.*

The proofs proceed by considering the cases X finite and X infinite separately. In the former, $G = (AX_2) \times X_{2'}$; and in the latter, $A \lhd G$. The arguments are straightforward, following the pattern established in Lemma 2.3 of [2], Lemma 2.3 of [3], Lemma 2.1 of [4] and Lemma 2.1 above, and may safely be omitted.

From (iii) we obtain

Corollary 3.2. *Let* $A = \langle a \rangle$ *be a cyclic quasinormal* 2*-subgroup of a group G. Then* $[A, G] = \langle [a, g] | g \in G \rangle$*.*

We have already reduced the proof of Theorem B to the case where $A = \langle a \rangle$ is a 2-group. When G is also a finite 2-group, then Theorem B is contained in Theorems 2.1 and 2.2 of [3]. The arguments given in Sections 2, 3 and 4 of [3], upto and including the proofs of Theorems 2.1 and 2.2, use the finiteness of G only for the purpose of applying induction. In the present context, following on from Lemma 3.1, whenever this is necessary, we may assume that G is finitely generated. Then, as in Lemma 2.7 above, we deduce that A^G is finite and this suffices for the induction arguments of [3] to apply here. The presence of elements of infinite order in G is not a problem, essentially because they always normalise A. Thus we can safely omit the remaining details of the proof of Theorem B, and thereby spare the reader much tedium.

In the same way, we can proceed as in Corollary 4.1 of [3] to obtain the following consequence of Theorem B.

Corollary 3.3. *Let A be a finite cyclic quasinormal subgroup of a group G satisfying* $(*)$*. Then* $A \cap [A, G] \leqslant Z([A, G])$*.*

We now pass to the proof of Theorem C. Here G is any group and A is a finite cyclic quasinormal subgroup of G. Our arguments reduce easily to the case where A is a 2-group. Then we have the following.

THEOREM 3.4. *Let $A = \langle a \rangle$ be a finite cyclic 2-subgroup, quasinormal in an arbitrary group G. Then*

(i) $[A, G, A] \lhd G$; *and*
(ii) $[A, G, A] = \{[u, a] | u \in [A, G]\}$.

THEOREM 3.5. *Let A be a finite cyclic 2-subgroup, quasinormal in an arbitrary group G, and put $B = [A, G, A]$. Then*

(i) $B \leqslant Z([A, G])$;
(ii) *A centralises* $[A, G]/B$;
(iii) *A acts by conjugation on B as a group of power automorphisms; and*
(iv) $[A, G]$ *is nilpotent of class at most* 2.

REMARK. As in Theorem 2.12, the power automorphisms here are universal.

When G is a finite 2-group, Theorem 3.5 is Theorem 1.3 of [4]. Using Lemma 3.1, Corollary 3.2 and Theorem 3.4 above, the argument of [4] applies here, without modification, to establish Theorem 3.5.

PROOF OF THEOREM C. We have

$$[A, G] = [A_2 \times A_{2'}, G] = [A_2, G] \times [A_{2'}, G],$$

because both factors are normal in G and are a 2-group and a $2'$-group respectively. So (i) follows.

By Theorem A, $[A_{2'}, G]$ is abelian and $A_{2'}$ acts on it as a group of power automorphisms. Since $(A_2)^G$ is a 2-group, $A_2[A_{2'}, G] = A_2 \times [A_{2'}, G]$. Therefore (ii) is true. Part (iii) follows from Theorem 3.5 (iv). For part (iv), $[A_2, G, A] = [A_2, G, A_2] = B$, say, since $A_{2'}$ centralises $[A_2, G]$. By Theorem 3.4 (i), $B \lhd G$; and by Theorem 3.5 (i), $B \leqslant Z([A, G])$.

Finally (v) follows from Theorem 3.5 (iii), again since $A_{2'}$ centralises B. □

It remains to prove Theorem 3.4. When G is a finite 2-group, this is Theorem 1.2 of [4]. For part (i) in [4], we were able to assume that G is generated by A and at most 2 other elements. That argument did not require G to be a 2-group or even finite, and so again we may assume that

$$G = \langle A, x, y \rangle. \tag{4}$$

By Lemma 2.7, A^G is finite. Thus we suppose that Theorem 3.4 (i) is false and choose a counter-example (4) with $|A^G|$ minimal. Let $A = \langle a \rangle$ and

$B = [A, G, A]$. Then $B_G = 1$ and $B \neq 1$. Two cases must be distinguished.

Case (i). *Suppose that* $\langle [a, x] \rangle \cap \langle [a, y] \rangle \neq 1$. If a commutes with $[a, x]$ and $[a, y]$, then, as in [4], it would follow that $B = 1$. (This is Lemma 4.2 below.) Thus we may assume that $[a, x, a] \neq 1$. Then

$$\Omega(\langle [a, x] \rangle) = \Omega(\langle [a, y] \rangle) \leqslant B \cap Z(G),$$

contradicting $B_G = 1$.

Case (ii). *Suppose that* $\langle [a, x] \rangle \cap \langle [a, y] \rangle = 1$. For each integer $i \geqslant 0$, write $A_i = A^{2^i}$. Choose i such that a acts non-trivially on $[A_i, G]$ and trivially on $[A_{i+1}, G]$. Then following the detailed analysis of the analogous situation in [4], we can show that a acts as a power automorphism on $[A_i, G]$. Thus suppose that a conjugates each element of $[A_i, G]$ to its n-th power and let $L = [A_i, G]$. Then

$$L^{n-1} \leqslant [L, A] \leqslant L^{n-1}$$

and so $[L, A] = L^{n-1} \lhd G$. But $1 \neq [L, A] \leqslant B$, again contradicting $B_G = 1$. This completes the proof of Theorem 3.4 (i).

For Theorem 3.4 (ii), again $A = \langle a \rangle$ is a finite cyclic 2-subgroup, quasinormal in an arbitrary group G. We must show that

(5) $$[A, G, A] = \{[u, a] | u \in [A, G]\}.$$

When G is a finite 2-group, we argued in [4] by induction on $|A|$ and we do the same here. When $|A| \leqslant 4$, then $|[a, g]| \leqslant 4$, for all $g \in G$, by Lemma 2.11. Thus a centralises all $[a, g]$, by Lemma 3.1 (iv), so $[A, G, A] = 1$ and (5) is true. Therefore we suppose that $|A| \geqslant 8$ and assume the usual induction hypothesis. This means that

$$[A^2, G, A^2] = \{[v, a^2] | v \in [A^2, G]\} = K,$$

say. By Theorem 3.4 (i), $K \lhd G$. Also by Lemma 3.1 (ii), A normalises each cyclic subgroup of K and hence A^G does the same. Thus K *is abelian* and a acts as a universal power automorphism on K. As in Theorem 3.5 (i), it follows that

(6) $$[A, G, K] = 1.$$

Following the analogous argument in Theorem 1.2 (ii) of [4] (without modification), we deduce from (6) that each element of K has the form $[u, a]$, with $u \in [A, G]$. Recall that by Lemma 2.8, either $A \lhd G$ (in which case (5) is trivial) or $A \leqslant Z_m(G)$, for some integer m. Thus we may assume the latter and so there is a central series of G between 1 and K. Then a simple induction on the length of this series allows us to assume that

$$K = 1.$$

Again as in Theorem 1.2 (ii) of [4], we can now deduce from Lemma 3.1 that

$$[A, G, A^2] = \langle [a, g, a^2] | g \in G \rangle = L,$$

say. In turn this leads to $L \subseteq \{[y, a] | y \in [A^2, G]\} \subseteq [A^2, G, A]$. But using Lemma 3.1, we can show that $L = [A, G, A]^2 \triangleleft G$, and so the Three Subgroup Lemma gives

$$[A^2, G, A] \leqslant L.$$

Thus we have equality here and in exactly the same way as we were able to assume that $K = 1$, we may also assume that

$$L = 1.$$

Using Lemma 3.1 (iv), this leads to $[A, G, A, [A, G]] = 1$, from which we deduce (5). This completes the proof of Theorem 3.4 (ii). □

REMARKS. If A is a *periodic locally cyclic* quasinormal subgroup of a group G, then, for each prime p, the p-component S of A is quasinormal in G (see [1], Proposition 1.6, or [11], Lemma 6.2.16). Moreover if $S^p = S$, then $S \triangleleft G$, as a consequence of Lemma 2.7. Then applying Theorems A, B and C to the p-components of A makes the structure of A^G quite transparent. Also all the subgroups of A are quasinormal in G, by Lemma 2.3; and if N is the join of all the normal p-components of A, then $A^G/N \leqslant Z_\omega(G/N)$, by Lemma 2.8.

4. Infinite cyclic quasinormal subgroups.

We move on now to consider an infinite cyclic quasinormal subgroup A of a group G. As for finite cyclic quasinormal subgroups, the structure of products AX, where X is also cyclic, is of fundamental importance (see Lemmas 2.1 and 3.1 above). Once again it turns out that these subgroups are metacyclic.

LEMMA 4.1. *Let $G = AX$, where $A = \langle a \rangle$ is an infinite cyclic quasinormal subgroup of G and $X = \langle x \rangle$ is also cyclic. Then G is metacyclic and $G' = \langle [a, x] \rangle$.*

PROOF. Clearly the second statement follows from the first. To show that G is metacyclic, we consider five cases.

Case 1. *Suppose that* $X \cong C_\infty$ *and* $A \cap X = 1$. Then by Lemma 2.6, $A \lhd G$ and there is nothing to prove.

Case 2. *Suppose that* $X \cong C_\infty$ *and* $A \cap X \neq 1$. We may assume that $A \ntrianglelefteq G$. Let $N = A_G$. Then $N \geqslant A \cap X$. Since x must either invert or centralise N, we can see that the latter must apply. So $N \leqslant Z(G)$. Let p be prime and let P/N be any non-trivial p-component of A/N. Then P/N *qn* G/N and hence

$$P/N \leqslant Z_m(G/N),$$

for some finite integer m, by Lemma 2.8. Therefore $P \leqslant Z_{m+1}(G)$. Since this applies for all primes p, we have $A \leqslant Z_n(G)$, for some finite integer n, and so G is nilpotent. The Hirsch length $h(G) = 1$ and the torsion subgroup T of G is finite. Moreover

$$\mathcal{L}(T) \cong \mathcal{L}(AT : A) \subseteq \mathcal{L}(G : A) \cong \mathcal{L}(X : A \cap X),$$

a distributive lattice, so T is cyclic, by [9] (see also [11], Corollary 1.2.4). Finally, since G/T is torsion-free, it must be cyclic and so G is metacyclic. It is easy to see that G is always nilpotent in this case.

Case 3. *Suppose that* $X \cong C_{p^n}$, *where* p *is an odd prime.* We claim that

$$X \lhd G \text{ and } G \text{ is nilpotent.} \tag{7}$$

To see this, we argue by induction on $|X|$. If $|X| = p$, then $A \lhd G$, by Lemma 2.5. So G is abelian and (7) is true. Therefore assume that $|X| \geqslant p^2$ and let $X_1 = \Omega(X)$. Then by the same argument, AX_1 is abelian and hence $X_1 \leqslant Z(G)$. Now by induction $X/X_1 \lhd G/X_1$, i.e. $X \lhd G$. Also G/X_1 is nilpotent and so G is nilpotent, establishing (7).

REMARK. The group G here does not have to be a finite cyclic extension of an infinite cyclic group. For example, this is not the case when $x^a = x^{p+1}$. But G is a modular group and all its subgroups are quasinormal, by [7] (see also [11], Theorem 2.4.11).

Case 4. *Suppose that* $X \cong C_{2^n}$. Here we claim that

$$X^2 \lhd G \text{ and } AX^2 \text{ is nilpotent.} \tag{8}$$

Consider first the case $|X| = 4$. We know that all the subgroups of A are quasinormal in G. So there are two possibilities.

(a) *Suppose that* $ax = x^i a$, *for some* i. Then $X \lhd G$ and G is abelian. So (8) is true.

(b) *Suppose that* $ax = x^i a^{-1}$, *for some* i. If $i = 1$, then $A \lhd G$ and $a^x = a^{-1}$. So $x^2 \in Z(G)$ and again (8) is true. On the other hand, if $i = -1$, then $(ax)^2 = 1$. But $G = A\langle ax\rangle$ and this would imply that $|G : A| = 2$, a contradiction.
We have shown that when $|X| = 4$,

(9) *either* $G = A \times X$ *or* $A \lhd G$ *and* x *inverts* A.

Now we suppose that $|X| \geqslant 8$ and argue by induction on $|X|$ to prove (8). Again let $X_1 = \Omega(X)$. Then $X_1 \leqslant Z(G)$, by (9). Therefore by induction $X^2/X_1 \lhd G/X_1$ and so $X^2 \lhd G$. Also AX^2/X_1 is nilpotent and so AX^2 is nilpotent. Thus (8) is true.

To prove that G is metacyclic, we may assume that $|X| \geqslant 8$, by (9). Since $X^2 \lhd G$, we also have $X^4 \lhd G$ and so (a) or (b) above applies to G/X^4. If (a) holds, then $X \lhd G$ and we are finished. Therefore we may assume that

$$ax = x^i a^{-1} \ mod\ X^4.$$

Here $i = \pm 1$. But if $i = -1$, then modulo X^4 we get the same contradiction as in (b). Therefore $i = 1$ and so we have

$$ax = x^{1+4j} a^{-1},$$

for some integer j. Then $x^{-1}ax = x^{4j}a^{-1}$ and $x^{-2}ax^2 = x^{4j}ax^{-4j}$. Hence x^{2+4j} commutes with a and so $[A, X^2] = 1$. Therefore

$$(ax^{-2j})^x = x^{4j}a^{-1}x^{-2j} = x^{2j}a^{-1}.$$

Thus x normalises $\langle ax^{-2j}\rangle$ and so does a. Then $\langle ax^{-2j}\rangle \lhd G$ and G is metacyclic, as required.

NOTE. In Case 4 we have shown that one of the following applies:

(i) $X \lhd G$; (ii) $A \lhd G$; (iii) $a^x = a^{-1}x^{4j}$, $x^2 \in Z(G)$ and $x^{4j} \neq 1$.

Example 5 in Section 5 shows that (iii) does occur.

Case 5. *Suppose that* X *is finite.* Let $|X| = n$, so we may assume that n is not a prime power. If n is odd, then $X \lhd G$, by Case 3. Therefore suppose that

n *is even, but not a power of* 2.

We may assume that $A \ntriangleleft G$ and $X \ntriangleleft G$.

Let $X_2 = \langle x_2\rangle$ be the 2-component of X. By the note above, (ii) or (iii) must apply to AX_2. We distinguish these possibilities.

(a) *Suppose that* $A \lhd AX_2$. Then

(10) $$a^{x_2} = a^{-1}.$$

Let $X_0 = \langle x_0 \rangle$ be the $2'$-component of X. Then $X_0 \lhd G$ and AX_0 is nilpotent, by Case 3. Also AX_0 is not abelian, since $A \ntriangleleft G$. Therefore

$$a^{x_0} = ax_0^i \text{ and } u = x_0^i \neq 1.$$

As a conjugate of A, $\langle au \rangle$ *qn* G. Consider $\langle au \rangle X_2$. With $\langle au \rangle$ the infinite cyclic quasinormal subgroup, (i) above cannot apply, otherwise $X_2 \lhd G$ and then $X \lhd G$. If (iii) applies, then

$$(au)^{x_2} = (au)^{-1} x_2^{4j} = a^{-1}u,$$

by (10). But then we obtain $x_2^{4j} = 1$, contradicting (iii). We are left with (ii). Then $(au)^{x_2} = (au)^{-1} = a^{-1}u$, again by (10). So $u^a = u^{-1}$ and this contradicts the nilpotency of AX_0.

(b) *Suppose that* (iii) *applies to* A *as a quasinormal subgroup of* AX_2. Then for some j,

$$a^{x_2} = a^{-1}x_2^{4j} \text{ and } x_2^{4j} \neq 1.$$

If $[A, X_0] = 1$, then $G = \langle ax_2^{-2j} \rangle \rtimes X$ and G is metacyclic. Thus suppose that $[A, X_0] \neq 1$. By Case 3,

(11) $$X_0 \lhd AX_0.$$

So again with $u = [a, x_0]$, we consider $\langle au \rangle$ *qn* $\langle au \rangle X_2$ and the three possiblities in the note after Case 4. Since $X \ntriangleleft G$, (i) cannot apply. If (ii) holds, then $\langle au \rangle$ is normalised by x_2 and hence the action is by inversion. So

$$u^{-1}a^{-1} = a^{-1}x_2^{4j}u,$$

i.e. $x_2^{4j} = 1$, a contradiction. Finally, suppose that (iii) applies. Then for some integer k,

$$(au)^{x_2} = u^{-1}a^{-1}x_2^{4k} = a^{-1}x_2^{4j}u.$$

But then by (11) we must have $x_2^{4k} = x_2^{4j}$ and $u^a = u^{-1}$, contradicting (7).

This completes the proof of Lemma 4.1. □

REMARK. Even in a finite group $G = AX$, with A and X cyclic and A *qn* G, it does not follow in general that G is metacyclic. (See Example 1 in Section 5.)

For the proof of Theorem D we need one more result.

LEMMA 4.2. *Let $A = \langle a \rangle$ be a cyclic quasinormal subgroup of a group $G = \langle a, x, y \rangle$. Then*

$$[A, G] = (A \cap [A, G])\langle [a, x] \rangle \langle [a, y] \rangle. \tag{12}$$

Proof. When G is a finite 2-group, this is Lemma 3.3 in [3]. Here G may be infinite, but we still need the case where A is finite. This is a routine generalisation of Lemma 3.3 of [3], already used in the proof of Theorem 3.4 above. For convenience we include the argument here.

Thus suppose first that A is finite. Then A^G is finite, by Lemma 2.7, and we can argue by induction on $|A^G|$. We may assume that $[A, G] \neq 1$. So let N $(\neq 1)$ be a normal subgroup of G contained in A or in $[A, G]$. By induction we have from G/N

$$[A, G]N = (A \cap [A, G])\langle [a, x] \rangle \langle [a, y] \rangle N. \tag{13}$$

Suppose that $A_G \neq 1$. Then there is a normal subgroup A_1 of G with $|A_1|$ prime and $A_1 \leqslant A$. Take $N = A_1$. If $N \leqslant [A, G]$, then $N \leqslant A \cap [A, G]$ and (13) becomes (12). On the other hand, if $N \cap [A, G] = 1$, then intersecting both sides of (13) with $[A, G]$ again gives (12).

Now suppose that $A_G = 1$. Then we may assume, without loss of generality, that $L = [A_1, x] \neq 1$, where A_1 is again a subgroup of prime order in A. Since A_1 *qn* G, we have $L \leqslant Z(G)$, by Lemma 2.4. So we can take $N = L$. But then $N \leqslant \langle [a, x] \rangle$ and again (13) becomes (12).

Finally suppose that A is infinite. By Lemma 2.7, $|A^G : A|$ is finite and so A^G/A_G is finite. Therefore $A_G \neq 1$. Suppose that $A \cap [A, G] = 1$. By the case when A is finite, with $N = A_G$ we have (13). Then intersecting with $[A, G]$ we get (12). But if $A \cap [A, G] \neq 1$, then there is a normal subgroup N of G with $1 \neq N \leqslant A \cap [A, G]$. Again (13) holds and this is (12). □

PROOF OF THEOREM D. Let $A = \langle a \rangle$ be an infinite cyclic quasinormal subgroup of a group G. Suppose that $[A, G]$ is periodic. Then clearly $A \cap [A, G] = 1$. By Theorem 2.2, $[A, G]$ is abelian and a acts on it as a power automorphism. Conversely, suppose that $A \cap [A, G] = 1$. Again $[A, G]$ is abelian by Theorem 2.2. By Lemma 4.1,

$$[A, G] = \langle [a, g] | g \in G \rangle.$$

Let $g \in G$ and $H = A\langle g \rangle$. By Lemma 2.7, $|A^H : A|$ is finite and therefore $|[a, g]|$ is finite. It follows that $[A, G]$ is periodic. This proves (ii).

Now suppose that $[A, G]$ is not periodic. By what we have already proved, this is equivalent to $A \cap [A, G] \neq 1$. By Lemma 2.5, A^G is generated by ascendant cyclic subgroups and so it is locally nilpotent, by [6], Theorem 2. Thus $[A, G]$ is locally nilpotent and so there is an element $g \in G$ such that $[a, g]$ has infinite order. But a normalises $\langle [a, g] \rangle$, by Lemma 4.1, and so a centralises $[a, g]$. This applies to any commutator $[a, g]$ of infinite order. Suppose that $[a, h]$ has finite order. Then

$$[a, hg] = [a, g][a, h]^g$$

has infinite order and so a centralises $[a, hg]$. Put $H = \langle a, g, h \rangle = \langle a, g, hg \rangle$. By Lemma 4.2,

$$[A, H] = (A \cap [A, H])\langle [a, g] \rangle \langle [a, hg] \rangle$$

and so a centralises $[A, H]$. Therefore a centralises $[a, h]$.

It follows that A centralises $[A, G]$. Thus A^G centralises $[A, G]$ and $[A, G]$ is abelian. Hence A^G is abelian. This completes the proof of Theorem D. □

The power automorphisms in (i) are not always universal. For example, let H be an abelian group of type p^∞ and let a be a p-adic integer such that $a \equiv 1 \; mod \; p$ ($a \equiv 1 \; mod \; 4$ if $p = 2$). Then the map

$$h \mapsto h^a, \tag{14}$$

for all $h \in H$, defines an automorphism of H. Let G be the split extension of H by the cyclic group A generated by a. Then every subgroup of G is quasinormal (see [11], Theorem 2.4.11). If $a \neq 1$, then $[A, G] = H$. Clearly the power automorphism (14) is not universal in general.

We end this Section, as we did in Section 3, with a brief discussion of how our results extend to *locally cyclic* subgroups. Let A be a torsion-free locally cyclic group. Then the automorphism group $\mathrm{Aut}(A)$ of A is isomorphic to a subgroup of the multiplicative group of the rationals. Also $\mathrm{Aut}(A)$ contains an element a of infinite order, if and only if we have $A^p = A$, for at least one prime p. In this case, no non-trivial cyclic subgroup of A is left invariant by a. (See [5].)

THEOREM 4.3. *Let A be a torsion-free locally cyclic quasinormal subgroup of a group G. Then the following conditions are equivalent.*

(i) *There is a non-trivial cyclic subgroup of A which is quasinormal in G.*

(ii) *Every subgroup of A is quasinormal in G.*

(iii) *For all $g \in G$, $|\langle g \rangle : C_{\langle g \rangle}(A)|$ is finite.*

PROOF. Assume that (iii) holds. Let $a \in A$, $a \neq 1$. Let $g \in G$ and put $T = \langle a, g \rangle$, $L = \langle A, g \rangle$ and $A_1 = A \cap T$. If $|L : A| = \infty$, then $A \lhd L$, by

Lemma 2.6, and so $A_1 = \langle a \rangle^{\langle g \rangle} \lhd T$. Since A_1 is finitely generated, we have $\langle a \rangle \leqslant A_1 = \langle a_1 \rangle \lhd T$ and so $\langle a \rangle \lhd T$. If $|L : A|$ is finite, then $\langle a \rangle \leqslant A_1$ *qn* T and $|T : A_1|$ is finite. Thus A_1 is finitely generated, hence cyclic, and so $\langle a \rangle \leqslant A_1 = \langle a_1 \rangle$ *qn* T. Then $\langle a \rangle$ *qn* T, by Lemma 2.3. It follows that $\langle a \rangle$ *qn* G. Therefore (iii) implies (ii).

Now suppose that (iii) fails, i.e. $|\langle g \rangle : C_{\langle g \rangle}(A)| = \infty$, for some element g. Then $A^g = A$, by Lemma 2.6, and g induces an automorphism of A of infinite order. But then $\langle a \rangle^g \neq \langle a \rangle$ for any $a \in A$, $a \neq 1$, as stated above. Thus $\langle a \rangle$ is not quasinormal in G, and (i) fails. □

THEOREM 4.4. *Let A be a torsion-free locally cyclic quasinormal subgroup of a group G. Then*

(i) $[A, G]$ *is abelian, A acts on it as a group of power automorphisms and A^G is a modular group;*
(ii) $[A, G]$ *is periodic if and only if $A \cap [A, G] = 1$;*
(iii) *if $A \cap [A, G] \neq 1$, then A^G is abelian; and*
(iv) *for every natural number n, A^n is quasinormal in G.*

The proofs of the above statements can safely be left as exercises, following the arguments of Lemma 4.2, Theorem D and Theorem 4.3. Of course (iv) holds for any abelian quasinormal subgroup A of any group G, provided n is odd or divisible by 4. (See [15], Theorem 1.)

5. Examples.

All the results that have been proved in the previous Sections have depended on showing that a group $G = AX$, with A and X cyclic subgroups and A quasinormal, is metacyclic, provided certain hypotheses are satisfied. We begin this final Section by showing that hypotheses *are* required here.

EXAMPLE 1. Let B and X be cyclic groups of orders 7 and 9, respectively, and let $B = \langle b \rangle$, $X = \langle x \rangle$. Form the split extension

$$H = B \rtimes X,$$

where $b^x = b^2$. Now let C be a cyclic group of order 3 generated by the element c and form the split extension

$$G = H \rtimes C,$$

where $b^c = b$ and $x^c = x^4$. The action of c on H has order 3 and the relations of H are preserved. So G exists and has order $3^3.7$.

Set $A = BC$, a cyclic group of order 21. We claim that

(15) *A is quasinormal in G.*

To see this, since $B \lhd G$, we may factor by B, i.e. we may assume that $B = 1$. Then $G = X \rtimes C$, which is a modular group with all subgroups quasinormal. Therefore (15) is true. However, *G is not metacyclic*. For, $G' = \langle b, x^3 \rangle \cong C_{21}$. If $G = NK$, with $N \lhd G$ and N and K both cyclic, then $G' \leqslant N$. Since $G/G' \cong C_3 \times C_3$, we must have $G' < N$. Thus $|N| = 3^2.7$ and so $N_3 \cong C_9$. But N_3 must centralise B, while $C_G(B) = B \times \langle x^3 \rangle \times C$ containing no cyclic subgroup of order 9.

Another key result was Lemma 2.9 showing that when $A = \langle a \rangle$ is a cyclic quasinormal subgroup of odd prime power order in G and $A \cap [A, G] = 1$, then

(16) $$[A, G] = \{[a, g] | g \in G\}.$$

This result is also true when A has 2-power order, provided condition $(*)$ holds. (It follows easily from [3], Lemma 2.7.) However, if $(*)$ is not satisfied, then (16) can fail to hold.

Example 2. Let H and X be cyclic groups of orders 8 and 16, respectively, and let $H = \langle h \rangle$ and $X = \langle x \rangle$. We form the split extension

$$G = H \rtimes X,$$

with $h^x = h^{-1}$. Let $a = hx^2$ and $A = \langle a \rangle$, a cyclic group of order 8. Then $C_G(A) = HX^2$. Also $A \cap H = A \cap X = 1$, so $G = AX$ and

$$[A, G] = \langle [h, x] \rangle = H^2 \cong C_4.$$

Thus $A \cap [A, G] = 1$. Also the only elements of form $[a, g]$ are 1 and h^{-2} ($= [a, x]$). Therefore

$$[A, G] \neq \{[a, g] | g \in G\}.$$

However,

(17) *A is quasinormal in G.*

For, any cyclic subgroup of G of form $\langle h^i x^{2j} \rangle$ commutes with A. On the other hand, $\langle h^i x^j \rangle$, with j odd, has order 16 (because $(h^i x^j)^2 = h^i x^j h^i x^{-j} x^{2j} = x^{2j}$) and intersects A trivially. So $|A\langle h^i x^j \rangle| = 2^7$ and hence $A\langle h^i x^j \rangle = G$. Thus (17) is true.

REMARK. Clearly the subgroup X is *not* quasinormal in G and so $(*)$ is not satisfied here. Also each element of $[A, G]$ has the form $[a^i, g]$, for some integer i and element g. For many purposes this is as useful as (16) and it would be interesting to know if it is *always* the case.

We know (see Lemma 2.6) that if A is *any* quasinormal subgroup of *any* group G and if X is an infinite cyclic subgroup of G such that $A \cap X = 1$, then X normalises A. It is natural to ask if this then implies that A is normal in G. In fact this is not the case, as is shown by the next two examples.

EXAMPLE 3. Let $X = \langle x \rangle$ be an infinite cyclic group and let $Y = \langle y \rangle$ be a cyclic group of order 8. We form the split extension

$$K = X \rtimes Y,$$

where $x^y = x^{-1}$. Then let $A = \langle a \rangle$ be a cyclic group of order 2 and form the split extension

$$G = K \rtimes A,$$

where $x^a = x$, $y^a = y^5$. This extension exists, because the relations $y^8 = 1$ and $x^y = x^{-1}$ are both preserved by the a-action (of order 2). Clearly

A is not normal in G.

However, A is quasinormal in G. For, consider an arbitrary cyclic subgroup $H = \langle x^i y^j a^k \rangle$, $0 \leqslant j \leqslant 7$, $0 \leqslant k \leqslant 1$. We have

$$(x^i y^j a^k)^a = x^i y^{5j} a^k.$$

Case (i). *Suppose that j is even.* Then $y^{5j} = y^j$ and A centralises H.

Case (ii). *Suppose that j is odd.* Then $[x^i y^j, a] = [y^j, a] = y^4 \in Z(G)$ and $\langle x^i y^j, a \rangle$ has nilpotency class 2. Thus $(x^i y^j a^k)^5 = (x^i y^j)^5 a^k$. Also $(x^i y^j)^2 = y^{2j}$ and $(x^i y^j)^5 = x^i y^{5j}$. So $(x^i y^j a^k)^5 = x^i y^{5j} a^k = (x^i y^j a^k)^a$ and A normalises H.

Therefore $A \leqslant norm(G)$ and so A *qn* G, as claimed.

The previous group G is isomorphic to $C_\infty \rtimes (C_8 \rtimes C_2)$ and involves a dihedral action. We show that the same result can be achieved using odd primes instead of the prime 2.

EXAMPLE 4. Let p be an odd prime and let Y be a cyclic group of order p^2 generated by y. Put $Y_1 = Y/Y^p$ and $y_1 = yY^p$. Let $R = \mathbb{Z}Y_1$, the integral group ring of Y_1 and let $I = R(y_1 - 1)$, the augmentation ideal of R.

Then I becomes a Y-module via the natural action $u^y = uy_1$, all $u \in I$. We form the split extension $K = I \rtimes Y$, a free abelian group of rank $p-1$ extended by C_{p^2}.

Let A be a cyclic group of order p generated by a and define an action of A on K by

$$u^a = u, \text{ all } u \in I, \; y^a = y^{1+p}.$$

Then form the split extension $G = K \rtimes A$. Thus A is not normal in G. But, as before, we claim that

$$A \leqslant norm(G). \tag{18}$$

For, A centralises $IY^pA = L$, say. A typical cyclic subgroup of G, not contained in L, is generated by an element of the form $g = y^i u a^j$, where $i \equiv 1$ mod p and $u \in I$. Then $g^a = y^{i(1+p)} u a^j$. But we have

$$g^{1+p} = y^{i(1+p)} u a^j. \tag{19}$$

For, $\langle y^i u, a\rangle$ is nilpotent of class 2, since its derived subgroup is Y^p, which is central in G. Therefore

$$g^{1+p} = (y^i u)^{1+p} a^j.$$

Also $(yu)^{1+p} = yu(yu)^p = yu(y^p u(1 + y_1 + y_1^2 + \cdots + y_1^{p-1})) = y^{1+p}u$. Thus $(y^i u)^{1+p} = y^{i(1+p)}u$ and (19) is true. Therefore (18) holds and *A is quasinormal in G.*

When proving Lemma 4.1, we considered in Case 4 a group $G = AX$, where $A = \langle a\rangle$ is an infinite cyclic quasinormal subgroup of G and $X = \langle x\rangle$ is cyclic of order 2^n. We showed that if neither X nor A is normal in G, then

$$a^x = a^{-1}x^{4j},$$

with $x^2 \in Z(G)$ and $x^{4j} \neq 1$. In fact this situation does occur.

EXAMPLE 5. Let $A = \langle a\rangle$ be an infinite cyclic group and let $Y = \langle y\rangle$ be a cyclic group of order 2^{n-1} ($n \geqslant 3$). Let $H = A \times Y$ and let j be an integer. Then H has an automorphism of order 2 defined by

$$a \mapsto a^{-1}y^{2j}, \qquad y \mapsto y.$$

So there is an extension G of H by a group of order 2 defined by

$$G = \langle a, x | a^x = a^{-1}x^{4j}, x^{2^n} = 1\rangle.$$

(See [12], Theorem 9.7.1 (ii).) Then $G = AX$, where $X = \langle x\rangle$ and $ax^2 = x^2 a$. We claim that

$$A \textit{ is quasinormal in } G. \tag{20}$$

For, a typical cyclic subgroup H of G is generated by an element $a^i x^k$. If k is even, then A centralises H. If k is odd, then $(a^i x^k)^2 = x^{2k+4ij}$, generating X^2 and $X^2 \leqslant Z(G)$. So $H^2 = X^2$ and $AH = G$. Thus (20) is true. Provided 2^{n-2} does not divide j, we have a group of type described in (iii) of Case 4 of Lemma 4.1.

Our final example also concerns the case when the quasinormal subgroup A is infinite cyclic. Situations where $[A, G]$ are periodic or torsion-free are familiar. However, the *mixed* case can also occur.

EXAMPLE 6. Let $H = \langle a, y | y^8 = 1, y^a = y^5 \rangle$. Then H has an automorphism of order 2 defined by

$$a \mapsto a^{-1}, \quad y \mapsto y^5. \tag{21}$$

So we can form a split extension $G = H \rtimes X$, where $X = \langle x \rangle$ is a cyclic group of order 2 with action on H defined by (21). We claim that $A = \langle a \rangle$ *is quasinormal in* G. To see this, since $A^2 \lhd G$, we may factor by A^2, i.e. assume that $a^2 = 1$. Then easy calculations show that a conjugates each element of G to its 5-th power. Therefore A *qn* G. However,

$$[A, G] = \langle a^2 \rangle \times \langle y^4 \rangle \cong C_\infty \times C_2.$$

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