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Longtime Behavior of Semilinear Reaction-Diffusion Equations on the Whole Space.

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ABSTRACT - We analyze a parabolic reaction-diffusion equation on the whole space. We prove existence, uniqueness, and continuous dependence results, and we investigate the longtime behavior of solutions. In particular, we show that the associated semigroup possesses a universal attractor. Specific difficulties arise here, due to the lack of compactness of the usual Sobolev embeddings in unbounded domains.

1. Introduction.

In this paper, we consider the following semilinear reaction-diffusion equation on $\mathbb{R}^3$:

$$\begin{aligned} &u_t - \Delta u + g(x, u) = f, \qquad (x, t) \in \mathbb{R}^3 \times (\tau, T], \\ &u = u(x, t), \\ &f = f(x, t), \\ &u(x, \tau) = u_0(x). \end{aligned} \tag{1.1}$$

In bounded domains, such equations have been widely investigated (see, e.g., [2, 7, 13] and the references therein), and sharp results concerning the longtime behavior of solutions have been proved, such as the existence of universal attractors of finite fractal dimension. In the whole space

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$\mathbb{R}^n$, problem (1.1) has been studied in weighted Hilbert spaces $L^2_\gamma(\mathbb{R}^n)$, $\gamma \in \mathbb{R}$, with norm

$$\|u\|_\gamma^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^\gamma u^2(x)\, dx \,.$$

Babin and Vishik [1] proved that for $\gamma > 0$, under some growth restrictions on the nonlinearity g, (1.1) possesses a universal attractor in $L^2_\gamma(\mathbb{R}^n)$. Incidentally, the growth restrictions on g seem unavoidable also for existence and uniqueness results. Indeed, nonuniqueness and nonexistence may occur in general (see [9, 14]). In [6] the same result is obtained for $\gamma > n/2$. Our scope is to analyze the case $\gamma = 0$, that is, absence of weight.

Existence of universal attractors of equations of parabolic type is usually proved showing the existence of a bounded absorbing set in a space which is compactly embedded in the space on which the semigroup acts. In bounded domains Ω, one typically finds an absorbing set in $H^1(\Omega)$, and exploits the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. In our case, we can still find an absorbing set in $H^1(\mathbb{R}^3)$, but the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is no longer compact. Thus, to get the desired result, we shall employ different techniques.

The plan of the paper is the following. In Section 2 we discuss the assumptions on the nonlinear term g. In Section 3 we prove existence, uniqueness, and continuous dependence results. In particular, we show that the solutions to (1.1) can be expressed by means of an evolution process. Section 4 is devoted to the existence of absorbing sets, which are uniform as the external source is allowed to vary in a suitable space. In Section 5 we show that, in the autonomous case, the semigroup associated to the problem possesses a universal attractor.

Finally, we mention that all the results of the paper are valid for systems on $\mathbb{R}^n$, that is, when $u = (u_1, \ldots, u_m)$, $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$ are m-dimensional vectors, and $x \in \mathbb{R}^n$. Indeed, all formulations and proofs can be easily generalized to this setting. Here, for brevity, we restrict our analysis to the scalar case on $\mathbb{R}^3$.

NOTATION. We set $H = L^2(\mathbb{R}^3)$, $V = H^1_0(\mathbb{R}^3) \equiv H^1(\mathbb{R}^3)$, and $V^* = H^{-1}(\mathbb{R}^3)$ (the dual space of V). As usual, we identify H with its dual space H^*. Given a space $\mathcal{X}$, we denote its inner product and its norm by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$, respectively. The symbol $\langle \cdot, \cdot \rangle$ denotes the duality map between V^* and V and between $L^p(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$. We also consider spaces of $\mathcal{X}$-valued functions defined on an interval I (possibly infinite), such as $C(I, \mathcal{X})$, $L^p(I, \mathcal{X})$, and $H^1(I, \mathcal{X})$, with the usual norms.

2. Conditions on the Nonlinear Term.

In order to solve problem (1.1) in a proper setting, we stipulate some growth and monotonicity conditions on the nonlinear term. Namely, we take $g = g(x, z) \in C(\mathbb{R}^4)$, with $g(x, \cdot) \in C^1(\mathbb{R})$ for every fixed $x \in \mathbb{R}^3$. Moreover, we assume that there exist two functions $\psi_1 \in H \cap C(\mathbb{R}^3)$, $\psi_2 \in L^1(\mathbb{R}^3)$, and positive constants c_1, c_2, c_3, and r_0, such that

(G1) $g(\cdot, 0) \in H$,

(G2) $\left| \dfrac{\partial g}{\partial z}(x, z) \right| \leq c_1(1 + |z|^{4/3}), \qquad \forall x \in \mathbb{R}^3,$

(G3) $\liminf\limits_{|z| \to \infty} \dfrac{g(x, z) - \psi_1(x)}{z} \geq 0, \quad$ uniformly as $x \in \mathbb{R}^3$,

(G4) $(g(x, z) - g(x, 0))\, z \geq c_2 z^2 - \psi_2(x), \qquad \forall z \in \mathbb{R},\ |x| \geq r_0,$

(G5) $\dfrac{\partial g}{\partial z}(x, z) \geq -c_3, \qquad \forall z \in \mathbb{R},\ \forall x \in \mathbb{R}^3.$

Notice that, upon replacing $f(x, t)$ with $f(x, t) - g(x, 0)$ in equation (1.1), and $\psi_1(x)$ with $\psi_1(x) - g(x, 0)$ in (G3), we may (and will) rewrite (G1) as

$$\text{(G1)} \quad g(\cdot, 0) = 0.$$

From (G1)-(G2) we see at once that

$$|g(x, z)| \leq c_1(|z| + |z|^{7/3}).$$

In force of the above inequality, if $u, w \in H \cap L^6(\mathbb{R}^3)$, the generalized Hölder inequality (with exponents 6, 3/2, 6) entails

$$|\langle g(\cdot, u), w\rangle| \leq c_1 \|u\|_H \|w\|_H + c_1 \|u\|_{L^6(\mathbb{R}^3)} \|u\|_H^{4/3} \|w\|_{L^6(\mathbb{R}^3)}.$$

Thus, if $u \in V$, due to the continuous embedding $V \hookrightarrow L^6(\mathbb{R}^3)$, and Young inequality, we conclude that

$$\|g(\cdot, u)\|_{V^*}^2 \leq c_4(1 + \|u\|_H^{8/3} + \|u\|_V^2) \tag{2.1}$$

for some $c_4 > 0$.

The next lemma will be crucial for the energy estimates.

LEMMA 2.1. *Assume* (G1), (G3)-(G4). *Then, for every* $\nu > 0$ *there are* $c(\nu) \geq 0$ *and* $\varrho(\nu) > 0$ *such that*

$$\langle g(\cdot, u), u \rangle - \varrho(\nu) \|u\|_H^2 \geq -\nu \|\nabla u\|_{H^3}^2 - c(\nu) \tag{2.2}$$

for every $u \in V$.

PROOF. Let $\nu > 0$. From (G4),

$$g(x, z)\, z \geq c_2 z^2 - \psi_2(x) \geq \min\{\nu, c_2\}\, z^2 - |\psi_2(x)|. \tag{2.3}$$

By the use of the Poincaré-Wirtinger inequality, it is easy to prove that there exists $c_5 > 0$ (depending on r_0) such that

$$\int_{|x| > r_0} w^2(x)\, dx \geq c_5 \|w\|_H^2 - \|\nabla w\|_{H^3}^2, \qquad \forall w \in V. \tag{2.4}$$

On setting $z = u(x)$ and $\varrho(\nu) = c_5 \min\{\nu/2, c_2/2\}$, integration of (2.3) over $|x| > r_0$ and (2.4) bear

$$\int_{|x| > r_0} g(x, u(x))\, u(x)\, dx \geq -\nu \|\nabla u\|_{H^3}^2 + 2\varrho(\nu) \|u\|_H^2 - \|\psi_2\|_{L^1(\mathbb{R}^3)}. \tag{2.5}$$

When $|x| \leq r_0$, using (G3), and the continuity of g, we find $c_6 > 0$ such that

$$g(x, z)\, z \geq -\frac{\varrho(\nu)}{2} z^2 + \psi_1(x)\, z - c_6. \tag{2.6}$$

Integration of (2.6) with $z = u(x)$ over $|x| \leq r_0$, and the Young inequality, give

$$\int_{|x| \leq r_0} g(x, u(x))\, u(x)\, dx \geq -\frac{\varrho(\nu)}{2} \|u\|_H^2 - |\langle \psi_1, u \rangle_H| - \frac{4}{3} \pi c_6 r_0^3 \geq \tag{2.7}$$

$$\geq -\varrho(\nu) \|u\|_H^2 - \frac{1}{2\varrho(\nu)} \|\psi_1\|_H^2 - \frac{4}{3} \pi c_6 r_0^3.$$

Addition of (2.5) and (2.7) implies (2.2). ∎

3. Existence and Uniqueness.

Firstly, we give the definition of a weak solution to problem (1.1).

DEFINITION 3.1. Set $I=[\tau, T]$, for $\tau \in \mathbb{R}$ and $T>\tau$. Let $u_0 \in H$, and

$$f \in L^1(I, H) + L^2(I, V^*).$$

A function

$$u \in L^2(I, V) \cap C(I, H) \cap H^1(I, V^*)$$

is a weak solution to problem (1.1) in the time interval I provided that

$$u(\tau) = u_0$$

and

$$\langle u_t, w\rangle + \langle \nabla u, \nabla w\rangle_H + \langle g(\cdot, u), w\rangle = \langle f, w\rangle$$

for all $w \in V$, and a.e. $t \in I$.

THEOREM 3.2. *Assume* (G1)-(G5). *Then, given any time interval* $I=[\tau, T]$, *problem* (1.1) *has a weak solution* u *on* I.

PROOF. We adopt a Faedo-Galerkin procedure. Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of H of compactly supported, sufficiently regular functions. For any integer n, we consider the subspace

$$H_n = \operatorname{Span}\{w_1, \ldots, w_n\} \subset V.$$

It is convenient to approximate f with a sequence $\{f_n\}_{n=1}^{\infty} \subset C^0(I, H)$ such that

$$f_n = f_n^1 + f_n^2$$

with

$$f_n^1 \to f^1 \text{ in } L^1(I, H) \quad \text{and} \quad f_n^2 \to f^2 \text{ in } L^2(I, V^*) \tag{3.1}$$

where (f^1, f^2) is some fixed decomposition of f, that is $f=f^1+f^2$. We now look for $t_n \in (\tau, T]$, and functions $a_j^n \in C^1([\tau, t_n])$, for $j=1, \ldots, n$, such that the function

$$u_n(t) = \sum_{j=1}^{n} a_j^n(t)\, w_j$$

fulfills the system

(3.2) $\langle \partial_t u_n, w_j \rangle_H + \langle \nabla u_n, \nabla w_j \rangle_{H^3} + \langle g(\cdot, u_n), w_j \rangle_H = \langle f_n^1 + f_n^2, w_j \rangle_H,$

for $t \in (\tau, t_n]$, with initial conditions

(3.3) $$\langle u_n(\tau), w_j \rangle_H = \langle u_0, w_j \rangle_H,$$

where $j = 1, \ldots, n$. System (3.2) can be easily put in normal form. Due to the regularity in space of u_n, an application of a standard fixed-point argument implies that (3.2) has a (unique) solution with t_n small enough.

The second step is to find uniform energy estimates for the u_n. Multiplying (3.2) times $a_j^n(t)$, and summing over $j = 1, \ldots, n$, we get

(3.4) $$\frac{1}{2}\frac{d}{dt}\|u_n\|_H^2 + \|\nabla u_n\|_{H^3}^2 + \langle g(\cdot, u_n), u_n \rangle_H = \langle f_n^1, u_n \rangle_H + \langle f_n^2, u_n \rangle_H .$$

By virtue of Lemma 2.1 and Young inequality, it is possible to find $\varepsilon > 0$ small enough and $c_7 > 0$ such that the inequality

(3.5) $$\frac{d}{dt}\|u_n\|_H^2 + \varepsilon\|u_n\|_H^2 + \varepsilon\|\nabla u_n\|_{H^3}^2 \leq c_7(1 + \|f_n^1\|_H \|u_n\|_H + \|f_n^2\|_{V^*}^2)$$

holds for a.e. $t \in (\tau, t_n)$. In force of (3.1) there is $c_8 > 0$ such that

$$\|f_n^1\|_{L^1(I, H)}^2 + \|f_n^2\|_{L^2(I, V^*)}^2 \leq c_8\|f^1\|_{L^1(I, H)}^2 + c_8\|f^2\|_{L^2(I, V^*)}^2, \qquad \forall n .$$

Thus, observing that $\|u_n(\tau)\|_H \leq \|u_0\|_H$, Gronwall lemma applied to (3.5), and (3.3), lead to

$$\|u_n(t)\|_H^2 \leq 2\|u_0\|_H + 2c_7(1 + c_8\|f^1\|_{L^1(I, H)}^2 + c_8\|f^2\|_{L^2(I, V^*)}^2)$$

for every $t \in (\tau, t_n)$. Being the above estimate independent of n (and t_n), we conclude that $t_n = T$. Thus

(3.6) u_n is uniformly bounded in $L^\infty(I, H)$.

From an integration in time of (3.5), with the aid of (3.6), we also learn that

(3.7) u_n is uniformly bounded in $L^2(I, V)$.

Finally, from (2.1), we get that

(3.8) $g(\cdot, u_n)$ is uniformly bounded in $L^2(I, V^*)$.

Using the uniform bounds (3.6)-(3.8), up to a subsequence, we get the convergences

(3.9) $u_n \to u$ weakly star in $L^\infty(I, H)$, weakly in $L^2(I, V)$,

(3.10) $g(\cdot, u_n) \to \chi$ weakly in $L^2(I, V^*)$.

We now fix an integer m, and we take $w \in H_m$. In force of (3.2), for $n \geqslant m$, we have

$$\frac{d}{dt}\langle u_n, w\rangle_H + \langle \nabla u_n, \nabla w\rangle_{H^3} + \langle g(\cdot, u_n), w\rangle_H = \langle f_n, w\rangle_H, \tag{3.11}$$

Due to (3.1), (3.9)-(3.10), we can pass to the limit in (3.11), and we find that

$$\frac{d}{dt}\langle u, w\rangle_H + \langle \nabla u, \nabla w\rangle_{H^3} + \langle \chi, w\rangle = \langle f, w\rangle, \qquad \forall w \in H_m, \tag{3.12}$$

in the distribution sense on (τ, T). Since H_m is dense in V, (3.12) is true in fact for every $w \in V$, and we conclude that the equality

$$u_t - \Delta u + \chi = f \tag{3.13}$$

holds in the weak sense. Moreover, by comparison in (3.13), we see that $u_t \in L^2(I, V^*) + L^1(I, H)$, which, together with (3.9), entails $u \in C(I, H)$. By a standard argument (see, e.g., [4]), we get that $u(0) = u_0$. We are left to show the equality $\chi = g(\cdot, u)$. To this aim, we apply the method of Minty-Browder (see, e.g., [10]). Define

$$\tilde{g}(x, z) = g(x, z) + c_3 z .$$

Notice that, due to (G.5),

$$\langle \tilde{g}(\cdot, w) - \tilde{g}(\cdot, v), w - v\rangle \geqslant 0, \qquad \forall w, v \in V. \tag{3.14}$$

Then, we integrate from 0 to $t \in (\tau, T]$ equation (3.4), and we take the limit for $n \to \infty$. Exploiting convergences (3.9)-(3.10), we have

$$\frac{1}{2}\|u(t)\|_H^2 + \int_0^t \|\nabla u(\vartheta)\|_{H^3}^2 d\vartheta + \liminf_{n\to\infty} \int_0^t \langle g(\cdot, u_n(\vartheta)), u_n(\vartheta)\rangle \, d\vartheta \leqslant \tag{3.15}$$
$$\leqslant \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle f, u(\vartheta)\rangle \, d\vartheta$$

for a.e. $t \in I$. Then we take the duality product of (3.13) with u, and integrate from 0 to t, to get

$$\frac{1}{2}\|u(t)\|_H^2 + \int_0^t \|\nabla u(\vartheta)\|_{H^3}^2 \, d\vartheta + \int_0^t \langle \chi, u(\vartheta)\rangle_H \, d\vartheta = \tag{3.16}$$

$$= \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle f, u(\vartheta)\rangle \, d\vartheta .$$

Substitution of (3.16) into (3.15) yields

$$\liminf_{n\to\infty} \int_0^t \langle g(\cdot, u_n(\vartheta)), u_n(\vartheta)\rangle \, d\vartheta \leqslant \int_0^t \langle \chi, u(\vartheta)\rangle \, d\vartheta \tag{3.17}$$

for a.e. $t \in I$. Hence, from (3.9)-(3.10), (3.14) and (3.17),

$$\int_0^t \langle \chi + c_3 u - \tilde{g}(\cdot, w(\vartheta)), u(\vartheta) - w(\vartheta)\rangle \, d\vartheta \geqslant \tag{3.18}$$

$$\geqslant \liminf_{n\to\infty} \left(\int_0^t \langle g(\cdot, u_n(\vartheta)) - g(\cdot, w(\vartheta)), u_n(\vartheta) - w(\vartheta)\rangle \, d\vartheta + \right.$$

$$\left. + \int_0^t \langle c_3 u(\vartheta) - c_3 w(\vartheta), u_n(\vartheta) - w(\vartheta)\rangle \, d\vartheta \right) \geqslant 0, \qquad \forall w \in L^2(I, V)$$

for a.e. $t \in I$. Choosing a sequence $t_n \to T$, for which (3.18) holds, we conclude that

$$\int_0^t \langle \chi + c_3 u - \tilde{g}(\cdot, w(\vartheta)), u(\vartheta) - w(\vartheta)\rangle \, d\vartheta \geqslant 0, \qquad \forall w \in L^2(I, V). \tag{3.19}$$

Appealing now to a standard argument (that is, setting $w = u - \lambda v$, with $v \in V$, in (3.19), and letting $\lambda \to 0$), we get $\chi + c_3 u = \tilde{g}(\cdot, u)$, which implies the desired equality $\chi = g(\cdot, u)$. ∎

THEOREM 3.3. *Assume* (G1)-(G5), *and let* $\{u_{0i}, f_i\}_{i=1,2}$, *with* $u_{0i} \in H$ *and* $f_i \in L^1(I, H) + L^2(I, V^*)$, *be two sets of data, and denote by* u_i *two corresponding solutions to problem* (1.1) *on* I. *Then the following esti-*

mate holds:

$$\|u_1(t) - u_2(t)\|_H^2 \leq C(\|u_{01} - u_{02}\|_H^2 + \|f_1 - f_2\|_{L^1(I,\,H) + L^2(I,\,V^*)}^2), \qquad \forall t \in I\,,$$

for some constant $C = C(T - \tau) > 0$. *In particular, problem* (1.1) *has a unique solution.*

PROOF. The variable $u = u_1 - u_2$ fulfills the equation

$$u_t - \Delta u + g(\cdot,\, u_1) - g(\cdot,\, u_2) = f_1 - f_2 \tag{3.20}$$

with initial value

$$u(\tau) = u_{01} - u_{02}.$$

The thesis easily follows taking the product of (3.20) with u, using (G5), and applying Gronwall lemma. ■

REMARK 3.4. Let $\{U_f(t,\, \tau),\, t \geq \tau,\, \tau \in \mathbb{R}\}$ be the two-parameter family of operators on H which associates to u_0 the solution to (1.1) at time t with initial data u_0 given at time τ. In virtue of Theorem 3.2 and Theorem 3.3, $U_f(t,\, \tau)$ is a strongly continuous process of continuous (nonlinear) operators on H with functional symbol f, according to the usual definition (see, e.g., [8], Chapter 6). When f is independent of time, $\{U_f(t,\, 0),\, t \geq 0\}$ is a strongly continuous semigroup of continuous operators on H.

4. Uniform absorbing sets.

To describe the asymptotic behavior of the solutions to (1.1), we need to introduce the Banach space $\mathcal{C}^1$ of $L^1_{\rm loc}$-*translation bounded* functions with values in H, namely,

$$\mathcal{C}^1 = \left\{ f \in L^1_{\rm loc}(\mathbb{R},\, H) \colon \|f\|_{\mathcal{C}^1} = \sup_{r \in \mathbb{R}} \int_r^{r+1} \|f(y)\|_H \, dy < \infty \right\}.$$

Similarly, we define the Banach space $\mathcal{C}^2$ of $L^2_{\rm loc}$-translation bounded functions with values in V^*, that is,

$$\mathcal{C}^2 = \left\{ f \in L^2_{\rm loc}(\mathbb{R},\, V^*) \colon \|f\|_{\mathcal{C}^2}^2 = \sup_{r \in \mathbb{R}} \int_r^{r+1} \|f(y)\|_{V^*}^2 \, dy < \infty \right\}.$$

THEOREM 4.1. *Assume* (G1)-(G5), *and let* $f \in \mathcal{C}^1 + \mathcal{C}^2$. *Then there are* $\varepsilon > 0$ *and a bounded increasing function* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$(4.1) \qquad \|U_f(t, \tau)\, u_0\|_H \leq 2\|u_0\|_H e^{-\varepsilon(t-\tau)} + \Phi(\|f\|_{\mathcal{C}^1 + \mathcal{C}^2}), \qquad \forall t \geq \tau .$$

PROOF. Let $f = f^1 + f^2$ be a fixed decomposition of f such that $\|f^i\|_{\mathcal{C}^i} \leq 2\|f\|_{\mathcal{C}^1 + \mathcal{C}^2}$. It is apparent that inequality (3.5) holds for u, f^1, f^2 as well, namely,

$$(4.2) \qquad \frac{d}{dt}\|u\|_H^2 + \varepsilon\|u\|_H^2 + \varepsilon\|\nabla u\|_{H^3}^2 \leq c_7(1 + \|f^1\|_H \|u\|_H + \|f^2\|_{V^*}^2).$$

A generalized form of Gronwall lemma (see, e.g., Lemma A.3 in [3] and [12]), implies that

$$\|u(t)\|_H^2 \leq 2\|u_0\|_H^2 e^{-\varepsilon(t-\tau)} + \frac{2c_7}{\varepsilon} + \frac{c_7^2 e^{\varepsilon}}{(1 - e^{-\varepsilon/2})^2}\|f^1\|_{\mathcal{C}^1}^2 + \frac{2c_7 e^{\varepsilon}}{1 - e^{-\varepsilon}}\|f^2\|_{\mathcal{C}^2}^2,$$

which gives (4.1). ■

We also have an L^2-control of the gradient norm.

PROPOSITION 4.2. *Assume* (G1)-(G5), *and let* $f \in \mathcal{C}^1 + \mathcal{C}^2$. *Then, there is a bounded function* $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, *increasing in both variables, such that the inequality*

$$(4.3) \qquad \int_t^{t+1} \|\nabla U_f(\vartheta, \tau)\, u_0\|^2 \, d\vartheta \leq \Psi(\|u_0\|_H, \|f\|_{\mathcal{C}^1 + \mathcal{C}^2})$$

holds for every $t \geq \tau$.

PROOF. Integrate (4.2) from t to $t+1$, and use (4.1). ■

As an immediate consequence of Theorem 4.1 we have

COROLLARY 4.3. *Let* (G1)-(G5) *hold, and let* $\mathcal{F} \subset \mathcal{C}^1 + \mathcal{C}^2$ *be a bounded set. Then there is a bounded absorbing set in* H *for the family of processes* $\{U_f(t, \tau), f \in \mathcal{F}\}$, *which is uniform as* $f \in \mathcal{F}$.

PROOF. Denote $M = \sup_{h \in \mathcal{F}} \|h\|_{\mathcal{C}^1 + \mathcal{C}^2}$, and let $\mathcal{B}$ be the ball of H centered at 0 and of radius $2\Phi(M)$. It is then clear from (4.1) that, given any

$R > 0$, there exists $t_0 \geqslant 0$ big enough such that

$$\bigcup_{\|w\|_H \leqslant R} \bigcup_{f \in \mathcal{F}} \bigcup_{\tau \in \mathbb{R}} \bigcup_{t \geqslant t_0 + \tau} U_f(t, \tau)\, w \subset \mathcal{B}$$

as desired. ■

In the sequel, we take $f \in H$ independent of time (so, in particular, $f \in \mathcal{C}^1$), and we set

$$S(t) = U_f(t, 0).$$

REMARK 4.4. It is straightforward to check that

$$\mathcal{B}_0 = \bigcup_{t \geqslant 0} S(t)\, \mathcal{B},$$

with $\mathcal{B}$ given by Corollary 4.3, is a connected, bounded absorbing set in H for the semigroup $S(t)$ which is invariant for $S(t)$ (that is, $S(t)\, \mathcal{B}_0 \subset \mathcal{B}_0$ for every $t \geqslant 0$).

Our next step is to prove the existence of an absorbing set for $S(t)$ in V. To do so, we have to ask a further condition on the nonlinear term:

$$\text{(G6)} \qquad \begin{aligned} &\text{(i)} \quad |g(x, z)| \leqslant c_9\, |z|(1 + |z|), \quad \forall x \in \mathbb{R}^3, \\ &\qquad \text{or} \\ &\text{(ii)} \quad \|D_x g(\cdot, u)\|_{(V^*)^3} \leqslant c_{10}(1 + \|u\|_V^2), \end{aligned}$$

for some $c_9, c_{10} > 0$, where D_x denotes differentiation with respect to the first three variables of g. It should be noticed that condition (ii) is trivially satisfied when g does not depend on x explicitly, or if $|D_x g(x, z)|$ fulfills an inequality like (i).

We shall also make use of the uniform Gronwall lemma (cf. [13]), which we quote below for reader's convenience.

LEMMA 4.5. *Let φ, m_1, m_2, be three non-negative locally summable functions on $\mathbb{R}^+$ satisfying*

$$\frac{d}{dt}\varphi(t) \leqslant m_1(t)\, \varphi(t) + m_2(t), \qquad \textit{for a.e.} \quad t \in \mathbb{R}^+,$$

and such that

$$\int_t^{t+1} m_j(s)\, ds \leqslant a_j \quad \text{and} \quad \int_t^{t+1} \varphi(s)\, ds \leqslant a_3$$

$(j = 1, 2)$ *for some positive constants* a_1, a_2, a_3. *Then,*

$$\varphi(t+1) \leqslant (a_2 + a_3)\, e^{a_1}, \qquad \forall\, t \in \mathbb{R}^+ .$$

THEOREM 4.6. *Assume* (G1)-(G6), *and let* $f \in H$, *independent of time. Then, there is a bounded set* $\mathcal{B}_1 \subset V$ *such that* $S(t)\,\mathcal{B}_0 \subset \mathcal{B}_1$ *for any* $t \geqslant 1$.

PROOF. Let $u_0 \in \mathcal{B}_0$, which implies $S(t)\, u_0 \in \mathcal{B}_0$ for every $t \geqslant 0$. Take the inner product in H of (1.1) and $-\Delta u$, and apply the Young inequality to the term $\langle f, \Delta u \rangle_H$, to get

$$\frac{d}{dt} \|\nabla u\|^2_{H^3} + \|\Delta u\|^2_H \leqslant \|f\|^2_H - 2\langle g(\cdot, u), \Delta u \rangle_H . \tag{4.4}$$

If (i) of (G6) holds, using the uniform boundedness of u in H as u_0 runs in $\mathcal{B}_0$ and the embedding $V \hookrightarrow L^4(\mathbb{R}^3)$, we have that

$$-2\langle g(\cdot, u), \Delta u \rangle_H \leqslant \|g(\cdot, u)\|^2_H + \|\Delta u\|^2_H \leqslant c_{11}(1 + \|\nabla u\|^4_{H^3}) + \|\Delta u\|^2_H, \tag{4.5}$$

for some $c_{11} > 0$. If (ii) of (G6) holds, we have that

$$-2\langle g(\cdot, u), \Delta u \rangle_H = -2\left\langle \frac{\partial g}{\partial z}(\cdot, u)\, \nabla u, \nabla u \right\rangle_{H^3} - 2\langle D_x g(\cdot, u), \nabla u \rangle_{H^3} .$$

Using (G5), we see that

$$-2\left\langle \frac{\partial g}{\partial z}(\cdot, u)\, \nabla u, \nabla u \right\rangle_{H^3} \leqslant 2 c_3 \|\nabla u\|^2_{H^3} . \tag{4.6}$$

Observe now that, since $u \in \mathcal{B}_0$, the inequality

$$\|\nabla u\|^2_{V^3} \leqslant c_{12}(1 + \|\Delta u\|^2_H)$$

holds for some $c_{12} > 0$. Hence, (ii) of (G6) and the Young inequality entail that

$$\begin{aligned} -2\langle D_x g(\cdot, u), \nabla u \rangle_{H^3} &\leqslant 2 \|D_x g(\cdot, u)\|_{(V^*)^3} \|\nabla u\|_{V^3} \\ &\leqslant c_{10}^2 c_{12} (1 + \|u\|^2_V)^2 + 1 + \|\Delta u\|^2_H . \end{aligned} \tag{4.7}$$

Adding (4.6) and (4.7), we discover that (4.5) holds in this case as well (upon redefining c_{11}). Thus, (4.4) reads

$$\frac{d}{dt}\|\nabla u\|_{H^3}^2 \leq c_{11}\|\nabla u\|_{H^3}^2\|\nabla u\|_{H^3}^2 + c_{11} + \|f\|_H^2.$$

From Proposition 4.2 and Lemma 4.5, with $m_1 = c_{11}\|\nabla u\|_{H^3}^2$ and $m_2 = c_{11} + \|f\|_H^2$, we conclude that there exists $c_{13} > 0$ such that

$$\|\nabla u(t)\|_{H^3}^2 \leq c_{13}, \qquad \forall t \geq 1.$$

The above inequality, and Remark 4.4, yield the thesis. ∎

5. The universal attractor.

In the sequel, we assume (G1)-(G6). Moreover, we take

$$f \in H \text{ independent of time}.$$

The aim of this last section is to show that the semigroup $S(t)$ associated to problem (1.1) has a universal attractor.

We first recall the classical definition of universal attractor; more details can be found in the books [2, 7, 13].

DEFINITION 5.1. A compact set $\mathcal{A} \subset H$ is said to be the *universal attractor* for the semigroup $S(t)$ if it enjoys the following properties:

(i) $\mathcal{A}$ is fully invariant for $S(t)$, that is, $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$;

(ii) $\mathcal{A}$ is an attracting set, namely, for any bounded set $\mathcal{B} \subset H$,

$$\lim_{t \to \infty} \operatorname{dist}(S(t)\mathcal{B}, \mathcal{A}) = 0$$

(where «dist» denotes the semidistance). We also recall the *Kuratowski measure of noncompactness* α (cf. [7]). Given a set $\mathcal{B} \subset H$, $\alpha(\mathcal{B})$ is defined by

$$\alpha(\mathcal{B}) = \inf\{d : \mathcal{B} \text{ has a finite cover of balls of } H, \text{ of diameter less than } d\}.$$

It is clear that $\mathcal{B}$ is relatively compact if and only if $\alpha(\mathcal{B}) = 0$.

We shall exploit the following theorem (cf. [7] and [11], Appendix).

THEOREM 5.2. *Assume the following hypotheses:*

(i) *there is an invariant, connected, bounded absorbing set* $\mathcal{B}_0 \subset H$;

(ii) *there is a sequence* $t_j \geq 0$ *such that* $\lim_{j \to \infty} \alpha(S(t_j)\, \mathcal{B}_0) = 0$.

Then, the ω*-limit set of* $\mathcal{B}_0$*, namely,* $\omega(\mathcal{B}_0) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)\, \mathcal{B}_0}$*, is the (connected) universal attractor of* $S(t)$.

In virtue of Remark 4.4, we already know that condition (i) of Theorem 5.2 is fulfilled. Thus we are left with proving that condition (ii) holds true. To this aim, we perform a decomposition of the solution to (1.1). For any $r \geq r_0$, we introduce two smooth positive functions $\varphi_i^r : \mathbb{R}^3 \to \mathbb{R}^+$, $i = 1, 2$, such that

$$\varphi_1^r(x) + \varphi_2^r(x) = 1\,, \qquad \forall x \in \mathbb{R}^3,$$

and

$$\varphi_1^r(x) = 0\,, \qquad \text{if } |x| \leq r\,,$$

$$\varphi_2^r(x) = 0\,, \qquad \text{if } |x| \geq r+1\,.$$

Since $g(x, z) - \psi_1(x)$ is continuous on $\mathbb{R}^4$, there is a constant $\nu_r \in (0, 1]$ such that

$$|g(x, z) - \psi_1(x)|\,|z| \leq \left(\frac{3}{8\pi(r+1)^3}\right)\frac{1}{r} \tag{5.1}$$

whenever $|z| \leq \nu_r$, and $|x| \leq r+1$. Moreover, from (G3), there is $c_r > 0$ such that

$$\frac{g(x, z) - \psi_1(x)}{z} \geq -c_r \tag{5.2}$$

as $|z| \geq \nu_r$ for every $x \in \mathbb{R}^3$. Decompose $-g(x, z) + f(x)$ as

$$-g(x, z) + f(x) = -g_1(x, z) - g_2(x, z) + f_1(x) + f_2(x)\,,$$

with

$$g_1(x, z) = g(x, z)\, \varphi_1^r(x) + (g(x, z) - \psi_1(x) + c_2 z + c_r z)\, \varphi_2^r(x)\,,$$

$$g_2(x, z) = -(c_2 z + c_r z)\, \varphi_2^r(x)\,,$$

and

$$f_1(x, z) = f(x)\, \varphi_1^r(x),$$
$$f_2(x, z) = (f(x) - \psi_1(x))\, \varphi_2^r(x).$$

The dependence on r of g_i and f_i is omitted for simplicity of notation. Notice that g_1 still fulfills (G1)-(G3) (with c_1 replaced by $c_1 + c_2 + c_r$). Condition (G4) and (5.1)-(5.2) entail

$$g_1(x, z)\, z \geq c_2 z^2 - \varphi_2^r(x) \left(\frac{3}{8\pi(r+1)^3} \right) \frac{1}{r} - \varphi_1^r(x) \psi_2(x), \tag{5.3}$$

for all $x \in \mathbb{R}^3$ and $z \in \mathbb{R}$. Concerning the other functions, we have

$$f_2(x) = g_2(x, z) = 0, \qquad \text{as } |x| \geq r + 1, \tag{5.4}$$

for all $z \in \mathbb{R}$. Moreover, it is clear that

$$\lim_{r \to \infty} \|f_1\|_H = 0, \tag{5.5}$$

$$\lim_{r \to \infty} \|\varphi_1^r \psi_2\|_{L^1(\mathbb{R}^3)} = 0. \tag{5.6}$$

At this point, following [11] (see also [5]), we decompose any solution to our system, with initial data $u_0 \in \mathcal{B}_0$ as the sum

$$u = v_r + w_r$$

where

$$\partial_t v_r = \Delta v_r - g_1(\cdot, v_r) + f_1, \tag{5.7}$$

with

$$v_r(0) = u_0, \tag{5.8}$$

and

$$\partial_t w_r = \Delta w_r - g_1(\cdot, u) + g_1(\cdot, v_r) - g_2(\cdot, u) + f_2, \tag{5.9}$$

with

$$w_r(0) = 0. \tag{5.10}$$

LEMMA 5.3. *Given any $\omega > 0$ there exist $t_\omega \geq 1$ (independent of r) and $r_\omega \geq r_0$ such that the solution v_r to (5.7)-(5.8) fulfills the estimate*

$$\|v_r(t_\omega)\|_H \leq \omega ,$$

for every $r \geq r_\omega$ and $u_0 \in \mathcal{B}_0$.

PROOF. Taking the inner product in H of the equation (5.7) and v_r, and using (5.3) and Young inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|v_r\|_H^2 + c_2\|v_r\|_H^2 \leq h(r),$$

having set

$$h(r) = \frac{1}{r} + \frac{1}{c_2}\|f_1\|_H^2 + \|\varphi_1^r \psi_2\|_{L^1(\mathbb{R}^3)}.$$

The Gronwall lemma and (5.8) lead to the inequality

$$\|v_r(t)\|_H^2 \leq e^{-c_2 t}\|u_0\|_H^2 + \frac{1}{c_2}h(r). \tag{5.11}$$

From (5.5)-(5.6) we see that $h(r) \to 0$ as $r \to \infty$. On recalling that $\sup_{w \in B_0} \|w\|_H^2 < \infty$, for any $\omega > 0$ it is immediate to find a couple t_ω, r_ω which fulfills the thesis. ■

For $r > 0$, we denote $B_r = \{x \in \mathbb{R}^3 : |x| \leq r\}$ and $B_r^C = \mathbb{R}^3 \setminus B_r$.

LEMMA 5.4. *Given any $\omega > 0$, there exist r_1 and r_2, with $r_1 \geq r_\omega$ and $r_2 = 2r_1 + 2$, such that the solution w_{r_1} to (5.9)-(5.10) fulfills the estimate*

$$\|w_{r_1}(t_\omega)\|_{L^2(B_{r_2}^C)} \leq \omega ,$$

for every $u_0 \in \mathcal{B}_0$, with t_ω given as in the previous lemma.

PROOF. Given $r \geq r_\omega$, let $\varrho : \mathbb{R}^3 \to [0, 1]$ be defined as

$$\varrho(x) = \begin{cases} 0, & |x| < r+1, \\ \dfrac{1}{r+1}(|x| - r - 1), & r+1 \leq |x| \leq 2r+2, \\ 1, & |x| > 2r+2. \end{cases}$$

We have the estimate

$$|\nabla \varrho^2(x)| \leqslant \frac{2}{r+1}\varrho(x), \qquad \forall x \in \mathbb{R}^3. \tag{5.12}$$

Take the inner product in H of (5.9) and $\varrho^2 w_r$, to get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \varrho^2 w_r^2\,dx = \int_{\mathbb{R}^3} \varrho^2 w_r \Delta w_r\,dx + \int_{\mathbb{R}^3} \varrho^2 (f_2 - g_2(\cdot, u))\,w_r\,dx - \int_{\mathbb{R}^3} \varrho^2 (g_1(\cdot, u) - g_1(\cdot, v_r))\,w_r\,dx\,. \tag{5.13}$$

From Remark 4.4 and (5.11), there exists $c_{14} > 0$ (independent of r and $u_0 \in \mathcal{B}_0$) such that $\|w_r\|_H^2 \leqslant c_{14}$. Hence, in virtue of the Young inequality and (5.12),

$$\int_{\mathbb{R}^3} \varrho^2 w_r \Delta w_r\,dx \leqslant -\int_{\mathbb{R}^3} \varrho^2 |\nabla w_r|^2\,dx + \frac{2}{r+1}\int_{\mathbb{R}^3} \varrho\,|w_r|\,|\nabla w_r|\,dx \leqslant \frac{c_{14}}{r+1}\,. \tag{5.14}$$

Due to (5.4),

$$\int_{\mathbb{R}^3} \varrho^2 (f_2 - g_2(\cdot, u))\,w_r\,dx = 0\,. \tag{5.15}$$

Finally, using (G5), we have that

$$\varrho^2 (g_1(\cdot,u) - g_1(\cdot, v_r))\,w_r \geqslant -c_3 \varrho^2 w_r^2 + \varrho^2 \varphi_2^r (c_2 + c_r)\,w_r^2 = -c_3 \varrho^2 w_r^2\,, \tag{5.16}$$

since $\varrho^2(x)\,\varphi_2^r(x) = 0$ for all $x \in \mathbb{R}^3$. On collecting (5.14)-(5.16), equality (5.13) transforms into

$$\frac{d}{dt}\int_{\mathbb{R}^3} \varrho^2 w_r^2\,dx \leqslant 2c_3 \int_{\mathbb{R}^3} \varrho^2 w_r^2\,dx + \frac{2c_{14}}{r+1}\,.$$

Applying the Gronwall lemma on the interval $[0, t_\omega]$, and recalling

(5.10), we get

$$\int_{B^C_{2r+2}} w_r^2(t_\omega)\,dx \leqslant \int_{\mathbb{R}^3} \varrho^2 w_r^2(t_\omega)\,dx \leqslant \frac{c_{15}}{r+1},$$

where we set

$$c_{15} = 2c_{14}\,t_\omega\,e^{2c_3 t_\omega}.$$

At this point, choosing r_1 large enough for $\sqrt{c_{15}/(r_1+1)} \leqslant \omega$, we conclude that

$$\|w_{r_1}(t_\omega)\|_{L^2(B^C_{r_2})} \leqslant \omega\ ,$$

as claimed. ■

According to Lemma 5.3 and Lemma 5.4, for any $\omega > 0$ there exist $t_\omega \geqslant 1$ and $r_2 = r_2(\omega) > 0$ such that

$$\|u(t_\omega)\|_{L^2(B^C_{r_2})} \leqslant \omega\ ,$$

for every $u_0 \in \mathcal{B}_0$. Moreover, from Theorem 4.6,

$$\|u(t_\omega)\|_{H^1(B_{r_2})} \leqslant c_{16},$$

for some $c_{16} > 0$ independent of r_2. Exploiting the compact embedding

$$H^1(B_{r_2}) \hookrightarrow L^2(B_{r_2}),$$

we conclude that $S(t_\omega)\,\mathcal{B}_0$ can be covered by finitely many balls of radius ω, hence

$$\alpha(S(t_\omega)\,\mathcal{B}_0) < 2\omega\ .$$

Letting $\omega \to 0$, it is readily seen that condition (ii) of Theorem (5.2) is satisfied as well. We have then proved our main result in this paper, namely,

THEOREM 5.5. *The semigroup* $S(t)$ *associated to problem* (1.1) *has a connected universal attractor* $\mathcal{A}$.

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