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On the Semi-Simplicity of Galois Actions.

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Let K be a finitely generated field and X be a smooth projective variety over K; let G_K denote the absolute Galois group of K and l a prime number different from $\operatorname{char} K$. Then we have

CONJECTURE 1 (Grothendieck-Serre). *The action of G_K on the l-adic cohomology groups $H^*(\overline{X}, \boldsymbol{Q}_l)$ is semi-simple.*

There is a weaker version of this conjecture:

CONJECTURE 2 ($S^n(X)$). *For all $n \geqslant 0$, the action of G_K on the l-adic cohomology groups $H^{2n}(\overline{X}, \boldsymbol{Q}_l(n))$ is «semi-simple at the eigenvalue 1», i.e. the composite map*

$$H^{2n}(\overline{X}, \boldsymbol{Q}_l(n))^{G_K} \hookrightarrow H^{2n}(\overline{X}, \boldsymbol{Q}_l(n)) \twoheadrightarrow H^{2n}(\overline{X}, \boldsymbol{Q}_l(n))_{G_K}$$

is bijective.

If K is a finite field, then Conjecture 2 implies Conjecture 1. This is well-known and was written-up in [8] and [4], manuscript notes distributed at the 1991 Seattle conference on motives. Strangely, this is the only result of op. cit. that was not reproduced in [10]. We propose here a simpler proof than those in [8] and [4], which does not involve Jordan blocks, representations of SL_2 or the Lefschetz trace formula.

We also show that Conjecture 2 for K finite implies Conjecture 1 for any K of positive characteristic. The proof is exactly similar to that in [3, pp. 212-213], except that it relies on Deligne's geometric semi-simplicity theorem [2, cor. 3.4.13]; I am grateful to Yves André for explaining it to me. This gives a rather simple proof of Zarhin's semi-simplicity theorem

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for abelian varieties (see Remark 8.1). There is also a small result for K of characteristic 0 (see Remark 8.2). Besides, this paper does not claim much originality.

In order to justify later arguments we start with a well-known elementary lemma:

LEMMA 3. *Let E be a topological field of characteristic* 0 *and G a topological group acting continuously on some finite-dimensional E-vector space V. Suppose that the action of some open subgroup of finite index H is semi-simple. Then the action of G is semi-simple.*

PROOF. Let $W \subseteq V$ be a G-invariant subspace. By assumption, there is an H-invariant projector $e \in \mathrm{End}(V)$ with image W. Then

$$e' = \frac{1}{(G:H)} \sum_{g \in G/H} geg^{-1}$$

is a G-invariant projector with image W. ∎

LEMMA 4. *Let K be a field of characteristic* 0, *A a finite-dimensional semi-simple K-algebra and M an A-bimodule. Let $\mathfrak{A}$ be the Lie algebra associated to A, and let $\mathfrak{M}$ be the $\mathfrak{A}$-module associated to M ($ad(a)m = am - ma$). Then $\mathfrak{M}$ is semi-simple.*

PROOF. Since K has characteristic 0, $A \otimes_K A^{\mathrm{op}}$ is semi-simple. We may reduce to the case where K is algebraically closed by a trace argument, and then to M simple (as a left $A \otimes_K A^{\mathrm{op}}$-module). Write $A = \prod_i End_K(V_i)$; then $A \otimes_K A^{\mathrm{op}} = \prod_{i,j} End_K(V_i \otimes V_j^*)$ and M is isomorphic to one of the $V_i \otimes V_j^*$. We distinguish two cases:

a) $i = j$. We may assume $A = End(V)$ ($V = V_i$). Then $\mathfrak{A} = \mathfrak{gl}(V) = \mathfrak{sl}(V) \times K$, and $\mathfrak{sl}(V)$ is simple. By [9, th. 5.1], to see that $\mathfrak{M} = V \otimes V^*$ is semi-simple, it suffices to check that the action of $K = Cent(\mathfrak{A})$ can be diagonalised. But $a \in \mathfrak{A}$ acts by

$$ad(a)(v \otimes w) = a(v) \otimes w - v \otimes {}^t a(w)$$

and if a is a scalar, then $ad(a) = 0$.

b) $i \neq j$. We may assume $A = End(V) \times End(W)$ ($V = V_i$, $W = V_j$). This time, $\mathfrak{A} = \mathfrak{gl}(V) \times \mathfrak{gl}(W) = \mathfrak{sl}(V) \times \mathfrak{sl}(W) \times K \times K$. The action of $\mathfrak{A}$

onto $\mathfrak{M} = V \otimes W^*$ is given by the formula

$$ad(a, b)(v \otimes w) = a(v) \otimes w - v \otimes {}^t b(w).$$

Hence the centre acts by $ad(\lambda, \mu) = \lambda - \mu$ and the conditions of [9, th. 5.1] are again verified. ■

PROPOSITION 5. *Let V be a finite-dimensional vector space over a field K of characteristic 0. For u an endomorphism of V, denote by $ad(u)$ the endomorphism $v \mapsto uv - vu$ of $End_K(V)$. Let A be a K-subalgebra of $End_K(V)$ and B its commutant. Consider the following conditions*:

(i) *A is semi-simple.*

(ii) $End_K(V) = B \oplus \sum_{a \in A} ad(a)\, End_K(V)$.

(iii) *B is semi-simple.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *and* (iii) $\Rightarrow$ (i) *if A is commutative.*

PROOF. (i) $\Rightarrow$ (ii): let $\mathfrak{A}$ be the Lie algebra associated to A. By lemma 4, $End_K(V)$ is semi-simple for the adjoint action of $\mathfrak{A}$. Then (ii) follows from [1, §3, prop. 6].

(ii) $\Rightarrow$ (iii): let us show that the radical J of B is 0. Let $x \in J$. For $y \in B$, we have $xy \in J$; in particular, xy is nilpotent, hence $Tr(xy) = 0$. For $z \in End_K(V)$, and $u \in A$, we have

$$Tr(x(uz - zu)) = Tr(xuz - xzu) = Tr(uxz - xzu) = 0.$$

Hence $Tr(xy) = 0$ for all $y \in End_K(V)$, and $x = 0$.

(iii) $\Rightarrow$ (i) supposing A commutative: let us show this time that the radical R of A is 0. Suppose the contrary, and let $r > 1$ be minimal such that $R^r = 0$; let $I = R^{r-1}$. Then $I^2 = 0$. Let $W = IV$: then W is B-invariant, hence B acts on V/W. Let

$$N = \{v \in B \mid v(V) \subseteq W\}$$

be the kernel of this action: then N is a two-sided ideal of B and obviously $NI = IN = 0$. Let $v, v' \in N$ and $x \in V$. Then there exist $y \in V$ and $w \in I$ such that $v(x) = w(y)$. Hence

$$v'\, v(x) = v'\, w(y) = 0$$

and $N^2 = 0$. Since B is semi-simple, this implies $N = 0$. But, since A is commutative, $I \subseteq N$, a contradiction. ■

THEOREM 6. *Let X be a smooth, projective variety of dimension d over a field k of characteristic $\neq l$. Let k_s be a separable closure of k, $G = Gal(k_s/k)$ and $\overline{X} = X \times_k k_s$. Consider the following conditions*:

(i) *For all $i \geq 0$, the action of G on $H^i(X, \boldsymbol{Q}_l)$ is semi-simple.*

(ii) *$S^d(X \times X)$ holds.*

(iii) *The algebra $H^{2d}(\overline{X} \times_{k_s} \overline{X}, \boldsymbol{Q}_l(d))^G$ is semi-simple.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *and* (iii) $\Rightarrow$ (i) *if k is contained in the algebraic closure of a finite field.*

PROOF. By the Künneth formula and Poincaré duality, we have the well-known Galois-equivariant isomorphism of $\boldsymbol{Q}_l$-algebras

$$H^{2d}_{\mathrm{cont}}(\overline{X} \times \overline{X}, \boldsymbol{Q}_l(d)) \simeq \prod_{q=0}^{2d} End_{\boldsymbol{Q}_l}(H^q_{\mathrm{cont}}(\overline{X}, \boldsymbol{Q}_l)).$$

For $q \in [0, 2d]$, let A_q be the image of $\boldsymbol{Q}_l[G]$ in $End_{\boldsymbol{Q}_l}(H^q_{\mathrm{cont}}(\overline{X}, \boldsymbol{Q}_l))$. Then condition (i) (*resp.* (ii), (iii)) of theorem 6 is equivalent to condition (i) (*resp.* (ii), (iii)) of proposition 5 for all A_q. The conclusion follows by remarking that the A_q are commutative if k is contained in the algebraic closure of a finite field. ■

I don't know how to prove (iii) $\Rightarrow$ (i) in general in theorem 6, but in fact there is something better:

THEOREM 7. *Let F be a finitely generated field over $\boldsymbol{F}_p$ and let X be a smooth, projective variety of dimension d over F. Let $\mathcal{O}$ be a valuation ring of F with finite residue field, such that X has good reduction at $\mathcal{O}$. Let Y be the special fibre of a smooth projective model $\widetilde{X}$ of X over $\mathcal{O}$. Assume that $S^d(Y \times Y)$ holds. Then the Galois action on the $\boldsymbol{Q}_l$-adic cohomology of X is semi-simple.*

PROOF. For the proof we may assume that X is geometrically irreducible. By Lemma 3 we may also enlarge F by a finite extension and hence, by de Jong [5, Th. 4.1], assume that it admits a smooth projective model T over $\boldsymbol{F}_p$. By the valuative criterion for properness, $\mathcal{O}$ has a centre u on T with finite residue field k. Up to extending the field of constants of T to k, we may also assume that u is a rational point. Now spread X to a smooth, projective morphism

$$f : \mathcal{X} \to U$$

over an appropriate open neighbourhood U of u (in a way compatible to $\tilde{X}$).

The action of G_F on $H^*(\overline{X}, \boldsymbol{Q}_l)$ factors through $\pi_1(U)$. Moreover u yields a section σ of the homomorphism $\pi_1(U) \to \pi_1(\operatorname{Spec} k)$; in other terms, we have a split exact sequence of profinite groups

$$1 \to \pi_1(\overline{U}) \to \pi_1(U) \to \pi_1(\operatorname{Spec} k) \to 1.$$

Let $i \geq 0$, $V = H^i(\overline{X}, \boldsymbol{Q}_l)$, $\Gamma = GL(V)$ and $\varrho : \pi_1(U) \to \Gamma$ the monodromy representation. Denote respectively by A, B, C the Zariski closures of the images of $\pi_1(\overline{U})$ $\pi_1(U)$ and $\sigma(\pi_1(\operatorname{Spec} k))$. Then A is closed and normal in B, and $B = AC$.

By [2, cor. 3.4.13], $\pi_1(\overline{U})$ acts semi-simply on V; this is also true for $\sigma(\pi_1(\operatorname{Spec} k))$ by the smooth and proper base change theorem and Theorem 6 applied to Y. It follows that A and C act semi-simply on V; in particular they are reductive. But then B is reductive, hence its representation on V is semi-simple and so is that of $\pi_1(U)$. ■

REMARKS 8. 1. If X is an abelian variety, we recover a result of Zarhin [11, 12]. Theorem 7 applies more generally by just assuming that Y is of abelian type in the sense of [6], for example is an abelian variety or a Fermat hypersurface [7]. (Recall, e.g. [6, Lemma 1.9], that the proof of semi-simplicity for an abelian variety X over a finite field boils down to the fact that Frobenius is central in the semi-simple algebra $End(X) \otimes \boldsymbol{Q}$.)

2. If F is finitely generated over $\boldsymbol{Q}$, this argument gives the following (keeping the notation of Theorem 7). *Let F_0 be the field of constants of F. Assume that $S^d(Y \times Y)$ holds and that, moreover, the action of $Gal(\overline{K}/K^{ab})$ on the $\boldsymbol{Q}_l$-adic cohomology of Z is semi-simple, where Z is the special fibre of $\tilde{X} \otimes_{\mathcal{O}} F_0\mathcal{O}$ and K is the residue field of $F_0\mathcal{O}$. Then the conclusion of theorem 7 still holds.*

To see this, enlarge F as before so that it has a regular projective model $g : T \to \operatorname{Spec} A$ (where A is the ring of integers of F_0), this time by [5, Th. 8.2]. Let u be the centre of $\mathcal{O}$ on T and U an open neighbourhood of u, small enough so that X spreads to a smooth projective morphism $f : \mathcal{X} \to U$. Let $S = g(U)$ and $s = g(u)$. Up to extending F_0 and then shrinking S, we may assume that $g : U \to S$ has a section σ such that $u = \sigma(s)$, that $\mu_{2l} \subset \Gamma(S, \mathcal{O}_S^*)$ and that $\mu_{l^\infty}(\kappa(s)) = \mu_{l^\infty}(S)$.

Let S_∞ be a connected component of $S \otimes_{\boldsymbol{Z}} \boldsymbol{Z}[\mu_{l^\infty}]$ and $U_\infty = U \times_S S_\infty$.

We then have two short exact sequences

$$1 \to \pi_1(\overline{U}) \to \pi_1(U_\infty) \to \pi_1(S_\infty) \to 1$$

$$1 \to \pi_1(U_\infty) \to \pi_1(U) \xrightarrow{\chi} \boldsymbol{Z}_l^*$$

where χ is the cyclotomic character. The first sequence is split by σ; the second one is almost split in the sense that $\chi(\pi_1(u)) = \chi(\pi_1(U))$. By assumption, $\pi_1(S_\infty)$ acts semi-simply on the cohomology of the generic geometric fibre of Z and (using theorem 6) $\pi_1(u)$ acts semi-simply on the cohomology of Y. Arguing as in the proof of theorem 7, we then get that $\pi_1(U_\infty)$, and then $\pi_1(U)$, act semi-simply on $H^*_{\text{cont}}(\overline{X}, \boldsymbol{Q}_l)$. (To justify applying the smooth and proper base change theorem to Y, note that $gf : \mathcal{X} \to S$ is smooth at s by the good reduction assumption.)

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