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Homogeneous Totally Real Submanifolds of Complex Projective Space (*).

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ABSTRACT - This paper is devoted to the study of a family of totally real submanifolds of complex projective space CP^n. This is the family of the so called R-spaces or real flag manifolds which have very natural immersions into certain complex projective spaces. The main result determines the geometric properties that characterize these immersions.

1. - A natural immersion.

Every R-space (real flag manifold) M^n has a natural totally real immersion into a complex projective space CP^{n+q}. This immersion arises naturally from the canonical embedding associated to the R-space. Let us recall that an R-space M^n, by definition, has a canonical embedding into R^{n+p} defined as follows. Let $\mathfrak{g}$ be a real semisimple Lie algebra without compact factors, $\mathfrak{k}$ a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus p$ the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{k}$. If we denote by B the Killing form of $\mathfrak{g}$ then $\mathfrak{p}$ can be considered a Euclidean space with the inner product defined by the restriction of B to $\mathfrak{p}$. Let $G = \mathrm{Int}(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$; whose Lie algebra may be identified with $\mathfrak{g}$. Let K be the analytic subgroup of G corre-

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sponding to $\mathfrak{k}$; then K is compact and acts on $\mathfrak{p}$ as an isometry group. The R-space M is the orbit of a non zero vector $E \in \mathfrak{p}$ i.e. $M = \mathrm{Ad}_G(K)E$. A particular case of this situation is that of the complex flag manifolds (see for instance [12]) where one takes a compact semisimple Lie algebra $\mathfrak{u}$ and consider $\mathfrak{g} = \mathfrak{u}^C = \mathfrak{u} \oplus i\mathfrak{u}$ as a real semisimple Lie algebra. Then $\mathfrak{u} \oplus i\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}$ [5][p. 185] and $U = \mathrm{Int}(\mathfrak{u})$ acting on $i\mathfrak{u}$ by the adjoint representation has orbits which are complex flag manifolds.

Let $\varrho: M^n \to (\mathfrak{p}, B)$ be the canonical embedding of an R-space. We may consider that it is isometric by taking on M^n the induced metric. It is clear that the image of the embedding ϱ lies on a sphere. This defines a new embedding

$$\varrho_1: M^n \to S^{n+q} \tag{1}$$

where $n+q+1$ is the dimension of $\mathfrak{p}$. We may take now the induced immersion

$$\varrho_2: M^n \to RP^{n+q} \tag{2}$$

which followed by the natural embedding $\sigma: RP^{n+q} \to CP^{n+q}$ yields an immersion

$$\varphi: M^n \to CP^{n+q}, \qquad \varphi = \sigma \circ \varrho_2. \tag{3}$$

which obviously is totally real.

In the present paper we determine the geometric properties that, in terms of the second fundamental form, characterize these immersions.

One of the motivations for this work is a result mentioned in [3] (specifically Theorem C in page 354) where the authors study minimal totally real submanifolds of CP^N obtaining a nice characterization of them in terms of properties of the second fundamental form of the immersion under a restriction on the Ricci curvature.

Another motivation is the known characterization of R-spaces as submanifolds of R^N with canonically parallel second fundamental form [8].

The paper is organized as follows. In Section 2 we find the relevant geometric properties of the immersion φ defined in (3). Section 3 contains the proof of the fact that these properties determine the totally real immersion φ.

2. – Geometric properties of the immersion φ.

We describe now the fundamental properties of the immersion φ.

THEOREM 1. *The second fundamental form α of the isometric immersion φ has the following properties:*

i) *α is canonically parallel,*

ii) *If J denotes the complex structure on CP^{n+q} at any point $p \in \varphi(M)$ then*

$$\langle \alpha_p(X, X), J_p X\rangle = 0\,, \qquad \forall X \in T_p(M^n).$$

PROOF. It is known, [8], that the canonical embedding ϱ of the manifold M^n into $\mathfrak{p} = R^{n+q+1}$ described above, has canonically parallel second fundamental form in the sense of [9] with respect to any one of the possible canonical connections that can be defined on M^n.

Recall that a linear connection ∇^c on M^n is said to be *canonical* if it satisfies the following properties

i) $\nabla^c\langle .\,, .\rangle = 0$;

ii) $\nabla^c D = 0$, where ∇ is the Riemannian connection of the induced metric $\langle .\,, .\rangle$ on M^n and $D = \nabla - \nabla^c$.

With the canonical connection ∇^c on M^n we can define as in [9] the *canonical covariant derivative of the second fundamental form* α_0 of the embedding ϱ by the formula

$$(4) \qquad (\nabla^c_X \alpha_0)(Y, Z) = \nabla^{\perp}_X (\alpha_0(Y, Z)) - \alpha_0(\nabla^c_X Y, Z) - \alpha_0(Y, \nabla^c_X Z)\,.$$

We say that α_0 is *canonically parallel* if

$$(\nabla^c_X \alpha_0)(Y, Z) = 0\,, \qquad \forall X, Y, Z \in T_p(M^n)\,, \qquad \forall p \in M^n.$$

Let α_1 be the second fundamental form of the embedding ϱ_1 defined in (1) and let α_2 be that of ϱ_2 in (2). It is easy to see that $(\nabla^c_X \alpha_0) = 0$ implies $(\nabla^c_X \alpha_1) = 0$ which in turn immediately yields $(\nabla^c_X \alpha_2) = 0$. Since σ is a totally geodesic embedding we have proved the first part of the proposition.

Let us prove (ii). We do this in two steps. Let us assume first that M^n is a *complex flag manifold* and let $\varrho\colon M^n \to \mathfrak{g}$ be one of its canonical embeddings. There exists on M^n an integrable almost complex structure J_1 (see for instance [1]) which commutes with the shape operators A_ξ of the imbedding ϱ (see [10] and references therein) i.e. for each $X \in T_p(M)$, $\xi \in T_p(M)^\perp$ and $p \in M^n$

$$A_\xi J_1 X = J_1 A_\xi X .$$

Then the second fundamental form α_0 satisfies

$$\alpha_0(J_1 X, J_1 Y) = \alpha_0(X, Y)$$

and since

$$\alpha_0(Y, Z) = \alpha_1(Y, Z) + h(Y, Z)\, \xi\,, \tag{5}$$

where h is a real valued symmetric bilinear form and ξ is a locally defined normal vector field, we clearly have

$$\alpha_1(J_1 X, J_1 Y) = \alpha_1(X, Y).$$

In turn α_2 and α satisfy the same equalities. On the other hand, since the immersion φ is totally real, its second fundamental form α has to satisfy the following identity where J is the complex structure in the complex projective space CP^N (see for instance [7, p. 431])

$$\langle \alpha(X, Y), JZ \rangle = \langle \alpha(X, Z), JY \rangle, \qquad \forall X, Y, Z, \in T_p(M), \qquad \forall p \in M^n.$$

Then we have

$$\langle \alpha(X, X), JX \rangle = \langle \alpha(J_1 X, J_1 X), JX \rangle = \langle \alpha(J_1 X, X), JJ_1 X \rangle = 0$$

because $\alpha(X, J_1 X) = 0$, $\forall X \in T_p(M)$ and therefore the identity (ii) holds in this case.

Let us consider now the case in which M^n is a real flag manifold (i.e. an R-space) *which is not a complex flag manifold.* We have then the data associated to the R-space M^n and its canonical imbedding namely a real semisimple Lie algebra $\mathfrak{g}$ without compact factors, $\mathfrak{k}$ a maximal compactly imbedded subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{k}$. Let B denote the Killing form of $\mathfrak{g}$; then $\mathfrak{p}$ can be considered a Euclidean space with the inner product defined by the restriction of B to $\mathfrak{p}$. Let $G = \operatorname{Int}(G)$ be the group of inner automorphisms of $\mathfrak{g}$. We may identify $\mathfrak{g}$ with the Lie algebra of G. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$; K is compact and acts on $\mathfrak{p}$ as a group of isometries. The R-space M^n is the orbit of a non zero vector $E \in \mathfrak{p}$ i.e. $M = \operatorname{Ad}(K)E$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace; we may assume $E \in \mathfrak{a}$ [5, p. 247]. Let us extend $\mathfrak{a}$, to a Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Consider the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ and let ω be the corresponding conjugation. Since $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition there exists a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_c$ such that

$$\omega\mathfrak{g}_u \subset \mathfrak{g}_u$$
$$\mathfrak{k}=\mathfrak{g}\cap\mathfrak{g}_u, \qquad \mathfrak{p}=\mathfrak{g}\cap i\mathfrak{g}_u$$
$$\mathfrak{g}_u=\mathfrak{k}\oplus i\mathfrak{p}\,.$$

Let G_c be the complex, simply connected, semisimple Lie group associated to the complex Lie algebra $\mathfrak{g}_c$ and let G_1 and G_u be the analytic subgroups of G_c corresponding to the subalgebras $\mathfrak{g}$ and $\mathfrak{g}_u$ respectively. They are closed in G_c by [5, p. 152, 4, (ii)]. Let K_1 be the analytic subgroup of G_c corresponding to $\mathfrak{k}$, clearly $K_1\subset G_1\cap G_u$ and $\mathrm{Ad}\colon G_1\to G$ is an analytic homomorphism onto G such that $\mathrm{Ad}_{G_1}(K_1)=K$.

The manifold $\mathrm{Ad}_{G_u}(K_1)(iE)\subset\mathfrak{g}_u$ is again our R-space M because the representations of K_1 in $\mathfrak{p}$ and $i\mathfrak{p}$ are equivalent. By [5][p. 180], if $X, Y\in$ $\in\mathfrak{p}$ then

$$-B_u(iX, iY) = -B_c(iX, iY) = B_c(X, Y) = B(X, Y)$$

and so, if we take on $i\mathfrak{p}$ the Euclidean metric induced by $-B_u$ then M^n and $\mathrm{Ad}_{G_u}(K_1)(iE)$ are isometric.

Let $M_c=\mathrm{Ad}_{G_u}(G_u)(iE)$; this is obviously a complex flag manifold, contains M and, if we consider on M_c the Riemannian metric induced by the inner product on $\mathfrak{g}_u$ defined by $-B_u$, it is clear that M^n is isometrically embedded in M_c.

Our construction of the immersion $\varphi\colon M^n\to CP^{n+q}$ is not changed if we use $i\varrho\colon M^n\to i\mathfrak{p}$ instead of $\varrho\colon M^n\to\mathfrak{p}$. Now we can associate to φ an immersion $\varphi^c\colon M_c\to CP^N$ defined as the bottom line in the next diagram ($\varphi^c=\sigma^c\circ\varrho_2^c$)

$$\begin{tikzcd}
 & S(i\mathfrak{p}) \arrow[dr, "\pi"] & & \\
M \arrow[ur, "i\varrho_1"] \arrow[rr, "\varrho_2"] & & RP(S(i\mathfrak{p})) \arrow[r, "\sigma"] & CP(S(i\mathfrak{p})) \\
\cap & & \cap & \cap \\
M_c \arrow[dr, "\varrho_1^c"'] \arrow[rr, "\varrho_2^c"] & & RP(S(\mathfrak{g}_u)) \arrow[r, "\sigma^c"'] & CP(S(\mathfrak{g}_u)) \\
 & S(\mathfrak{g}_u) \arrow[ur, "\pi^c"'] & &
\end{tikzcd}$$

where both spheres have radius $\|iE\|$. In fact $S(i\mathfrak{p}) = i\mathfrak{p} \cap S(\mathfrak{g}_u)$, $M = i\mathfrak{p} \cap M_c$ and $\varphi = (\varphi^c \,|M)$. We keep the notation above (α_1 is the second fundamental form of ϱ_1, or $i\varrho_1$, α_2 that of ϱ_2, etc.) we just add a «c» to indicate the corresponding second fundamental forms for M_c. Let γ denote the second fundamental form of M in M_c.

Then, for X, $Y \in T_p(M)$, we have

$$\alpha_j(X, Y) = \gamma(X, Y) + \alpha_j^c(X, Y) \qquad \text{for } j = 1, 2$$

and hence

$$\alpha(X, Y) = \gamma(X, Y) + \alpha^c(X, Y).$$

Then, for $X \in T_p(M)$ and the complex structure J in $CP(S(\mathfrak{g}_u))$, we have that

$$\langle \alpha(X, X), JX \rangle = \langle \gamma(X, X), JX \rangle + \langle \alpha^c(X, X), JX \rangle.$$

In view of the fact that (ii) holds for complex flag manifolds we have $\langle \alpha^c(X, X), JX \rangle = 0$ and since M_c is totally real in $CP(S(\mathfrak{g}_u))$, it is clear that $\langle \gamma(X, X), JX \rangle = 0$. Therefore we obtain

$$\langle \alpha(X, X), JX \rangle = 0$$

and the proof of the theorem is complete. ∎

3. – The characterization.

In this section we prove that the conditions of Theorem 1 essentially determine the immersion φ. In fact we have

THEOREM 2. *Let M^n be a simply connected, compact, Riemannian manifold with a canonical connection and f: $M^n \to CP^{n+q}$ an isometric immersion which satisfies the following conditions.*

i) *f is totally real.*

ii) *The second fundamental form α of the immersion f is canonically parallel* ($\nabla^c \alpha = 0$).

iii) *If J is the complex structure on CP^{n+q} then*

$$\langle \alpha_p(X, X), J_p X \rangle = 0\,, \qquad \forall p \in f(M^n)5, \qquad X \in T_p(M).$$

Then M^n is a real flag manifold (R-*space*) *and there is a vector $E \in \mathfrak{p}$ such that $i\varrho_1(hK_E) = \mathrm{Ad}(h)\, iE$ and f is of the form $f = \sigma \circ \pi \circ i\varrho_1 = \varphi$.*

PROOF. The proof will be divided into several lemmas.

Let us take a point $p \in M^n$ *which we shall keep fixed for the rest of the proof.* We shall use the following notation.

$$N_p(M) = \mathrm{Span}_R\{\alpha(X, Y): X, Y \in T_p(M)\}$$

is the *first normal space at* p and

$$O_p(M) = T_p(M) \oplus N_p(M),$$

the *first osculating space at p.* Also $N_p(M)^\perp$ will denote the orthogonal complement of $N_p(M)$ in $T_p(M)^\perp$. Since M is totally real in CP^{n+q} we have $J(T_p(M)) \subset T_p(M)^\perp$.

As it is well known, the way to construct symmetric subspaces (i.e. totally geodesic submanifolds) of a symmetric space, such as CP^{n+q}, is to consider a Lie triple system in the tangent space at a point $p \in CP^{n+q}$.

The following lemma is a modification, to our conditions ($\nabla^c \alpha = 0$ instead of $\nabla \alpha = 0$), of [6, p. 101, 13].

LEMMA 3. *Under the conditions of Theorem* 2, *if* $\overline{R}$ *denotes the curvature tensor in* $CP^{n+q}(c)$ *then*

A) $\overline{R}_p(T_p(M), T_p(M))\, T_p(M) \subset T_p(M)$,

B) $\overline{R}_p(T_p(M), T_p(M))\, N_p(M) \subset N_p(M)$.

PROOF. Since f is totally real, (A) follows from [2, p. 260, 3.1]. To prove (B) we take $X, Y \in T_p(M)$, $H \in N_p(M)$ and $\xi \in N_p(M)^\perp$. Clearly $A_\xi \equiv 0$ on $T_p(M)$.

Let us prove that $R^\perp(X, Y)\, H \in N_p(M)$. We may assume that $H = \alpha(U, V)$ for some $U, V \in T_p(M)$; then

$$\nabla_X^\perp(\alpha(U, V)) = \alpha(\nabla_X^c U, V) + \alpha(U, \nabla_X^c V)$$

because $\nabla_X^c \alpha = 0$. This shows that $R^\perp(X, Y)\, H$ may be written in terms of elements belonging to $N_p(M)$ and then $R^\perp(X, Y)\, H \in N_p(M)$.

Now

$$\langle \overline{R}(X, Y)\, H, \xi \rangle = \langle R^\perp(X, Y)\, H, \xi \rangle + \langle [A_H, A_\xi]\, X, Y \rangle = 0$$

and therefore $\overline{R}(X, Y)\, H \in O_p(M)$.

Now for $Z \in T_p(M)$ we have

$$\langle \overline{R}(X, Y)\, Z, H \rangle + \langle Z, \overline{R}(X, Y)\, H \rangle = 0$$

and by part (A) the first term is zero so $\overline{R}(X, Y)H \in N_p(M)$ and part (B) is proved. ■

LEMMA 4. *Under the conditions of Theorem 2, at the chosen point $p \in M^n$, $J_p(T_p(M)) \subset N_p(M)^{\perp}$.*

PROOF. $J_p(T_p(M)) \subset T_p(M)^{\perp} = N_p(M) \oplus N_p(M)^{\perp}$. For a totally real immersion, the second fundamental form satisfies the following identity [7, p. 431]. For every $X, Y, Z \in T_p(M)$,

$$\langle \alpha_p(X, Y), J_p Z\rangle = \langle \alpha_p(X, Z), J_p Y\rangle .$$

Then the three-linear function $\langle \alpha_p(X, Y), J_p Z\rangle$ is symmetric on $T_p(M)$. By condition (iii) of Theorem 2 it vanishes on the diagonal of $T_p(M) \times T_p(M) \times T_p(M)$ and hence it is identically zero. This clearly means that

$$J_p(T_p(M)) \subset N_p(M)^{\perp}$$

as was to be proven. ■

LEMMA 5. *Under the hypothesis of Theorem* 2

C) $\overline{R}_p(T_p(M), N_p(M))\,T_p(M) \subset N_p(M)$,

D) $\overline{R}_p(T_p(M), N_p(M))\,N_p(M) \subset O_p(M)$.

PROOF. According to Lemma 4 we have $J(T_p(M)) \subset N_p(M)^{\perp}$.

The proof of the present lemma is *formally analogous to that of* [7, p. 433]. The proof is omitted. ■

LEMMA 6. *Under the hypothesis of Theorem* 2

E) $\overline{R}_p(N_p(M), N_p(M))\,T_p(M) \subset O_p(M)$,

F) $\overline{R}_p(N_p(M), N_p(M))\,N_p(M) \subset O_p(M)$.

PROOF. Let X, Y, Z be vector fields tangent to M^n and H a section of $N(M)$. Since CP^{n+q} is symmetric we have $(\overline{\nabla}_Z \overline{R})(X, H)Y = 0$. Then

$$\begin{aligned}\overline{\nabla}_Z(\overline{R}(X, H)\,Y) &= \overline{R}(\overline{\nabla}_Z X, H)\,Y + \overline{R}(X, \overline{\nabla}_Z H)\,Y + \overline{R}(X, H)\,\overline{\nabla}_Z Y = \\ &= \overline{R}(\nabla_Z X, H)\,Y + \overline{R}(\alpha(Z, X), H)\,Y - \overline{R}(X, A_H(Z))\,Y + \\ &\quad + \overline{R}(X, \nabla_Z^{\perp} H)\,Y + \overline{R}(X, H)\,\nabla_Z Y + \overline{R}(X, H)\,\alpha(Z, Y).\end{aligned}$$

We may assume, as above, that $H = \alpha(U, V)$. Then, since $\nabla^c \alpha = 0$, we have

$$\nabla_Z^\perp H = \nabla_Z^\perp (\alpha(U, V)) = \alpha(\nabla_Z^c U, V) + \alpha(U, \nabla_Z^c V)$$

and therefore

$$\begin{aligned} &\bar{R}(X, \nabla_Z^\perp H)\, Y \in N_p(M) && \text{by (C)},\\ &\bar{R}(\nabla_Z X, H)\, Y \in N_p(M) && \text{by (C)},\\ &\bar{R}(X, A_H(Z)) \in T_p(M) && \text{by (A)},\\ &\bar{R}(X, H)\, \nabla_Z Y \in N_p(M) && \text{by (C)},\\ &\bar{R}(X, H)\, \alpha(Z, Y) \in N_p(M) && \text{by (D)},\\ &W = \bar{R}(X, H)\, Y \in N_p(M) && \text{by (C)}. \end{aligned}$$

We may proceed then as we did with H and hence

$$\bar{\nabla}_Z(W) = -A_W(Z) + \nabla_Z^\perp W \in T_p(M) \oplus N_p(M).$$

We conclude that

$$\bar{R}(\alpha(Z, X), H)\, Y \in O_p(M),$$

which yields (E).

Now we prove (F). Let X, Y be vector fields on M^n and H, W, sections of $N(M)$. We have $(\bar{\nabla}_Y \bar{R})(X, H)\, W = 0$ and then

$$\begin{aligned} \bar{\nabla}_Y(\bar{R}(X, H)\, W) &= \bar{R}(\bar{\nabla}_Y X, H)\, W + \bar{R}(X, \bar{\nabla}_Y H)\, W + \bar{R}(X, H)\, \bar{\nabla}_Y W = \\ &= \bar{R}(\nabla_Y X, H)\, W + \bar{R}(\alpha(Y, X), H)\, W - \bar{R}(X, A_H(Y))\, W + \\ &\quad + \bar{R}(X, \nabla_Y^\perp H)\, W + \bar{R}(X, H)\, A_W Y + \bar{R}(X, H)\, \nabla_Y^\perp W. \end{aligned}$$

As in the proof of (E) we may show that $\nabla_Y^\perp H$ and $\nabla_Y^\perp W$ belong to $N(M)$ and therefore

$$\begin{aligned} &\bar{R}(\nabla_Y X, H)\, W \in O_p(M) && \text{by (D)},\\ &\bar{R}(X, A_H Y)\, W \in N_p(M) && \text{by (B)},\\ &\bar{R}(X, \nabla_Y^\perp H)\, W \in O_p(M) && \text{by (D)},\\ &\bar{R}(X, H)\, A_W Y \in N_p(M) && \text{by (C)},\\ &\bar{R}(X, H)\, \nabla_Y^\perp W \in O_p(M) && \text{by (D)},\\ &\bar{R}(X, H)\, W \in O_p(M) && \text{by (D)}. \end{aligned}$$

We may write $\overline{R}(X, H)\,W = U + \xi$ with $U \in T(M)$ and $\xi \in N(M)$ and therefore

$$\overline{\nabla}_Y(U + \xi) = \nabla_Y U + \alpha(Y, U) - A_\xi Y + \nabla_Y^\perp \xi$$

which clearly belongs to $O_p(M)$.

It follows that $\overline{R}(\alpha(X, Y), H)\,W \in O_p(M)$; this proves (F) and completes the proof of the lemma. ■

The contents of Lemmas 3, 5 and 6 clearly yield the following

LEMMA 7. *Under the hypothesis of Theorem* 2, *the first osculating space* $O_p(M)$ *is a Lie triple system in* $T_p(CP^{n+q})$. ■

Then there exists a totally geodesic submanifold $L \subset CP^{n+q}$ such that $T_p(L) = O_p(M)$. The submanifold L is complete.

Since $c > 0$ we have two possibilities for L: either

a) $L = CP^{n+s}$ for $s \leqslant q$ or

b) $L = RP^{n+s}(c/4)$ for $s \leqslant q$.

By Lemma 4 the manifold L can not be of type $CP^{n+s}(c)$.

We have to show now that $f(M) \subset L$ and to that end we need

THEOREM 8. *Let* $M \xrightarrow{f} M_1$ *be an isometric immersion of a compact connected Riemannian manifold* M *into a Riemannian manifold* M_1 *and assume that*:

i) M *admits a canonical connection.*

ii) *The second fundamental form of the immersion* f *is canonically parallel i.e.* $\nabla^c \alpha \equiv 0$.

Then for each point $p \in M$ *and each unitary vector* $X \in T_p(M)$, *if* γ *is the* ∇^c*-geodesic on* M^n *defined by* X, *then* $c(t) = f(\gamma(t))$ *is a W-curve in* M_1, [4, p. 57], *of osculating rank* $r \leqslant \dim M_1$ *such that for* $j = 1, \dots, r$ *the j-th element of the Frenet frame can be written as* $V_j(t) = P_j(t) + Q_j(t)$ *where* P_j *and* Q_j *are the tangent and normal components respectively and satisfy*

$$\nabla^c_{\gamma'} P_j(t) = 0 = \nabla^\perp_{\gamma'} Q_j(t). \quad ■$$

REMARK. This is a generalization of a part of Theorem 13 of

[11] which is proven in that article for $M_1 = R^N$. The extension of the proof given there, to the general case, is straightforward.

Now the fact that $\nabla^c \alpha \equiv 0$ implies that $Q_1(0), \ldots, Q_r(0) \in N_p(M)$ and therefore $V_1(0), \ldots, V_r(0) \in T_p(L)$. Let $b(t)$ be the Frenet curve on L of osculating rank r with initial conditions $c(0), V_1(0), \ldots, V_r(0)$ and the constant curvatures $k_1, \ldots, k_r$ of the W-curve $c(t)$. The curve $b(t)$ is a Frenet curve in the ambient space $CP^{n+q}(c)$ because L is totally geodesic in $CP^{n+q}(c)$. But, since $c(t)$ and $b(t)$ satisfy the same equations and have the same initial conditions in $CP^{n+q}(c)$ we have $c(t) = b(t)$, $\forall t$. Since in a manifold with a linear connection every pair of points can be joined by a broken geodesic, we have proved

LEMMA 9. *$f(M) \subset L$.* ∎

We observe that for each $q \in M^n$ we have $T_q(L) = O_q(M)$ because due to Lemma 9 we have $T_q(L) \supset O_q(M)$ and since the dimension of $O_q(M)$ is constant along M^n (recall that $\nabla^c \alpha \equiv 0$) and at the chosen point $p \in M^n$ we have $T_p(L) = O_p(M)$ we have that they coincide everywhere in M^n.

We have

$$M^n \to RP^{n+s}\left(\frac{c}{4}\right) \to CP^{n+q}.$$

Since M^n is simply connected we have a lifting

$$M^n \xrightarrow{\mu} S^{n+s}\left(\frac{c}{4}\right) \xrightarrow{\pi} RP^{n+s}\left(\frac{c}{4}\right) \to CP^{n+q}$$

and since M^n has canonically parallel second fundamental form in $CP^{n+q}(c)$ and $RP^{n+s}(c/4)$ is totally geodesic in $CP^{n+q}(c)$ then M^n has canonically parallel second fundamental form in $RP^{n+s}(c/4)$ and hence the immersion μ has the same property. It is easy to show that in this situation the immersion μ considered as an immersion in R^{n+s+1} has canonically parallel second fundamental form. In view of [8, p. 195] the proof of Theorem 2 is now complete. ∎

REMARK. If the manifold M^n is not simply connected we need an extra condition to get the existence of the lifting μ. This condition is obviously $f_*[\pi_1(M^n)] = \{1\}$ in $\pi_1(RP^{n+s})$. To have this we may ask, instead of the simple connectedness of M^n, that the immersion f be such that:

For every totally geodesic submanifold $N \subset CP^{n+q}$ *such that* $f(M^n) \subset$ $\subset N \subset CP^{n+q}$

$$f_*[\pi_1(M^n)] = \{1\} \quad in \ \pi_1(N).$$

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