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On the Group of Automorphisms of Finite Wreath Products

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Dedicated to Guido Zappa on his 90th birthday

ABSTRACT - In this paper we study the structure of the group of automorphisms of the wreath product $A\, wr\, C_2$, where A is a finite nilpotent group, and C_2 is the cyclic group of order 2. In particular we give necessary and sufficient conditions for $\mathrm{Aut}\,(A\, wr\, C_2)$ to be supersolvable depending only on $\mathrm{Aut}\,(A)$ and on the Remak decomposition of A.

Introduction and statement of the main result.

This work is a contribution to the study of the group of automorphisms of a (restricted) wreath product $A\, wr\, B$ of two non-trivial groups A and B. Throughout this paper we denote with G the group $A\, wr\, B$. Particular interest concerns the relationships among the structures and the group-theoretical properties of the groups A and B and the ones of $\mathrm{Aut}\,(G)$.

For instance, if $\mathrm{Aut}\,(G)$ is a nilpotent group much is known. In fact in this case G is clearly nilpotent and for a result of G. Baumslag ([2]) we have that both A and B are nilpotent p-groups (for the same prime p), where A has finite exponent and B has finite order. Moreover, if $\mathrm{Aut}\,(G)$ is supposed to be finite (and nilpotent), then $\mathrm{Aut}\,(A)$ and $\mathrm{Aut}\,(B)$ are finite p-groups (see [9]). Conversely, if A and B are finite p-groups (for $p \neq 2$) with $\mathrm{Aut}\,(A)$ and $\mathrm{Aut}\,(B)$ p-groups too, then $\mathrm{Aut}\,(A\, wr\, B)$ is a p-group (see M.V. Khoroshevskii [4]).

Concerning supersolvability, in a recent paper ([7]) G. Corsi Tani and R. Brandl proved the following

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THEOREM. *If* Aut(G) *is supersolvable then* A *is nilpotent. Moreover if either* G *is supposed to be infinite or if* $(|A|, |B|) = 1$, *then* $B \simeq C_2$, *the cyclic group of order two.*

In this paper we study the "inverse problem" of the aforementioned result. We suppose that A is a finite nilpotent group and give necessary and sufficient conditions for Aut$(A \; wr \; C_2)$ to be supersolvable.

Our main Theorem is proved in section 3 and it can be stated as follows.

THEOREM 1. *Let* A *be a finite nilpotent group with* Aut(A) *supersolvable and let* $G := A \; wr \; C_2$. *Then* Aut(G) *is supersolvable if and only if either no two distinct factors in a Remak decomposition of* A *are isomorphic, or the only possible ones are 2-groups and in this case* $O_2(A)$ *is not isomorphic to* $C_2 \times C_2$.

Clearly the supersolvability of Aut(A) is a necessary condition since Aut(A) is, up to isomorphism, a subgroup of Aut(G).

The paper is organized as follows. In the first two sections we analyze what happens if A is a finite p-group, where p is any prime number. In the former we focus our attention on the inner structure of Aut(G), while in the latter on the problem of supersolvability. Finally, in the last section we generalize the results to any finite nilpotent group A.

1. The structure of Aut(G).

Let A be a finite p-group, C_2 be the cyclic group of order two and $G := A \; wr \; C_2$ the restricted wreath product of A and C_2. Then, by [3], if A itself is not the group of order two, the base group F of G is characteristic in G and, if we fix a complement $C_2 = \langle t \rangle$, we can express Aut(G) as a product

$$\mathrm{Aut}\,(G) = K \cdot I,$$

where K is the subgroup of Aut(G) consisting of those automorphisms which fix t and I is the subgroup of Aut(G) consisting of those inner automorphisms which correspond to conjugations by elements of F. Moreover I is normal in Aut(G), and since I is isomorphic to a quotient of F, I is a p-group. Let us give now another interpretation of K. We identify the base subgroup F with $A \times A$, the direct product of two copies of A. Denote as usual the elements of F with couples (x, y), where $x, y \in A$, and with δ the involution of F defined by: $\delta(x, y) = (y, x)$ for each $x, y \in A$. Let $C_{\mathrm{Aut}\,(F)}(\delta)$ be the centralizer of δ in $\mathrm{Aut}\,(F)$.

LEMMA 1. *The groups K and $C_{\mathrm{Aut}\,(F)}(\delta)$ are isomorphic.*

PROOF. An isomorphism is given by the map that sends $\eta \in K$ into $\pi_F \eta i_F \in K$ where i_F is the canonical injection of F in G and π_F the canonical projection of G on F. The inverse of this map is the application that sends $\varphi \in K$ into the automorphism Φ of G so defined: $\Phi(a_1 a_2 t^{\varepsilon}) := i_G(\varphi(a_1, a_2))t^{\varepsilon}$, for each $a_1 a_2 t^{\varepsilon} \in G$. $\square$

We note that in this situation the automorphism δ of F is indeed the conjugation by the element t that generates C_2 in G.

To study more in detail the structure of the group K, we give some more notation that will be used through all the paper.
We indicate with:

$\mathrm{Aut}_Z(A), \mathrm{Aut}_Z(F)$ the groups of the central automorphisms of A and of F respectively (that is the groups of the automorphisms which act trivially on the central factor groups $\dfrac{A}{Z(A)}$ and $\dfrac{F}{Z(F)}$ respectively).

$K_Z := K \cap \mathrm{Aut}_Z(F)$ and similarly if X is any subgroup of $\mathrm{Aut}\,(F)$ we use X_Z for $X \cap \mathrm{Aut}_Z(F)$.

$\Delta(Y) := \{(y, y) \mid y \in Y\}$ for each Y subgroup of A (we use simply Δ for $\Delta(A)$).

$\nabla(Y) := \langle (y, y^{-1}) \mid y \in Y \rangle$ for each Y subgroup of A (we use simply ∇ for $\nabla(A)$).

$H := C_K(\Delta)$ the centralizer of Δ in K.

$L := C_K(\nabla)$ the centralizer of ∇ in K.

$A^* := \{a_f \mid f \in \mathrm{Aut}\,(A)\}$ where a_f denotes the automorphism of F defined by $a_f(a_1, a_2) = (f(a_1), f(a_2))$, for each $(a_1, a_2) \in F$. (Clearly $A^* \simeq \mathrm{Aut}\,(A)$).

Given any element φ of $\mathrm{Aut}\,(F)$, define two endomorphisms φ_1, φ_2 of A by

$$\varphi(x, 1) := (\varphi_1(x), \varphi_2(x)), \forall x \in A.$$

As a Lemma we collect now some elementary facts. We omit the easy proofs.

LEMMA 2. (i) *If $\varphi \in K$, then $\forall x, y \in A$*

$$\varphi(1, x) = (\varphi_2(x), \varphi_1(x)) \quad \textit{and} \quad \varphi(x, y) = (\varphi_1(x)\varphi_2(y), \varphi_2(x)\varphi_1(y))$$

(ii) $[\operatorname{Im}\varphi_1, \operatorname{Im}\varphi_2] = 1$ *and* $A = (\operatorname{Im}\varphi_1)(\operatorname{Im}\varphi_2)$ *(in particular* $\operatorname{Im}\varphi_1 \cap \cap \operatorname{Im}\varphi_2 \leq Z(A)$ *and* $\operatorname{Im}\varphi_i$ *are normal subgroups of* A*).*

(iii) *Any element of* K *fixes the subgroups* Δ *and* ∇*, in particular it permutes the generators* (x, x^{-1}) *of* ∇.

(iv) *The subgroup* H *is the kernel of the group epimorphism* $\Phi : \varphi \in K \longmapsto \varphi_1 + \varphi_2 \in \operatorname{Aut}(A)$ *(where* $\varphi_1 + \varphi_2$ *is the map* $a \mapsto \varphi_1(a)\varphi_2(a)$ *for each* $a \in A$*) and* $K = [H]A^*$ *is the semidirect product of* H *and* A^*.

(v) $\dfrac{H}{H_Z}$ *is an elementary abelian 2-group.*

(vi) *The map* $\tilde{}: \varphi \in H_Z \mapsto \tilde{\varphi} \in \operatorname{Aut}_Z(A)$ *defined by* $\tilde{\varphi}(a) := \varphi_1(a^2)a^{-1}$ *is a homomorphism of groups.*

At this point we distinguish the two cases p even and p odd.

Let us suppose first that A is a finite 2-group.
In this situation the map $\tilde{}$ defined in Lemma 2 (vi) is in general not injective. In fact its kernel is constituted by the elements φ of H_Z such that $\varphi_1(a^2) = a^2$ for each $a \in A$, and so, since $\varphi_1 + \varphi_2 = id_A$, $\varphi_2(a^2) = 1$ for each $a \in A$. Then

$$\operatorname{Ker}\tilde{} = \{\varphi \in H_Z \mid \varphi_{|A^2 \times A^2} = id_{|A^2 \times A^2}\}.$$

Lemma 3. *The kernel of the map* $\tilde{}$ *is an elementary abelian 2-group.*

Proof. Let φ be a non trivial element of the kernel of $\tilde{}$ and a an element of A. Then

$$\varphi^2(a, 1) = (\varphi_1^2(a)\varphi_2^2(a), 1)$$

and so $(\varphi^2)_1 = \varphi_1^2 + \varphi_2^2$ and $(\varphi^2)_2 = 0$ the null endomorphism of A. Since $\varphi^2 \in H$, then $(\varphi^2)_1 + (\varphi^2)_2 = id_A$ and so $(\varphi^2)_1 = id_A$ and φ has order 2. □

Lemmas 2, 3 and the fact that $\operatorname{Aut}(G) = K \cdot I$ imply the following

Theorem 2. *If* A *is a 2-group, then the composition factors of* $\operatorname{Aut}(G)$ *are either composition factors of* $\operatorname{Aut}(A)$ *or cyclic groups of order* 2.

The same result can be proved for $p \neq 2$, in fact if p is odd, the map $\tilde{}$ of Lemma 2 (vi) is a monomorphism, therefore H_Z is a normal subgroup of H that can be embedded in $\operatorname{Aut}(A)$. So, according to Theorem 2, we can state

THEOREM 3. *If A is a p-group, with p any prime, then the composition factors of the group K are either composition factors of* Aut (A), *or cyclic groups of order* 2.

We next show that in the situation $p \neq 2$ we can lift the subgroup H_Z up to H, and this will be of fundamental importance for our purposes.

Fix a Remak decomposition of A, say $A = A_1 \times A_2 \times \ldots \times A_n$, where A_i are indecomposable direct factors of A. For each subset I of the set $\{1, 2, \ldots, n\}$, we define $X_I := \langle A_i \mid i \in I \rangle$, $Y_I := \langle A_j \mid j \notin I \rangle$, so that $A = X_I \times Y_I$, and denote with $\varphi_{(X_I, Y_I)}$ or φ_I the automorphism of F defined by

$$\varphi_{(X_I, Y_I)}(xy, x_1 y_1) := \varphi_I(xy, x_1 y_1) := (xy_1, x_1 y)$$

for each $x, x_1 \in X_I$ $y, y_1 \in Y_I$.

It is immediate to verify that $\varphi_I \in H$. We can then consider the subset of H $Q := \{\varphi_I \mid I \subseteq \{1, 2, \ldots, n\}\}$. It is not difficult to prove that Q is an elementary abelian 2-subgroup of H that can be generated by the elements $\varphi_{\{i\}}$, $i = 1, 2, \ldots, n$ (we leave the proofs to the reader).

Note that the subgroup Q depends on the Remak decomposition, say $\mathcal{A}$, of A chosen at first, so that it is convenient to write $Q^{\mathcal{A}} := := \{\varphi_I^{\mathcal{A}} \mid I \subseteq \{1, 2, \ldots, n\}\}$ instead of $Q := \{\varphi_I \mid I \subseteq \{1, 2, \ldots, n\}\}$. If we choose another direct decomposition of A, say $\mathcal{B}$, then we obtain in general another elementary abelian 2-subgroup, say $Q^{\mathcal{B}}$. The relation between $Q^{\mathcal{A}}$ and $Q^{\mathcal{B}}$ is explicitated in the next

LEMMA 4. *$Q^{\mathcal{A}}$ and $Q^{\mathcal{B}}$ are conjugate in K by an element of A_Z^*.*

PROOF. Suppose the two different Remak decompositions of A are $\mathcal{A} := \{A_i\}_1^n$ and $\mathcal{B} := \{B_i\}_1^n$. By the Krull-Schmidt theorem there exists a central automorphism f of A such that for each $i = 1, 2, \ldots, n$, $f(A_i) = B_i$. Consider then the automorphism a_f of F. a_f is a central automorphism of F and for each $I \subseteq \{1, 2, \ldots, n\}$,

$$a_f^{-1} \varphi_I^{\mathcal{B}} a_f = \varphi_I^{\mathcal{A}},$$

and so $Q^{\mathcal{A}}$ and $Q^{\mathcal{B}}$ are conjugate by an element of A_Z^*. □

These arguments allow us to give the following description of the structure of the subgroup H.

LEMMA 5. *Chosen a Remak decomposition $\mathcal{A}$ of A, the subgroup H of K is the product $H = H_Z \cdot Q^{\mathcal{A}}$.*

PROOF. By Lemma 2 we already know that H_Z is normal in K and that $\frac{H}{H_Z}$ is an elementary abelian 2-group. Suppose that φ is an element of H of order a power of 2, say 2^m. If we call $M := \operatorname{Ker}\varphi_2$ and $N := \operatorname{Im}\varphi_2$, we have that $A = M \times N$. In fact by Lemma 2, M and N are both normal subgroups of A. In order to prove that their intersection is trivial, consider an element $a \in M \cap N$, say $a = \varphi_2(x)$. By an induction argument we have that $\varphi^k(x, 1) = (\varphi_1^k(x), \varphi_2(x)^k)$, for each $k \geq 1$. In particular for $k = 2^m$ we obtain $\varphi_2(x)^{2^m} = 1$ and since 2 does not divide $|A|$, we have that $x \in \operatorname{Ker}\varphi_2$, i.e. $a = 1$. So $A = M \times N$ and this decomposition of A allows us to consider the automorphism $\varphi_{(M,N)}$. A simple computation shows that $\varphi\varphi_{(M,N)} \in H_Z$. In this way, according to the fact that the order of $\frac{H}{H_Z}$ is a power of 2, we have that for any φ in H there exists an element of H of the form $\varphi_{(R,S)}$ (for some decomposition (R, S) of A) such that $\varphi\varphi_{(R,S)} \in H_Z$.

Now we fix a Remak decomposition $\mathcal{A}$ of A and call Q the subgroup $Q^{\mathcal{A}}$ of H. If φ is any element of H, we proved that there exist $\sigma \in H_Z$ and $\varphi_{(R,S)} \in Q^{\mathcal{B}}$, for some decomposition $\mathcal{B}$ of A, that is a refinement of (R, S), such that $\varphi = \sigma\varphi_{(R,S)}$. Call f the central automorphism of A such that $Q = a_f^{-1}Q^{\mathcal{B}}a_f$, then $\varphi^{a_f} = \sigma^{a_f}\varphi_{(R,S)}^{a_f} \in H_Z Q$. But now we can write the element φ as $\varphi = [\varphi^{-1}, a_f^{-1}]\varphi^{a_f}$, and, since H and K_Z are normal subgroups of K, $[\varphi^{-1}, a_f^{-1}] \in H_Z$ and so $\varphi \in H_Z Q$, i.e. $H = H_Z Q$. □

2. Supersolvability of K.

We now concentrate our study on the problem of the supersolvability of the group K. We of course assume that $\operatorname{Aut}(A)$ is supersolvable.
As before, we consider first the case $p = 2$.

With the considerations of section 2.1 it is not difficult now to establish when $\operatorname{Aut}(G)$ is supersolvable. Making use of [8], we know that the automorphism group of a 2-group, not $C_2 \times C_2$, is supersolvable if and only if it is itself a 2-group. Therefore we have the following

THEOREM 4. *Let A be a finite 2-group. If A is not isomorphic to $C_2 \times C_2$, then* $\operatorname{Aut}(G)$ *is supersolvable if and only if* $\operatorname{Aut}(A)$ *is supersolvable. If $A \simeq C_2 \times C_2$, then* $\operatorname{Aut}(G)$ *is not supersolvable.*

PROOF. For the case A not isomorphic to $C_2 \times C_2$, Theorem 2 and the results in [8] tell us that $\operatorname{Aut}(G)$ is a 2-group and so it is supersolvable.

If $A \simeq C_2 \times C_2$, Aut(G), being not a 2-group, is not supersolvable. (In this case the subgroup $I \cdot A$ of Aut(G) is isomorphic to the symmetric group S_4). □

The case p odd requires more work.

We recall that a group T is said to be *strictly p-closed* (where p is any prime number) if $T'T^{p-1}$ is a p-subgroup of T, or equivalently if $T = [O_p(T)]S$ with S an abelian group of exponent that divides $p-1$. Strictly p-closed groups are supersolvable (see Baer [1]), and the automorphism group of a finite p-group ($p \neq 2$) is supersolvable if and only if it is strictly p-closed (see G. Corsi Tani [8]).

Before making any other hypothesis than the supersolvability of Aut(A), we prove a relevant result concerning the subgroup K_Z of K.

Lemma 6. *If A is a p-group ($p \neq 2$), then, with the same notations as before:*

(i) $K_Z = [H_Z]A^*_Z = [L_Z]A^*_Z$, *semidirect products.*

(ii) $H_Z \cap L_Z = 1$.

(iii) $H_Z \simeq L_Z \simeq$ *a subgroup of A^*_Z normal in A^*.*

(iv) *If* Aut(A) *is supersolvable, then K_Z is supersolvable.*

Proof. *(i)* It is immediate by the fact that the applications Φ_1 and Φ_2 from K_Z to Aut$_Z(G)$ defined respectively by $\Phi_1(\varphi) := \varphi_1 + \varphi_2$ and $\Phi_1(\varphi) := \varphi_1 - \varphi_2$ (where $\varphi_1 + \varphi_2(a) = \varphi_1(a)\varphi_2(a)$ and $\varphi_1 - \varphi_2(a) = \varphi_1(a)(\varphi_2(a))^{-1}$ for each $a \in A$) are both epimorphisms of kernels respectively H_Z and L_Z.

(ii) Let $\varphi \in H \cap L$, then $\varphi(a^2, 1) = \varphi(a, a)\varphi(a, a^{-1}) = (a, a)(a, a^{-1}) = (a^2, 1)$ and since 2 does not divide $|A|$, φ is the identity map, and so $H \cap L = 1$.

(iii) We have already proved that H_Z can be embedded in A^*. Moreover, using a better argument, one can consider the subgroup $T := H_Z \times L_Z$. T is subgroup of K_Z, which is normal in K, so $T \cap A^*$ is a subgroup of A^*_Z normal in A^*. Using *(i)* and Dedekind's modular law, we obtain that $T \cap A^* \simeq H_Z \simeq L_Z$.

(iv) By *(i)* we obtain that $\dfrac{K_Z}{H_Z} \simeq \dfrac{K_Z}{L_Z} \simeq A^*_Z$, and since $H_Z \cap L_Z = 1$, if Aut(A) is supersolvable, K_Z is a supersolvable group. □

In particular Lemma 6 applies when A is abelian and Aut(A) supersolvable. In this case Aut$_Z(F) =$ Aut(F) and so $K = K_Z$ is a supersolvable group.

Let us now concentrate on the case A *non*-abelian, and study more in detail the structure of the subgroup K_Z, with particular interest on the action of K on K_Z.

LEMMA 7. *Let A be a non abelian finite p-group ($p \neq 2$), with* $\mathrm{Aut}\,(A)$ *supersolvable. Then:*

(i) $O_{p'}(K_Z) = 1$ *and* $K_Z = [O_p(K_Z)]S$, *where S is an abelian Hall $p'-$subgroup of K_Z.*

(ii) $K = K_Z C_K(S)$.

(iii) $[K_Z, K] \leq O_p(K_Z)$.

PROOF. *(i)* Since $\mathrm{Aut}\,(A)$ is supersolvable, another result of G. Corsi ([8]) implies that $O_{p'}(\mathrm{Aut}\,(A)) = 1$, and so, as A^*_Z is normal in A^* and $A^* \simeq \mathrm{Aut}\,(A)$, $O_{p'}(A^*_Z) = 1$. Now the map Φ_1 defined in Lemma 6 *(i)* is a homomorphism with kernel H_Z, and so we deduce that $O_{p'}(K_Z) = O_{p'}(H_Z)$. By Lemma 6 again, H_Z is isomorphic to a normal subgroup of A^*_Z and so $O_{p'}(H_Z) = 1$, i.e. $O_{p'}(K_Z) = 1$ and the Fitting subgroup $\mathrm{Fit}(K_Z)$ coincides with $O_p(K_Z)$. The supersolvability of K and Schur-Zassenhaus theorem complete the proof of this step.

(ii) Take S a Hall $p'-$subgroup of K_Z. Using Frattini's argument we have that $K = K_Z N_K(S)$. In order to prove that $N_K(S) = C_K(S)$, consider a decomposition of A as $A = X \times Y$, where X and $Y \neq 1$ are respectively the product of the abelian and non-abelian direct factors of A. Since S is in particular a $p'-$subgroup of $\mathrm{Aut}_Z(F)$ we have, according to [5], $F = C_F(S) \times [F, S]$, where $C_F(S)$ is the centralizer of S in F and $[F, S] := \langle w^{-1}g(w) \mid g \in S, w \in F\rangle$ is abelian. Moreover, up to conjugation, we can suppose that S contains the automorphisms α and β defined by $\alpha(xy, \tilde{x}\tilde{y}) := (\tilde{x}y, x\tilde{y})$, $\beta(xy, \tilde{x}\tilde{y}) := (x^{-1}y, \widetilde{x^{-1}}\tilde{y})$, for each $x, \tilde{x} \in X$, $y, \tilde{y} \in Y$. Consider now the subgroups $\Delta(X)$ and $\nabla(X)$. One can easily prove that $\Delta(X) \simeq \nabla(X) \simeq X$, $X \times X = \Delta(X) \times \nabla(X)$, $[F, \alpha] = \nabla(X)$, and $[F, \beta] = X \times X$. From this, we obtain that $[F, S] = X \times X$. Then $N_K(S)$ fixes the subgroups $\Delta(X)$, $\nabla(X)$ and $Y \times Y$, and so any element f of $N_K(S)$ can be seen as a triple $f = (f_0, f_1, f_2)$, where $f_0 = f_{|Y \times Y}$ is an automorphism of $Y \times Y$ and $f_1 = f_{|\Delta(X)}$, $f_2 = f_{|\nabla(X)}$ are automorphisms of X. In particular the elements of S are of the form $\varphi = (id, \varphi_1, \varphi_2)$. Now taken any $f \in N_K(S)$ and any $\varphi \in S$ we have that $\varphi^f = (id, \varphi_1^{f_1}, \varphi_2^{f_2}) \in S$ and $\varphi_1, \varphi_1^{f_1}, \varphi_2$ and $\varphi_2^{f_2}$ can be seen as $p'-$elements of $\mathrm{Aut}\,(X)$. $\mathrm{Aut}\,(X)$ is a supersolvable group, since it is, up to isomorphism, a subgroup of $\mathrm{Aut}\,(A)$, in particular $\mathrm{Aut}\,(X)$ is strictly p-closed. Therefore $[\varphi_1, f_1]$ lies in $O_p(\mathrm{Aut}\,(X))$, but φ_1 lies in a Hall p'-subgroup of $\mathrm{Aut}\,(X)$ and f_1 normalizes this subgroup and so $[\varphi_1, f_1]$ must be also a p'-element, therefore $[\varphi_1, f_1] = 1$.

Similarly $[\varphi_2, f_2] = 1$, and so f centralizes S, and $K = K_Z C_K(S)$.
(iii) Using the previous two steps we obtain

$$[K_Z, K] = [O_p(K_Z)S, O_p(K_Z)C_K(S)] \leq O_p(K_Z). \qquad \square$$

As a consequence of this Lemma we obtain the following result on the subgroup H.

LEMMA 8. *The subgroup H of K is strictly p-closed.*

PROOF. Using the previous Lemmas we obtain that

$$H' = [H_Z Q, H_Z Q] \leq O_p(H_Z) \leq O_p(H).$$

So we just have to prove that $\dfrac{H}{O_p(H)}$ has exponent that divides $p-1$.
Note that $\dfrac{H}{H_Z}$ is an elementary abelian 2-group and so $O_p(H) = O_p(H_Z)$.
Using the facts that $H = H_Z Q$, Q is an elementary abelian 2-group, and $\dfrac{H}{O_p(H)}$ is abelian, we deduce that $\left(\dfrac{H}{O_p(H)}\right)^{p-1} = \left(\dfrac{H_Z}{O_p(H_Z)}\right)^{p-1}$, and since $H_Z \lesssim \mathrm{Aut}\,(A)$, H_Z is strictly p-closed and $\left(\dfrac{H}{O_p(H)}\right)^{p-1} = 1$.

In particular we proved that $[H, H] \leq O_p(H)$. As $[A^*, A^*] \leq O_p(A^*)$, we now just have to consider the action of A^* on H. At this point we make hypotheses on the structure of A. Suppose first of all that *A has no pairs of isomorphic direct factors in any of its Remak decompositions.* Note that this hypothesis is consistent in the case A abelian, in fact a result of G. Corsi [8] shows that any abelian p-group ($p \neq 2$) with supersolvable automorphism group necessary satisfies this condition.
A first consequence of this assumption is contained in the next

LEMMA 9. *Let A be a p-group ($p \neq 2$), with* Aut (A) *supersolvable and such that A has no isomorphic direct factors in its Remak decompositions, then*

(i) $N_K(Q) = C_K(Q)$.
(ii) $K = K_Z C_K(Q)$.

PROOF. *(i)* We let f be an element of $N_K(Q)$ and prove that f centralizes Q by showing that f centralizes the generators $\varphi_{\{i\}}$ of Q for $i = 1, 2, \ldots, n$. Let us begin by observing that f permutes the elements $\varphi_{\{i\}}$. In fact if Q is associated to the following Remak decomposition of A, $A = X_1 \times$

$\times X_2 \times \ldots \times X_n$, the automorphism $\varphi_{\{i\}}$ interchanges the elements of $X_i \times X_i$ in $A \times A$ and acts like the identity on the others. Similarly $f^{-1}\varphi_{\{i\}}f$ changes the positions of the elements of $f^{-1}(X_i) \times f^{-1}(X_i)$ and acts trivially on the rest. Since X_i is indecomposable, $f^{-1}(X_i)$ is to be such, and so $f^{-1}\varphi_{\{i\}}f$ must be an element of Q that acts not trivially only on one pair of direct indecomposable factors, i.e. $f^{-1}\varphi_{\{i\}}f = \varphi_{\{j\}}$ for some $j = 1, 2, \ldots, n$. Therefore f permutes the elements $\varphi_{\{i\}}$.
In order to prove that $i = j$, since A has no isomorphic direct factors, it is enough to show that $X_i \simeq X_j$. From the previous observation we immediately deduce that $|X_i| = |X_j|$. Let us prove now that $f(X_j' \times X_j') = X_i' \times X_i'$.

We denote with $[\delta F, \varphi_{\{j\}}]$ the subgroup $\langle \delta(u^{-1})\varphi_{\{j\}}(u) \mid u \in F\rangle$; then

$$[\delta F, \varphi_{\{j\}}] = \langle (x, x^{-1}) \mid x \in X_j\rangle = \nabla(X_j)$$

and

$$X_j' \times X_j' \leq \nabla(X_j) \cap F' \leq (X_j \times X_j) \cap F' = X_j' \times X_j'$$

and so $\nabla(X_j) \cap F' = X_j' \times X_j'$. Similarly $\nabla(X_i) \cap F' = X_i' \times X_i'$. Now we claim that $f(\nabla(X_j)) = \nabla(X_i)$. In fact let $(x, x^{-1}) \in \nabla(X_j)$, then by Lemma 2 (iii), $f(x, x^{-1}) = (y, y^{-1})$ for some $y \in A$. Let $y = uv$ with $u \in X_i$, $v \in Y_i$, then

$$(uv, u^{-1}v^{-1}) = f(x, x^{-1}) = f\varphi_j(x, x^{-1}) = \varphi_i f(x, x^{-1}) = (uv^{-1}, u^{-1}v)$$

and so, since 2 does not divide the order of A, $f(x, x^{-1}) = (u, u^{-1})$. Since $|X_i| = |X_j|$, we have proved that $f(\nabla(X_j)) = \nabla(X_i)$, and from this we obtain that

$$\begin{aligned} X_i' \times X_i' &= F' \cap \nabla(X_i) = f(F') \cap f(\nabla(X_j)) = \\ &= f(F' \cap \nabla(X_j)) = f(X_j' \times X_j') \end{aligned}$$

Now for the sake of simplicity we let
$X := \Delta(X_i) := \{(x, x) \mid x \in X_i\}$,
$Y := \Delta(Y_i) := \{(y, y) \mid y \in Y_i\}$,
$M := f(\Delta(X_j))$ and
$L := f(\Delta(Y_j))$.

Since f fixes $\Delta = \Delta(A)$, $M \leq \Delta \cap f(X_j \times X_j) = \Delta(f(X_j \times X_j))$ and by reasons of orders, $M = \Delta(f(X_j \times X_j))$. Similarly $L = \Delta(f(Y_j \times Y_j))$. Moreover, $A = X_i \times Y_i = X_j \times Y_j$ and this implies $M \simeq X_j, L \simeq Y_j$ and $\Delta = X \times Y = M \times L$.

Let now π_X and π_M be the projections from Δ respectively on X and on M, and set $\Phi := \pi_M \circ \pi_X$. Φ restricted to M is an endomorphism of M, so there exists $n \geq 1$ such that $\Phi^n(M) = \Phi^{n+1}(M) := R$. Now

$$M' = (\Delta f(X_j \times X_j))' = \Delta f(X_i \times X_i)' = \Delta(X_i') = X',$$

and so

$$\Phi(M') = \pi_M \pi_X(X') = \pi_M(X') = \pi_M(M') = M'.$$

Therefore $R = \Phi^n(M) \geq \Phi^n(M') = M'$, in particular $R \trianglelefteq M$.

Now call $S = \operatorname{Ker} \Phi^n$, then $S \trianglelefteq M$, $R \cap S = 1$ and $M = RS$. Therefore $M = R \times S$ and S is abelian as $R \geq M'$. Since $M \simeq X_j$, M is indecomposable, so either $M = R$ or $M = S$, in both cases we have $X_j \simeq X_i$. In fact if $M = R$, then $M = \Phi(M)$, so $M \simeq \pi_X(M)$. But $|M| = |X|$, and so $X = \pi_X(M) \simeq M$, i.e. $X_i \simeq X_j$. Otherwise if $M = S$ then M is cyclic, so X_j too. Using the fact that $f(X_j' \times X_j') = X_i' \times X_i'$, we have that

$$X_i' \times X_i' = f(X_j' \times X_j') = 1$$

so X_i is abelian and since it is indecomposable, it is cyclic. By the fact X_i and X_j are of the same order we deduce they are isomorphic.

(ii) As a consequence of Remak-Krull-Schmidt theorem, we have that for each $a_f \in A^*$ there exists a $a_{f_c} \in A_Z^*$ such that $Q^{a_f} = Q^{a_{f_c}}$, but this means that $a_f a_{f_c}^{-1} \in N_K(Q)$, and so $A^* = N_{A^*}(Q) A_Z^*$. Therefore $K = HA^* = H_Z Q N_{A^*}(Q) A_Z^* = N_K(Q) K_Z$, and by the previous Lemma $K = C_K(Q) K_Z$.

Lemma 10. $[A^*, H] \leq O_p(H)$.

Proof. Using Lemmas 6 and 9, and the fact that D_Z centralizes H we obtain that $K = A_Z^* C_K(Q)$. In particular $A^* = A_Z^* C_A^*(Q)$. Then

$$[A^*, H] = [A^*, H_Z Q] \leq [A^*, H_Z]^Q [A^*, Q] \leq [K, K_Z][A^*, Q] \leq$$

$$\leq [K, K_Z][A_Z^* H_{A^*}(Q), Q] \leq [K, K_Z][A_Z^*, Q] \leq [K, K_Z] \leq O_p(K_Z),$$

and since H is normal in K, $[A^*, H] \leq O_p(H)$. □

Now we are in condition to prove a result about K.

Theorem 5. *The group K is supersolvable if and only if* $\operatorname{Aut}(A)$ *is supersolvable and either A has no isomorphic direct factors (and in this case K is strictly p-closed), or $A = A_1 \times A_2$ with $A_1 \simeq A_2$ indecomposable.*

PROOF. Suppose first that in any Remak decomposition of A there are no pairs of isomorphic direct factors. Then from the previous Lemmas it is clear that $O_p(K) = O_p(H)O_p(A)$ and $K' \leq O_p(K)$. Since $\frac{K}{O_p(K)} = \frac{O_p(H)A^*}{O_p(K)} \times \frac{HO_p(A^*)}{O_p(K)}$, we have that the exponent of $\frac{K}{O_p(K)}$ divides the exponent of $\frac{\mathrm{Aut}\,(A)}{O_p(\mathrm{Aut}\,(A))}$ and so also $(p-1)$. It follows that K is strictly p-closed and thus supersolvable.

Suppose now that $A = A_1 \times A_2$ is the direct product of two isomorphic indecomposable groups. Since $\mathrm{Aut}\,(A)$ is supersolvable, this implies that the A_i are not abelian ([8]). In particular F has no abelian direct factors too, and this implies that K_Z is a p-subgroup of K. In fact if S is any p'-subgroup of K_Z, using [5] we have that $F = C_F(S) \times [F, S]$, with $[F, S]$ central, then Krull-Schmidt theorem implies $[F, S] = 1$, i.e. $S = 1$. So we have that $K_Z \leq O_p(K)$ and $[K_Z, K] \leq O_p(K)$. Using the same notations as before, the subgroup Q now reduces to $\langle \varphi_1, \varphi_2 \rangle$, an elementary abelian group of order 4. We indicate with l any isomorphism from A_1 to A_2; l induces the following involution in A $L(g_1 g_2) := l^{-1}(g_2)l(g_1)$ for each $g_1 \in A_1, g_2 \in A_2$. Moreover, L induces $a_L \in A^* \leq K$. We have that $|a_L| = 2$ and $N_K(Q) = \langle a_L \rangle C_K(Q)$. From these we obtain that $[A^*, H] \leq O_p(K)\langle \delta \rangle$, and so, since $[A^*, A^*] \leq O_p(A^*) \leq O_p(K)$ (as A^* is supersolvable with $O_{p'}(A^*) = 1$) and $[H, H] \leq [K_Z, K] \leq O_p(K)$, we have that $K' \leq O_p(K)\langle \delta \rangle \leq$ $\leq \mathrm{Fit}(K)$, the Fitting subgroup of K. Then we have that $\frac{K}{\mathrm{Fit}(K)} = \frac{\mathrm{Fit}(H)A^*}{\mathrm{Fit}(K)} \times \frac{HO_p(A^*)}{\mathrm{Fit}(K)}$ and so $\exp\left(\frac{K}{\mathrm{Fit}(K)}\right)$ divides $(p-1)$. This is enough to deduce the supersolvability of K.

Finally we consider the case $A = A_1 \times A_2 \times B$ with $A_1 \simeq A_2$ indecomposable groups and $B \neq 1$. In this situation we prove that K is not supersolvable by showing that there is an element of order two in the Fitting subgroup of K which does not commute with all the elements of order p, and this is a contradiction, since p is the largest prime number that divides $|K|$. Using the same notation as in the previous step, let us consider the following subgroup of K, $R := \langle \varphi_1 a_L, \varphi_2 \rangle$, (where a_L now acts like the identity automorphism on the elements of K). It is easy to see that R is a dihedral group of order 8 and that its centre consists of the subgroup generated by $\delta\varphi_{\{1,2\}} = [\varphi_1, a_L]$. If K were supersolvable then, $O_p(K)$ would be a Sylow p-subgroup of K, moreover $K' \leq \mathrm{Fit}(K)$, and so $\delta\varphi_{\{1,2\}} \in \mathrm{Fit}(K)$ would commute with every element of order p. Now let θ be a non trivial homomorphism from $\frac{A_1}{A_1'}$ to $Z(K)$ (remember that in this situation $\mathrm{Aut}\,(A)$ supersolvable and $A_1 \simeq A_2$ imply that A_1 is non-abelian). θ defines the following automorphism of A: $\Theta(g) := g\theta(g)$. Θ has order p and it induces $a_\Theta \in K$. Finally, a simple computation shows that $[\delta\varphi_{\{1,2\}}, a_\Theta] \neq 1$ and this contradiction completes the proof. □

3. **Supersolvability of** Aut (G).

This section is devoted to prove Theorem 1.

We first consider the case that A is a finite p-group and then derive the result for finite nilpotent groups as a consequence.

THEOREM 6. *Let A be a finite p-group with* $\mathrm{Aut}\,(A)$ *supersolvable. If $p = 2$,* $\mathrm{Aut}\,(A\ wr\ C_2)$ *is supersolvable if and only if $A \not\simeq C_2 \times C_2$. If $p \neq 2$,* $\mathrm{Aut}\,(A\ wr\ C_2)$ *is supersolvable if and only if A has no isomorphic direct factors in any of its Remak decomposition.*

PROOF. If $p = 2$ the result follows from Theorem 4.

Let now $p \neq 2$ and suppose first that A has no isomorphic direct factors. Then $\mathrm{Aut}\,(A\ wr\ C_2) = K \cdot I$, by Theorem 5, is an extension of a strictly p-closed group by a normal p-group and so it is strictly p-closed and supersolvable.

If otherwise A has isomorphic direct factors, in order to have $\mathrm{Aut}\,(G)$ supersolvable, we must require that K is supersolvable, so Theorem 5 implies $A = A_1 \times A_2$ with $A_1 \simeq A_2$ indecomposable factors. Now the conjugation γ_a (where a is the generator of C_2) is not in the Fitting subgroup of $\mathrm{Aut}\,(A\ wr\ C_2)$, otherwise it will be a central element. Therefore if we consider the dihedral subgroup $R := \langle \varphi_1 a_L, \varphi_2 \rangle$ (with the same construction used in the second step of Theorem 5), we have that $C_R(I) = 1$. Then $R \cap \mathrm{Fit}(\mathrm{Aut}\,(A\ wr\ C_2)) = 1$, and so $\frac{\mathrm{Aut}\,(A\ wr\ C_2)}{\mathrm{Fit}(\mathrm{Aut}\,(A\ wr\ C_2))}$ is not abelian and $\mathrm{Aut}\,(A\ wr\ C_2)$ not supersolvable. □

PROOF OF THEOREM 1. The key idea of the proof consists in observing that $\mathrm{Aut}\,(G)$ is a central product of the groups $\mathrm{Aut}\,(A_p\ wr\ C_2)$, where A_p is the p-component of A and p varies on the set of prime divisors $\pi(A)$ of $|A|$. Use induction on $|\pi(A)|$.

If $|\pi(A)| = 1$, then A is a prime power group and there is nothing to prove. Let $|\pi(A)| \geq 2$ and write $A = P \times Q$, where $P = O_p(A)$ and

$Q = O_{p'}(A)$ $(p \in \pi(A))$. Using the same notations as in the previous sections, we have again that F is characteristic in G and

$$\mathrm{Aut}\,(G) = I \cdot K.$$

Moreover $I = I_p \times I_{p'}$, where

$$I_p := \{\gamma_{(p_1,p_2)} \in \mathrm{Inn}(A\ wr\ C_2) \mid (p_1, p_2) \in P \times P\}$$

($I_{p'}$ is defined in a similar way), and $K := C_{\mathrm{Aut}\,(F)}(\delta)$. Since $F = A \times A$ is nilpotent $\mathrm{Aut}(F) = \mathrm{Aut}\,(P \times P) \times \mathrm{Aut}\,(Q \times Q)$ and we can write

$$K = K_p \times K_{p'}$$

where K_p is the subgroup of $\mathrm{Aut}(W)$ of all the elements which fix each element of $Q \times Q$ and commute with δ_p (δ_p is the involution of F that interchanges the elements of $P \times P$ and acts like the identity on the rest). In a similar way is defined $K_{p'}$. Finally we have that

$$[I_p, K_{p'}] = [I_{p'}, K_p] = 1.$$

So we obtain that $\mathrm{Aut}\,(G)$ is a central product of the groups $I_p \cdot K_p \simeq \mathrm{Aut}(P\ wr\ C_2)$ and $I_{p'} \cdot K_{p'} \simeq \mathrm{Aut}(Q\ wr\ C_2)$. The result follows from Theorem 6 and the inductive hypothesis. □

REFERENCES

[1] R. Baer, *Supersolvable immersion*, Canad. J. Math. **II** (1959), pp. 353–369.
[2] G. Baumslag, *Wreath products and p–groups*, Proc. Cambridge Philos. Soc. **55** (1959), pp. 224–231.
[3] C.H. Houghton, *On the automorphism group of certain wreath products*, Publ. Math. **9** (1962), pp. 453–460.
[4] M.V. Khoroshevskii, *On the automorphism group of finite p–groups*, Algebra i Logika **10** (1971), pp. 81–88.
[5] D.J. McCaughan - M.J. Curran, *Central automorphisms of finite groups*, Bull. Austral. Math. Soc. **34** (1986), pp. 191–198.
[6] P.M. Neumann, *On the structure of standard wreath products of groups*, Math. Z. **84** (1964), pp. 343–373.
[7] G. Corsi Tani - R. Brandl, *Wreath products with a supersolvable automorphism group*, Algebra Colloquium **5**, no. 2 (1998), pp. 135–142.
[8] G. Corsi Tani, *Su una congettura di J.R. Durbin e M. McDonald*, Rend. Acc. Naz. Lincei **LXIX** (1980), pp. 106–110.
[9] J.H. Ying, *On finite groups whose automorphism groups are nilpotent*, Arch. Math. **29** (1977), pp. 41–44.

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