

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 101 (1999), p. 1-18

http://www.numdam.org/item?id=RSMUP_1999__101__1_0

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On Positive Solutions of Some Periodic Parabolic Eigenvalue Problem with a Weight Function.

T. GODOY(*)(**) - A. GUERIN(*)(***) - S. PACZKA(*)(* *)

ABSTRACT - Let Ω be a bounded domain in $\mathbb{R}^n$ and let m be a T -periodic function such that its restriction to $\Omega \times (0, T)$ is in $L^r(\Omega \times (0, T))$ for some $r > n + 2$. We find necessary and sufficient conditions, on m , for the existence, uniqueness and simplicity of the principal eigenvalue for the Dirichlet and Neumann periodic parabolic eigenvalue problem with weight m .

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{2+\theta}$ boundary ($0 < \theta < 1$), and $\{a_{i,j}(x, t)\}_{1 \leq i, j \leq n}$ $\{a_j(x, t)\}_{1 \leq j \leq n}$ two families of $(\theta, \theta/2)$ -Hölder continuous and T -periodic in t functions on $\overline{\Omega} \times \mathbb{R}$. We also assume that $a_{i,j} = a_{j,i}$ and that $c \sum_i \xi_i^2 \leq \sum_{i,j} a_{i,j}(x, t) \xi_i \xi_j$ for some $c > 0$ and all $(x, t) \in \overline{\Omega} \times \mathbb{R}$, $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

Let $m(x, t)$ be a T -periodic in t real function on $\Omega \times \mathbb{R}$. Our aim is to consider, in a suitable weak sense, the following periodic parabolic

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(**) Partially supported by CONICOR and SECYTUNC.

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eigenvalue problem on $\Omega \times \mathfrak{R}$

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i,j} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_j a_j \frac{\partial u}{\partial x_j} = \lambda m u, \\ B(u) = 0, \\ u(x, t) = u(x, t + T), \end{cases}$$

where either $B(u) = u|_{\partial\Omega \times \mathfrak{R}}$ (Dirichlet condition) or $B(u) = (\partial u / \partial \nu)|_{\partial\Omega \times \mathfrak{R}}$ (Neumann condition). If $m \in C^{\theta, \theta/2}(\overline{\Omega} \times \mathfrak{R})$ and $B(u) = u|_{\partial\Omega \times \mathfrak{R}}$, Beltramo and Hess, in [B-H], found necessary and sufficient conditions on m for the existence, uniqueness and simplicity of the positive principal eigenvalue of the above problem. Beltramo extended these results to more general boundary conditions (that include the Neumann condition) in [B]. The case m continuous and $B(u) = (\partial u / \partial \nu)|_{\partial\Omega \times \mathfrak{R}}$ is studied in [P] and the case $m \in L^\infty(\Omega \times \mathfrak{R})$ is treated in [G-L-P] under the additional hypothesis $a_{i,j} \in C^1(\overline{\Omega} \times \mathfrak{R})$, $1 \leq i, j \leq n$. Our purpose is to obtain, under this additional assumption, similar results if $\left(\int_{\Omega \times (0, T)} |m|^r \right)^{1/r} < \infty$.

2. Notation and preliminaries.

Let Ω , $a_{i,j}$, a_j ; be as above with $a_{i,j} \in C^1(\overline{\Omega} \times \mathfrak{R})$, $1 \leq i, j \leq n$. We fix, for the whole paper, p, q and r such that $n + 2 < p < q < r$. We consider, for $u \in C^{2,1}(\overline{\Omega} \times \mathfrak{R})$, $L(u) = \partial u / \partial t + A(x, t, D)u$ where $A(x, t, D)u = -\sum a_{i,j}(x, t) D_{i,j}u - \sum a_j(x, t) D_j u$.

Let E be a vector space of functions on $\Omega \times \mathfrak{R}$, we set

$$E_B = \{u \in E \cap D(B): Bu = 0\},$$

where $D(B)$ is the domain of the boundary condition. For $1 \leq s \leq \infty$, let $L_T^s(\Omega \times \mathfrak{R})$ denote the space of the measurable functions $f: \Omega \times \mathfrak{R} \rightarrow \mathbb{C}$ such that $f(x, t) = f(x, t + T)$ a.e. $(x, t) \in \Omega \times \mathfrak{R}$ and $\|f\|_s < \infty$, where $\|f\|_s = \left(\int_{\Omega \times (0, T)} |f|^s \right)^{1/s}$ if $s < \infty$ and $\|f\|_\infty = \text{ess sup}_{(x, t) \in \Omega \times (0, T)} |f(x, t)|$. If S is a bounded operator from $L_T^p(\Omega \times \mathfrak{R})$ into $L_T^q(\Omega \times \mathfrak{R})$, we write $\|S\|_{p,q}$ for the norm operator with respect to the above norms. If a is a real or complex valued function on $\Omega \times \mathfrak{R}$, we still denote by a the operator multiplication by a .

Let $X = L^p(\Omega)$. If $a(x, t)$ is a T periodic function in $C^{\theta, \theta/2}(\Omega \times \mathfrak{R})$ and satisfies $a \geq 0$ if $B(u) = u|_{\partial\Omega \times \mathfrak{R}}$ and $a \geq 0$, $a \neq 0$ if $B(u) =$

$= (\partial u / \partial \nu)_{|\partial \Omega \times \mathfrak{R}}$, we consider $A_a(t): W_B^{2,p}(\Omega) \subset L^p(\Omega) \rightarrow L^p(\Omega)$ defined by $A_a(t)u = A(\cdot, t, D)u + a(\cdot, t)u$. Once we fix $k \in \mathfrak{R}$, $k > 1 + \|a\|_\infty$, we put $A = A_{a+k}(0)$ and for $\alpha \in [0, 1]$ -let A^α be defined as in [H,1]. Let X_α be the domain of A^α . Then X_α is a Banach space with the norm $\|x\|_\alpha = \|A^\alpha x\|_{L^p(\Omega)}$. We have $X_0 = L^p(\Omega)$, $X_1 = W_B^{2,p}(\Omega)$ and $X_\alpha \subset X_\beta$ whenever $0 \leq \beta < \alpha \leq 1$, the inclusion being compact. Moreover, for $1/2 + n/2p < \alpha < 1$, we have $X_\alpha \subset C_B^{1+\gamma}$ for some $\gamma = \gamma(\alpha) < 1$ and this inclusion is compact (see [H,1] or [A]).

For $\omega > 0$, $f \in C^\sigma([0, T + \omega], X)$, $\sigma \in (0, 1]$ the linear evolution equation

$$\frac{du}{dt} + A_{a+k}(t) u(t) = f(t), \quad u(0) = u_0$$

has a unique solution $u \in C([0, T + \omega], X) \cap C^1((0, T + \omega], X)$ if $u_0 \in X_0$ and $u \in C^1([0, T + \omega], X)$ if $u_0 \in X_1$. This solution is given by

$$u(t) = U_{a+k}(t, 0)u_0 + \int_0^t U_{a+k}(t, \tau) f(\tau) d\tau,$$

where $U_{a+k}(t, \tau)$ is the associated evolution operator. The change $u(t) = e^{-kt}v(t)$ reduces the problem

$$\frac{dv}{dt} + A_a(t) v(t) = f(t), \quad v(0) = u_0$$

to the above problem and gives us

$$v(t) = U_a(t, 0)u_0 + \int_0^t U_a(t, \tau) f(\tau) d\tau,$$

where $U_a(t, \tau) = e^{k(t-\tau)} U_{a+k}(t, \tau)$.

REMARK 2.1. For the rest of the paper we fix $\alpha \in (1/2 + n/2p, 1 - 1/p)$, let k be as above and set

$$K_a = U_a(T, 0)|_{X_\alpha}: X_\alpha \rightarrow X_\alpha.$$

Let $C_T^\gamma(\mathfrak{R}, X_\alpha)$, $C_T(\overline{\Omega} \times \mathfrak{R})$ and $C_T^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})$ denote the subspace of T -periodic functions in $C^\gamma(\mathfrak{R}, X_\alpha)$, $C(\overline{\Omega} \times \mathfrak{R})$ and $C_B^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})$ respectively. We identify $L_T^p(\Omega \times \mathfrak{R})$ with $L_T^p(\mathfrak{R}, L^p(\Omega))$ in the obvious way. Then, if a is as above, (see e.g. [G-L-P], lemma 3.1), there exists

$\gamma \in (0, 1)$ such that the operator $S_a: L^p(\Omega \times [0, T + \omega]) \rightarrow C^\gamma([0, T + \omega], X_a)$ defined by

$$(S_a(g))(t) = U_a(t, 0)(I - K_a)^{-1} \int_0^T U_a(T, \tau) g(\tau) d\tau + \int_0^t U_a(t, \tau) g(\tau) d\tau$$

is injective, positive and bounded. Moreover $S_a(g)$ has a unique T -periodic extension to $\Omega \times \mathfrak{R}$, still denoted by $S_a(g)$ such that for $q \geq p$

$$S_a(g)|_{L_T^q(\Omega \times \mathfrak{R})}: L_T^q(\Omega \times \mathfrak{R}) \rightarrow L_T^q(\Omega \times \mathfrak{R})$$

is a compact operator. Moreover, for some $\gamma^* \in (0, 1)$, the same is true for

$$S_a(g)|_{C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})}: C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R}) \rightarrow C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R}).$$

This follows from the observation that the following inclusions are continuous and that the second is compact

$$C_T^\gamma(\mathfrak{R}, X_a) \subset C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R}) \subset C_T(\overline{\Omega} \times \mathfrak{R}) \subset L_T^q(\Omega \times \mathfrak{R}).$$

REMARK 2.2. We now take $\lambda > 0$ and define $W = S_\lambda(L_T^p(\Omega \times \mathfrak{R}))$ and $\tilde{L}: W \rightarrow L_T^p(\Omega \times \mathfrak{R})$ by $\tilde{L} = S_\lambda^{-1} - \lambda I$. Then W and $\tilde{L}$ do not depend on λ (see e.g. [G-L-P], remark 3.5), $\tilde{L}$ is an extension of L and $(\tilde{L} + \lambda)^{-1}: L_T^p(\Omega \times \mathfrak{R}) \rightarrow W$ is a positive operator (see e.g. [G-L-P] Lemma 3.7). Note that, if we consider on W the topology induced by $C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})$, then $\tilde{L}$ is a closed operator.

LEMMA 2.3. $\lim_{\delta \rightarrow +\infty} \|(\tilde{L} + \delta)^{-1}\|_{r, \infty} = 0$.

PROOF. For $\delta > 0$, we have $U_a(t, \tau) = e^{\delta(t-\tau)} U_{a+\delta}(t, \tau)$. For $g \in L_T^q(\Omega \times \mathfrak{R})$, $0 \leq \tau \leq T$, we set $G_\delta(\tau) = e^{\delta\tau} g(\tau)$, Hölder inequality gives us

$$\|G_\delta\|_{L^p(\Omega \times [0, t])} \leq c\delta^{-(1/p - 1/q)} e^{\delta t} \|g\|_{L^q(\Omega \times [0, t])}.$$

For $0 \leq t \leq T$ we have $(\tilde{L} + \delta + 1)^{-1}(g)(t) = S^{(1)}(g)(t) + S^{(2)}(g)(t)$, where

$$S^{(1)}(g)(t) = e^{-\delta t} \int_0^t U_1(t, \tau) G_\delta(\tau) d\tau$$

and

$$S^{(2)}(g)(t) = e^{-\delta(t+T)} U_1(t, 0)(I - e^{-\delta T} K_1)^{-1} \int_0^t U_1(T, \tau) G_\delta(\tau) d\tau$$

reasoning as in lemma 3.1 in [G-L-P], we get

$$\begin{aligned} \|S^{(1)}(g)(t)\|_{L^\infty(\Omega)} &\leq c \|S^{(1)}(g)(t)\|_\alpha \leq c e^{-\delta t} \|G_\delta\|_{L^p(\Omega \times (0, t))} \leq \\ &\leq c \delta^{-(1/p-1/q)} \|g\|_{L^q(\Omega \times [0, T])}. \end{aligned}$$

So

$$\|S^{(1)}(g)(t)\|_{L^\infty(\Omega \times (0, T))} \leq c \delta^{-(1/p-1/q)} \|g\|_{L^q(\Omega \times (0, T))}.$$

The maximum principle implies

$$\|U_1(t, 0)\|_{C(\mathfrak{D}), C(\mathfrak{D})} \leq 1, \quad \|K_1^{-1}\|_{C(\mathfrak{D}), C(\mathfrak{D})} \leq 1,$$

then

$$\begin{aligned} \left\| e^{-\delta(t+T)} U_1(t, 0)(I - e^{-\delta T} K_1)^{-1} \int_0^t U_1(T, \tau) G_\delta(\tau) d\tau \right\|_{L^\infty(\Omega \times (0, T))} &\leq \\ &\leq c e^{-\delta T} \left\| \int_0^t U_1(T, \tau) G_\delta(\tau) d\tau \right\|_{L^\infty(\Omega \times (0, T))} \leq \\ &\leq c e^{-\delta T} \left\| \int_0^T U_1(T, \tau) G_\delta(\tau) d\tau \right\|_\alpha \leq c \delta^{-(1/p-1/q)} e^{\delta T} \|g\|_{L^q(\Omega \times [0, T])} \end{aligned}$$

and the lemma follows. ■

REMARK 2.4. By lemma 2.3 there exists a non increasing function $\delta_0: \mathfrak{R}^{>0} \rightarrow \mathfrak{R}^{>0}$ such that $\|(\tilde{L} + \delta)^{-1}\|_{r, \infty} < \varepsilon$ if $\delta > \delta_0(\varepsilon)$. We set $W^q = S_\lambda(L_T^q(\Omega \times \mathfrak{R}))$. Note that W^q does not depend on the choice of λ . , moreover $W = W^p$. We have

LEMMA 2.5. *If $a \in L_T^r(\Omega \times \mathfrak{R})$ and $\delta \in \mathfrak{R}$, $\delta > \delta_0(1/\|a\|_r)$, then*

- i) $(\tilde{L} + a + \delta)|_{W^q}$ is a bijection between W^q and $L_T^q(\Omega \times \mathfrak{R})$.
- ii) $((\tilde{L} + a + \delta)|_{W^q})^{-1}: L_T^q(\Omega \times \mathfrak{R}) \rightarrow L_T^q(\Omega \times \mathfrak{R})$ is a compact operator, moreover it is positive if $a \geq 0$.

PROOF. To see that $\tilde{L} + a + \delta$ is injective we note that $(\tilde{L} + a + \delta) \cdot w = 0$, $w \in W^q$, implies $(I + (\tilde{L} + \delta)^{-1}a)w = 0$ (since $\tilde{L} + \delta$ is injective). Now Lemma 2.3 give us the injectivity. Note also that, for $w \in W^q$, $u \in L_T^q(\Omega \times \mathfrak{R})$, $(\tilde{L} + a + \delta)w = u$ is equivalent to

$$w + (\tilde{L} + \delta)^{-1}(aw) = (\tilde{L} + \delta)^{-1}u$$

then Lemma 2.3 implies that for $u \in L_T^q(\Omega \times \mathfrak{R})$ this equation has an unique solution w in $L_T^\infty(\Omega \times \mathfrak{R})$. Moreover, the solution is given by

$$w = (\tilde{L} + \delta)^{-1}u - (\tilde{L} + \delta)^{-1}(aw)$$

then $w \in W^q$ and so $(\tilde{L} + a + \delta)|_{W^q}$ is bijective. On the other hand, Hölder inequality gives us $\|(\tilde{L} + \delta)^{-1}a\|_{\infty, \infty} < 1$. Therefore

$$I + ((\tilde{L} + \delta)^{-1}a)|_{C_T(\Omega \times \mathfrak{R})}: C_T(\Omega \times \mathfrak{R}) \rightarrow C_T(\Omega \times \mathfrak{R})$$

has a bounded inverse. Since $(\tilde{L} + \delta)|_{L_T^1(\Omega \times \mathfrak{R})}^{-1}$ is a compact operator on $L_T^q(\Omega \times \mathfrak{R})$, the first statement of (ii) follows from the identity

$$((\tilde{L} + a + \delta)|_{W^q})^{-1} = ((I + (\tilde{L} + \delta)^{-1}a)|_{C_T(\Omega \times \mathfrak{R})})^{-1}(\tilde{L} + \delta)^{-1}.$$

Now we take a sequence $\{a_j\}_{j \in \mathbb{N}}$ of nonnegative and T periodic Hölder continuous functions with support contained in $\Omega \times \mathfrak{R}$ that converges to a in $L_T^r(\Omega \times \mathfrak{R})$, then the sequence $((\tilde{L} + a_j + \delta)|_{W^q})^{-1}$ converges to $((\tilde{L} + a + \delta)|_{W^q})^{-1}$ in the norm topology on $B(L_T^q(\Omega \times \mathfrak{R}))$. Indeed

$$\begin{aligned} & \|((\tilde{L} + a + \delta)|_{W^q})^{-1} - ((\tilde{L} + a_j + \delta)|_{W^q})^{-1}\|_{q, q} \leq \\ & \leq \|((\tilde{L} + a + \delta)|_{W^q})^{-1}\|_{r, q} \|a_j - a\|_{\infty, r} \|((\tilde{L} + a_j + \delta)|_{W^q})^{-1}\|_{q, \infty} \leq \\ & \leq 2 \|((\tilde{L} + a + \delta)|_{W^q})^{-1}\|_{r, q} \|a_j - a\|_r \|((\tilde{L} + \delta)|_{W^q})^{-1}\|_{q, \infty}. \end{aligned}$$

Since each $((\tilde{L} + a_j + \delta)|_{W^q})^{-1}$ is a positive operator, the lemma follows. ■

LEMMA 2.6. *Let (λ_1, λ_2) be an open interval with $\lambda_1 > 0$. Suppose $m \in L_T^r(\Omega \times \mathfrak{R})$ and let $s_0 = s_0(\lambda_1, \lambda_2, m) = (1/\lambda_1) \delta_0(1/(4\lambda_2 \|m\|_{\infty, r}))$, where δ_0 is defined as in remark 2.4. Then for $\lambda \in (\lambda_1, \lambda_2)$ and $s > s_0$ we have*

- (i) $(\tilde{L} + \lambda(s - m))|_{W^q}: W^q \rightarrow L_T^q(\Omega \times \mathfrak{R})$ is a bijection.
- (ii) $\|((\tilde{L} + \lambda(s - m))|_{W^q})^{-1}\|_{\infty, \infty} \leq 4 \|((\tilde{L} + \lambda s)|_{W^q})^{-1}\|_{\infty, \infty}$.

(iii) $((\tilde{L} + \lambda(s - m))|_{W^q})^{-1}: L_T^q(\Omega \times \mathfrak{R}) \rightarrow L_T^q(\Omega \times \mathfrak{R})$ is a compact and positive operator.

PROOF. We write, as usual, $m = m^+ - m^-$ where $m^+ = \max\{m, 0\}$ and $m^- = -\min\{m, 0\}$. Suppose that $(\tilde{L} + \lambda(s - m))w = 0$ for some $w \in W$, then $(I - \lambda(\tilde{L} + \lambda s)^{-1}m)w = 0$ and so $w = 0$, since

$$\lambda\|(\tilde{L} + \lambda s)^{-1}m\|_{\infty, \infty} \leq \lambda\|(\tilde{L} + \lambda s)^{-1}\|_{r, \infty} \|m\|_{\infty, r} < 1.$$

By Lemma 2.5 $((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}$ is a compact and positive operator on $L_T^q(\Omega \times \mathfrak{R})$. We have that

$$(2.7) \quad \|((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}|m|\|_{\infty, \infty} \leq 2\|(\tilde{L} + \lambda s)|_{W^q})^{-1}|m|\|_{\infty, \infty} < \frac{1}{2}.$$

Then $I - ((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}|m|$, as operator on $L_T^\infty(\Omega \times \mathfrak{R})$, has a bounded inverse. Since

$$(2.8) \quad \begin{aligned} & \left(I - ((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}|m| \right)^{-1} = \\ & = \sum_{j \geq 0} \left(((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}|m| \right)^j \end{aligned}$$

this inverse is positive.

If $u \in L_T^q(\Omega \times \mathfrak{R})$, let

$$(2.9) \quad w = \left(I - ((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}|m| \right)^{-1} ((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1} u.$$

Then

$$w - ((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}(|m|w) = ((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}u.$$

Thus $w \in W$ and $(\tilde{L} + \lambda(s - m))w = u$. Therefore (i) holds. From (2.8), (2.9) and (2.7), we obtain (ii). The compactness stated in (iii) follows from (2.8), since (by Lemma 2.5) $((\tilde{L} + \lambda(s + 2m^-))|_{W^q})^{-1}$ is a compact operator on $L_T^q(\Omega \times \mathfrak{R})$. ■

REMARK 2.10. If m is a T -periodic function in $C^\infty(\Omega \times \mathfrak{R})$ then for s large enough, $(L + \lambda(s - m))|_{C_T^1(\Omega \times \mathfrak{R})}^{-1}$ is a bounded operator on $C_T(\Omega \times \mathfrak{R})$. Let ϱ denote its spectral radius and let $\mu_m: \mathfrak{R} \rightarrow \mathfrak{R}$ be defined as in [H,1], p. 38, then for $\lambda \in \mathfrak{R}$, $\mu_m(\lambda)$ is the unique real number such that there exists $u_\lambda > 0$, $u_\lambda \in C^{2,1}(\Omega \times \mathfrak{R})$ satisfying $Lu_\lambda = \lambda mu_\lambda +$

$+ \mu_m(\lambda) u_\lambda$, therefore

$$\frac{1}{\lambda s + \mu_m(\lambda)} = \varrho$$

DEFINITION 2.11. *Let S be a bounded operator on $L^q(\Omega \times [0, T])$ and let E be a measurable subset of $\Omega \times [0, T]$. As in [S], we say that E is invariant relative to S if $Sf = 0$ a.e. in E whenever $f = 0$ a.e. on E and that S is irreducible if there are no nontrivial invariant subsets relative to S .*

In the following we identify $L_T^q(\Omega \times \mathfrak{R})$ with $L^q(\Omega \times [0, T])$.

LEMMA 2.12. (i) *Suppose $a \in L^r(\Omega \times [0, T])$ and let δ be a positive real number as in lemma 2.5, then $((\tilde{L} + a + \delta)|_{W^q})^{-1}$ is irreducible on $L^q(\Omega \times [0, T])$*

(ii) *Let (λ_1, λ_2) be a finite open interval with $\lambda_1 > 0$. Suppose $m \in L_T^r(\Omega \times \mathfrak{R})$ and let $s_0 = s_0(\lambda_1, \lambda_2, m)$ be defined as in lemma 2.6. Then for $\lambda \in (\lambda_1, \lambda_2)$ and $s > s_0$*

$$((\tilde{L} + \lambda(s - m))|_{W^q})^{-1}: L_T^q(\Omega \times \mathfrak{R}) \rightarrow L_T^q(\Omega \times \mathfrak{R})$$

is irreducible on $L^q(\Omega \times [0, T])$.

PROOF. To prove (i) we take $b \in C_T^\infty(\Omega \times \mathfrak{R})$ such that

$$\|b - a - \delta\|_r < \|((\tilde{L} + a + \delta)|_{W^q})^{-1}\|_{r, \infty}^{-1}$$

and note that

$$\begin{aligned} ((\tilde{L} + b)|_{W^q})^{-1} &= \\ &= \left[\sum_{j=0}^{\infty} (-1)^j ((\tilde{L} + a + \delta)|_{W^q})^{-1} (b - a - \delta)^j \right] ((\tilde{L} + a + \delta)|_{W^q})^{-1}. \end{aligned}$$

Then if $E \subset \Omega \times [0, T]$ is invariant relative to $((\tilde{L} + a + \delta)|_{W^q})^{-1}$ it is also invariant relative to $((\tilde{L} + b)|_{W^q})^{-1}$. But this last operator on $L_T^q(\Omega \times \mathfrak{R})$ is irreducible, indeed, pick $c \in \mathfrak{R}$ such that $b \leq c$. If $f \in L_T^q(\Omega \times \mathfrak{R})$ and $f > 0$ then $(\tilde{L} + b)^{-1}f \geq (\tilde{L} + c)^{-1}f$ and the right hand side of this inequality is positive a.e.

To prove (ii) we take $b \in L_T^\infty(\Omega \times \mathfrak{R})$ such that $\|b - m\|_r < 1$. Then, for

$\lambda \varepsilon[\lambda_1, \lambda_2]$, we have $\|\lambda((\tilde{L} + \lambda(s - m))_{|W^q})^{-1}\|_{\infty, \infty} \leq 1/2$. Thus

$$\begin{aligned} ((\tilde{L} + \lambda(s - b))_{|W^q})^{-1} &= \\ &= \sum_{j=0}^{\infty} (-1)^j \lambda^j \left(((\tilde{L} + \lambda(s - m))_{|W^q})^{-1} (m - b) \right)^j ((\tilde{L} + \lambda(s - m))_{|W^q})^{-1}. \end{aligned}$$

then (ii) follows from (i) and the identity

$$\begin{aligned} ((\tilde{L} + \lambda s)_{|W^q})^{-1} &= \\ &= \sum (-1)^j \lambda^j \left(((\tilde{L} + \lambda(s - b))_{|W^q})^{-1} b \right)^j ((\tilde{L} + \lambda(s - b))_{|W^q})^{-1}. \quad \blacksquare \end{aligned}$$

COROLLARY 2.13. *Let $s_0(m, \lambda_1, \lambda_2)$ be as in Lemma 2.6, suppose $s > s_0(m, \lambda_1, \lambda_2)$, and consider the operator $S = ((\tilde{L} + \lambda(s - m))_{|W^q})^{-1}$ on $L^q(\Omega \times (0, T))$, then its spectral radius ϱ is an algebraically simple positive eigenvalue which has an a.e. positive eigenfunction, and no other eigenvalue has a positive eigenfunction. Moreover it is also an eigenvalue with an a.e. positive eigenfunction for the adjoint operator.*

PROOF. S is compact, irreducible and positive and so is its adjoint $S^*: L^{q'}(\Omega \times (0, T)) \rightarrow L^{q'}(\Omega \times (0, T))$. Then the spectral radius of S and S^* are positive (See [Z], p. 410) and the theorem follows from theorem 8 and lemma 16 in [S]. $\blacksquare$

DEFINITION 2.14. *Given $m \in L_T^r(\Omega \times \mathfrak{R})$ and $\lambda > 0$. Pick λ_1, λ_2 such that $0 < \lambda_1 < \lambda < \lambda_2$. Let $s_0(m, \lambda_1, \lambda_2)$ be as in Lemma 2.6 and pick $s > s_0(m, \lambda_1, \lambda_2)$. We define $\mu_m(\lambda)$ by $1/(\lambda s + \mu(\lambda)) = \varrho$, where ϱ is the spectral radius of*

$$((\tilde{L} + \lambda(s - m))_{|W^q})^{-1} \in B(L_T^q(\Omega \times \mathfrak{R})).$$

It is easy to check that $\mu_m(\lambda)$ does not depend on the particular λ_1, λ_2 and s chosen. Additionally we define $\mu_m(0) = 0$ for the Neumann boundary condition and $\mu_m(0)$ equal to the principal eigenvalue of L^{-1} for the Dirichlet boundary condition.

REMARK 2.15. Suppose $B(u) = u_{|\partial\Omega \times \mathfrak{R}}$. Let $\lambda_0 > 0$ be the principal eigenvalue of L and let $u_0 > 0$ be such that $Lu_0 = \lambda_0 u_0$. Consider $\tilde{L}^{-1}$ as operator on $L_T^q(\Omega \times \mathfrak{R})$. Then (see [S], Theorem 8), λ_0^{-1} is a simple eigen-

value of $\tilde{L}^{-1*}: L_T^{q'}(\Omega \times \mathfrak{R}) \rightarrow L_T^{q'}(\Omega \times \mathfrak{R})$ with a unique positive eigenfunction Ψ_D satisfying $\int_{\Omega \times [0, T]} \Psi_D = 1$. Similarly, for the Neumann condition, let Ψ_N be the positive eigenfunction of $(\tilde{L} + 1)^{-1*}: L_T^{q'}(\Omega \times \mathfrak{R}) \rightarrow L_T^{q'}(\Omega \times \mathfrak{R})$ with eigenvalue 1 satisfying $\int_{\Omega \times [0, T]} \Psi_N = 1$.

LEMMA 2.16. *Suppose $0 < \lambda_1 < \lambda_2$ and let $\{m_j\}_{j=1}^\infty$ be a sequence in $C_T^\infty(\overline{\Omega} \times \mathfrak{R})$ such that m_j converges to m in $L_T^r(\Omega \times \mathfrak{R})$. Then $\{\mu_{m_j}\}_{j=1}^\infty$ converges to μ_m uniformly on $[\lambda_1, \lambda_2]$.*

PROOF. We choose $s > s_0(\|m\|_r, \lambda_1, 2\lambda_2)$. For each $j \in N$ and $\lambda \in [\lambda_1, 2\lambda_2]$ there exists $u_{j,\lambda} \in C^{2,1}(\overline{\Omega} \times \mathfrak{R})$, $u_{j,\lambda}$ real analytic in λ , $u_{j,\lambda} > 0$ in $\Omega \times \mathfrak{R}$ such that

$$\begin{cases} (L + \lambda(s - m_j))u_{j,\lambda} = (\lambda s + \mu_{m_j}(\lambda))u_{j,\lambda}, \\ Bu_j(\lambda) = 0, \quad \|u_{j,\lambda}\|_\infty = 1 \end{cases}$$

(see [H,1], Lemma 15.1). Since $((L + \lambda(s - m_j))|_{W^q})^{-1}$ is positive, we have $\mu_{m_j}(\lambda) \geq -\lambda s$. Suppose the Dirichlet condition and let Ψ_D be as in remark 2.15. We derive with respect to λ at $\lambda = 0$ the identity

$$\langle u_{j,\lambda}, \Psi_D \rangle = \langle L^{-1}(\lambda m_j + \mu_{m_j}(\lambda))u_{j,\lambda}, \Psi_D \rangle$$

to obtain $d\mu_{m_j}/d\lambda|_{\lambda=0} = -\langle m_j u_0, \Psi_D \rangle / \langle u_0, \Psi_D \rangle$ where u_0 is a positive eigenfunction of L with eigenvalue λ_0 . Analogously we get $d\mu_{m_j}/d\lambda|_{\lambda=0} = -\langle m_j, \Psi_N \rangle / \langle 1, \Psi_N \rangle$ for the Neumann case. In either case the sequence $\{d\mu_{m_j}/d\lambda|_{\lambda=0}\}_{j=1}^\infty$ is bounded from above. Also, since μ_{m_j} is concave and satisfies $\mu_{m_j}(\lambda) \geq -\lambda s$ on $[\lambda_1, 2\lambda_2]$, it is easy to see that $\{d\mu_{m_j}/d\lambda|_{\lambda=\lambda_2}\}_{j=1}^\infty$ is bounded from below. Then $\{(d\mu_{m_j}/d\lambda)(\lambda)\}_{j \in N, \lambda \in [\lambda_1, \lambda_2]}$ is bounded, so, by the Ascoli-Arzelà theorem there exists subsequence $\{\mu_{m_{j_k}}\}_{k \in N}$ that converges uniformly on $[\lambda_1, \lambda_2]$. Let $\sigma(\lambda) = \lim_{k \rightarrow \infty} \mu_{m_{j_k}}(\lambda)$, then $\mu_m(\lambda) = \sigma(\lambda)$ for $\lambda \in [\lambda_1, \lambda_2]$. Indeed, let $v_{j,\lambda} = u_{j,\lambda} / \|u_{j,\lambda}\|_\infty$. From

$$(2.17) \quad v_{j,\lambda} = L^{-1}(\mu_{m_j}(\lambda) + \lambda m_j)v_{j,\lambda}$$

and since $\tilde{L}^{-1}: L_T^p(\Omega \times \mathfrak{R}) \rightarrow C_T(\Omega \times \mathfrak{R})$ is compact, there exists an $C_T(\Omega \times \mathfrak{R})$ convergent subsequence $\{v_{j_k,\lambda}\}_{k \in N}$. Let $v_\lambda = \lim_{k \rightarrow \infty} v_{j_k,\lambda}$, we note that $v_\lambda \neq 0$. Taking limits in (2.17) we get $v_\lambda = \tilde{L}^{-1}(\sigma(\lambda) + \lambda m)v_\lambda$, and so, Corollary 2.13 implies $\mu_m(\lambda) = \sigma(\lambda)$. Finally we observe that the

above argument shows that any subsequence of $\{\mu_{m_j}(\lambda)\}$ has a subsequence that converges to $\mu_m(\lambda)$ and so the sequence itself converges to $\mu_m(\lambda)$. ■

LEMMA 2.18. *If $m \in L_T^r(\Omega \times (0, T))$ then μ_m is continuous on $[0, \infty)$ and analytic on $(0, \infty)$.*

PROOF. Let $\{m_j\}_{j=1}^\infty$ be a sequence in $C_T^\infty(\overline{\Omega} \times \mathfrak{R})$ that converges to m in $L_T^r(\Omega \times \mathfrak{R})$, we pick $s > s_0(m, \lambda_1, \lambda_2)$. By lemma 2.14 μ_m is continuous on $(0, \infty)$. The continuity of μ_m at 0 follows from the facts that μ_m is the pointwise limit of a sequence of concave functions and that $\{d\mu_{m_j}/d\lambda|_{\lambda=0}\}_{j=1}^\infty$ has an upper bound.

It remains to see the analyticity. Let J be the inclusion from W^q into $L_T^q(\Omega \times \mathfrak{R})$. We consider W^q as a Banach space, with the topology inherited from the graph norm, then $\tilde{L} + \lambda(s - m): W^q \rightarrow L_T^q(\Omega \times \mathfrak{R})$ is a Banach space isomorphism and $\tilde{L} + \lambda(s - m) - (\lambda s + \mu_m(\lambda))J$ is a compact perturbation of it, therefore it is a Fredholm operator of index 0, then

$$\begin{aligned} \dim \text{Ker}(\tilde{L} + \lambda(s - m) - (\lambda s + \mu_m(\lambda))J) &= \\ &= \text{co dim}(\text{Im}(\tilde{L} + \lambda(s - m) - (\lambda s + \mu_m(\lambda))J)). \end{aligned}$$

Corollary 2.13 implies that $\lambda s + \mu_m(\lambda)$ is a J simple eigenvalue of $\tilde{L} + \lambda(s - m)$. It follows from Lemma 1.3 in [C-R] that there exists $\varepsilon > 0$ such that if $U \in B(W^q, L_T^q(\Omega \times \mathfrak{R}))$ and $\|\tilde{L} + \lambda(s - m) - U\| < \varepsilon$ then U has a unique eigenvalue $\varrho(U)$ satisfying $|\varrho(U) - (\lambda s + \mu_m(\lambda))| < \varepsilon$. Moreover, $\varrho(U)$ is a J simple eigenvalue and the application $U \rightarrow \varrho(U)$ is analytic. For λ' close enough to λ , $\lambda' s + \mu_m(\lambda')$ is a J simple eigenvalue of the operator $\tilde{L} + \lambda'(s - m)$. Since μ_m is continuous on $(0, \infty)$ we must have $\varrho(\tilde{L} + \lambda'(s - m)) = \mu_m(\lambda')$, so the analyticity follows. ■

REMARK 2.19. For $m \in L_T^r(\Omega \times \mathfrak{R})$ and $t \in \mathfrak{R}$, let $\tilde{m}(t) = \text{ess sup}_{x \in \Omega} m(x, t)$, $m^*(t) = \text{ess inf}_{x \in \Omega} m(x, t)$, and $P(m) = \int_0^T \tilde{m}(t) dt$. If m is independent of x , then $\mu_m(\lambda) = \mu_m(0) - \lambda(P(m)/T)$. Indeed, this is true if in addition $m \in C_T^\infty(\Omega \times \mathfrak{R})$ (see [H,1], Lemma 15.3) and so, by Lemma 2.16, for an arbitrary $m \in L_T^r(\Omega \times \mathfrak{R})$.

3. – The Main results.

LEMMA 3.1. *Let $m_1 > m_2$ be functions in $L_T^r(\Omega \times \mathfrak{R})$. Then $\mu_{m_1}(\lambda) < \mu_{m_2}(\lambda)$, for all $\lambda > 0$.*

PROOF. If $\lambda > 0$ we pick $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 < \lambda < \lambda_2/2$. We also pick $s > \max\{s_0(m_1, \lambda_1, \lambda_2), s_0(m_2, \lambda_1, \lambda_2)\}$. We set $T_1, T_2: L_T^q(\Omega \times \mathfrak{R}) \rightarrow L_T^r(\Omega \times \mathfrak{R})$ defined by $T_i = ((\tilde{L} + \lambda(s - m_i))|_{W^q})^{-1}$. Then T_1 and T_2 are positive operators and, by (iii) of Lemma 2.6

$$T_1 - T_2 = \lambda((\tilde{L} + \lambda(s - m_1))|_{W^q})^{-1}(m_1 - m_2)((\tilde{L} + \lambda(s - m_2))|_{W^q})^{-1} > 0.$$

Thus $\varrho(T_1) \geq \varrho(T_2)$. Suppose for contradiction that $\varrho(T_1) = \varrho(T_2)$ and let ϱ denote this value. Let $u \in W^q$ $u > 0$ such that $T_2 u = \varrho u$, then $u(x, t) > 0$ a. e. $(x, t) \in \Omega \times \mathfrak{R}$ (see [S], Lemma 16) and then

$$\begin{aligned} \frac{1}{\varrho} u &= (\tilde{L} + \lambda(s - m_2)) u = \\ &= (\tilde{L} + \lambda(s - m_1)) u + \lambda(m_1 - m_2) u > (\tilde{L} + \lambda(s - m_1)) u. \end{aligned}$$

Thus $T_1 u > \varrho u$. Contradiction. ■

Let $\Gamma \in C^2(\mathfrak{R}, \Omega)$ be a T -periodic curve in Ω and Ω_0 a domain in $\mathfrak{R}^n$ with C^∞ boundary such that $\Gamma(t) + \Omega_0 \subset \Omega$ for every $t \in \mathfrak{R}$. We define

$$B_{\Gamma, \Omega_0} = \{(\Gamma(t) + w, t): w \in \Omega_0, t \in [0, T]\}$$

and

$$P_{\Gamma, \Omega_0}(m) = \int_{B_{\Gamma, \Omega_0}} m.$$

THEOREM 3.2. *Let $\Gamma \in C^2(\mathfrak{R}, \Omega)$ be a T -periodic curve in Ω and Ω_0 a domain in $\mathfrak{R}^n$ with C^∞ boundary such that $\Gamma(t) + \Omega_0 \subset \Omega$ for every $t \in \mathfrak{R}$. Suppose $m \in L_T^r(\Omega \times \mathfrak{R})$. Assume in addition that $a_{i,j}$ has continuous spacial derivates $\partial a_{i,j} / \partial x_i$, $1 \leq i, j \leq n$. Then we have*

(i) *If $P_{\Gamma, \Omega_0}(m) > 0$, then there exists $\lambda^D > 0$ and $u^D > 0$, solution of the periodic eigenvalue Dirichlet problem $\tilde{L}u = \lambda mu$ in $\Omega \times \mathfrak{R}$, $u|_{\partial\Omega \times \mathfrak{R}} = 0$.*

(ii) *Suppose the Neumann boundary condition and $\tilde{m} \neq m^*$. Let*

Ψ_N be defined as in remark 2.15. If $P_{\Gamma, \Omega_0}(m) > 0$ and $\langle \Psi^N, m \rangle < 0$ then there exists $\lambda^N > 0$ and $u^N > 0$ that solve the problem $\tilde{L}u = \lambda mu$ in $\Omega \times \mathfrak{R}$, $\partial u / \partial \nu|_{\partial \Omega \times \mathfrak{R}} = 0$.

PROOF. We pick $c \in \mathfrak{R}$ such that $P_{\Gamma, \Omega_0}(m) > c > 0$. Let $\{m_j\}_{j \in N}$ be a sequence of functions in $C_T^\infty(\mathfrak{R}^n \times \mathfrak{R})$ such that $\text{supp } m_j \subset \Omega \times \mathfrak{R}$, $\lim_{j \rightarrow \infty} m_j = m$ in $L_T^r(\Omega \times \mathfrak{R})$. Without lost of generality we can assume that $P_{\Gamma, \Omega_0}(m_j) > c$ for all $j \in N$. In the Neumann case we can also assume that $\langle \Psi, m_j \rangle < 0$ and $\tilde{m}_j \not\equiv m_j^*$. Let λ_j^D, u_j^D (λ_j^N, u_j^N) be the principal eigenvalue and the corresponding positive eigenvector for the Dirichlet (Neumann) boundary condition (see [H,1], Theorems 16.1 and 16.3) corresponding to the weight m_j such that $\|u_j^D\|_\infty = 1, (\|u_j^N\|_\infty = 1)$.

We first consider the Dirichlet case. We introduce the change of coordinates given by $\Phi: \Omega \times \mathfrak{R} \rightarrow \mathfrak{R}^n \times \mathfrak{R}$ where $\Phi(w, t) = (w - \Gamma(t), t)$. In the new coordinates the equation $Lu_j = \lambda_j m_j u_j$ on $\Omega \times \mathfrak{R}$ becomes $L^\Phi u_j^\Phi = \lambda_j m_j^\Phi u_j^\Phi$ on $\Phi(\Omega \times \mathfrak{R})$, where $m_j^\Phi = m_j \circ \Phi^{-1}$ and $u_j^\Phi = u_j \circ \Phi^{-1}$.

Take $\sigma_j > 0$ and $v_j > 0$ satisfying $L^\Phi v_j = \sigma_j m_j v_j$ on $\Omega_0 \times \mathfrak{R}$, v_j T periodic and $v_j|_{\partial \Omega_0 \times \mathfrak{R}} = 0$ and $\|v_j\|_\infty = 1$. Since $\partial a_{i,j} / \partial x_i \in C(\bar{\Omega} \times \mathfrak{R})$ and $\int_{\Omega_0 \times \mathfrak{R}} m_j^\Phi > c$ we can apply proposition 3.1 in [H,2] to obtain that the sequence $\{\sigma_j\}$ is bounded. Reasoning as in [H,1] Lemma 15.4, we see that $\lambda_j < \sigma_j$ and so $\lambda_j \leq c$ for some $c > 0$ and all $j \in N$. Then we can find a subsequence (which we still denote $\{\lambda_j\}$) that converges to some $\lambda^D \geq 0$. Use that $\{\lambda_j^D m_j u_j^D\}_{j \in N}$ is bounded in $L_T^r(\Omega \times \mathfrak{R})$, $u_j^D = L^{-1}(\lambda_j m_j u_j^D)$ and that $\tilde{L}^{-1}$ is a compact operator on $L_T^r(\Omega \times \mathfrak{R}) \rightarrow C_T(\Omega \times \mathfrak{R})$ to conclude that there exists a subsequence $u_{j_k}^D$ that converges to some u^D in $L_T^\infty(\Omega \times \mathfrak{R})$. Then $u^D = \tilde{L}^{-1}(\lambda^D m u^D)$. Moreover, $\lambda^D > 0$, otherwise $(\tilde{L} + 1)^{-1}$ would have 2 positive eigenvalues with positive eigenfunctions.

Let's consider the Neumann case. It follows from $\lambda_j^N < \lambda_j^D$ that there exists a convergent subsequence $\lambda_{j_k}^N$. Let $\lambda^N = \lim \lambda_{j_k}^N$. Moreover, we can assume that $u_{j_k}^N$ converges (in $L_T^\infty(\Omega \times \mathfrak{R})$) to some u^N . Then we get, as above, $u^N \in W$ and $\tilde{L}u^N = \lambda^N m u^N$. To see that $\lambda^N > 0$ assume for contradiction that $\tilde{L}u^N = 0$. Then $u_N \equiv 1$, and we have

$$\langle u_j^N, \Psi \rangle = \langle (\tilde{L} + 1)^{-1}(\lambda_j^N m_j + 1) u_j^N, \Psi \rangle = \langle \lambda_j^N m_j u_j^N, \Psi \rangle + \langle u_j^N, \Psi \rangle.$$

Thus $\langle m_j u_j^N, \Psi^N \rangle = 0$ and then $\langle m, \Psi^N \rangle = 0$. Contradiction. ■

THEOREM 3.3. *Let Γ and Ω_0 be as above and $m \in L_T^r(\Omega \times \mathfrak{R})$. We have*

(i) *Assume the Dirichlet Boundary condition and $P_{\Gamma, \Omega_0}(m) > 0$. Then there exists at most one $\lambda^D > 0$ such that $\mu_m(\lambda^D) = 0$.*

(ii) *Suppose the Neumann boundary condition, $P_{\Gamma, \Omega_0}(m) > 0$, $\langle \Psi^N, m \rangle < 0$ and $\tilde{m} \neq m^*$ then there exists at most one $\lambda^N > 0$ such that $\mu_m(\lambda^N) = 0$.*

PROOF. (i) follows from the facts that μ_m is concave on $[0, \infty)$ and $\mu_m(0) > 0$. To see (ii) suppose that there exist $\lambda_1 > 0$, $\lambda_2 > 0$ such that $\mu_m(\lambda_1) = \mu_m(\lambda_2) = 0$. Since μ_m is concave and analytic we must have $\mu \equiv 0$ on $[0, \infty)$. Since $m^* \neq \tilde{m}$, $m < \tilde{m}$ then given $\varepsilon > 0$ there exists $h \in L_T^r(\Omega \times \mathfrak{R})$, $h > 0$ such that $m + h < \tilde{m}$ and $\|h\|_r < \varepsilon$. Moreover we can choose h such that $m + h$ is not function of t alone. For ε small enough we must have $P(m + h) > 0$ and $\langle \Psi, m + h \rangle < 0$ so by theorem 3.2 there exists $\lambda > 0$ such that $\mu_{m+h}(\lambda) = 0$, but by Lemma 3.1 $\mu_{m+h}(\lambda) < \mu_m(\lambda) = 0$. Contradiction. ■

REMARK 3.4. Let m , $a_{i,j}$ $1 \leq i, j \leq n$ be as in Theorem 3.2 and let $\{m_j\}_{j=1}^\infty$ be a sequence in $C_T^\infty(\Omega \times \mathfrak{R})$, $\sup p m_j \subset K_j \times \mathfrak{R}$ for some compact subset $K_j \subset \Omega_j$. Then the sequence $\{\lambda_j\}$ of principal eigenvalues associated to the weights m_j converges to the principal eigenvalue λ corresponding to the weight m . Indeed, for every subsequence $\{\lambda_{j_k}\}$ we can prove, as in theorem 3.2, that there exists a subsequence $\{\lambda_{j_{k_s}}\}$ convergent to some λ satisfying $\mu_m(\lambda) = 0$. So the assertion follows from lemma 3.8. A diagonal process gives us the following

COROLLARY 3.5. *Let m , $a_{i,j}$ $1 \leq i, j \leq n$ be as in Theorem 3.2 and let $\{m_j\}_{j=1}^\infty$ be a sequence in $L_T^r(\Omega \times \mathfrak{R})$ such that m_j converges to m in $L_T^r(\Omega \times \mathfrak{R})$. Then the sequence $\{\lambda_j\}$ of principal eigenvalues associated to the weights m_j converges to the principal eigenvalue λ corresponding to the weight m .*

We set $\pi: \mathfrak{R}^n \times \mathfrak{R} \rightarrow \mathfrak{R}$ defined by $\pi(x, t) = t$. If $B \subset \mathfrak{R}^n \times \mathfrak{R}$ and $t \in \mathfrak{R}$ we put $B_t = \{x \in \mathfrak{R}^n: (x, t) \in B\}$. If $\Omega \subset \mathfrak{R}^n$ we define for $\delta > 0$, $\Omega_\delta = \{x \in \Omega: \text{dist}(x, \partial\Omega) > \delta\}$.

LEMMA 3.6. *Suppose that $m \in L_T^r(\Omega \times \mathfrak{R})$ has an upper bound and that $\int_a^b \tilde{m}(t) dt > c$. Suppose also that $\delta > 0$ is such that $\Omega_\delta \neq \emptyset$. Then*

there exists a finite family $\{Q_r\}_{1 \leq r \leq N}$ of pairwise disjoint congruent open cubes with edges of length l and parallel to the coordinate axis such that

(1) $l \leq (\delta/2(n+1))$ and $Q_r \subseteq \Omega_{\delta/2} \times [a, b]$ for $1 \leq r \leq N$.

(2) The family $\{\pi(Q_r)\}_{r=1}^N$ is pairwise disjoint.

(3) $\sum_{r=1}^N |\pi(Q_r)| = b - a$.

(4) $\int_{\bigcup_{1 \leq r \leq N} Q_r} m > cl^n$.

PROOF. Without loss of generality we can assume that $m \leq 1$. Let $\tilde{m}_j$ defined as in remark 2.19. It is easy to see that $\tilde{m}_j$ is a measurable function on $[a, b]$. Also $\tilde{m}_j(t) \leq \tilde{m}_{j+1}(t)$ and $\lim_{j \rightarrow \infty} \tilde{m}_j(t) = \tilde{m}(t)$. Then we can fix k large such that $\int_a^b \tilde{m}_j(t) dt > c$, $|\Omega_k| \geq |\Omega|/2$ and $k \geq 1/\delta$.

For $0 < \theta < \delta < \eta$ we define

$$E(\eta, \theta) = \{(x, t) \in \Omega_{1/k} \times [a, b]: m(x, t) \geq \tilde{m}_k(t) - \eta + \theta\}$$

then $m - (\tilde{m} - \eta) \geq \theta$ on $E(\eta, \theta)$. Let $E^d(\eta, \theta)$ be the set of the points $(x, t) \in E(\eta, \theta)$ such that (x, t) is a Lebesgue point for $m(x, t) - (\tilde{m}_k(t) - \eta)$ and, for $r > 0$, let $E^{(r)}(\eta, \theta)$ be the set of the points (x, t) in $E^d(\eta, \theta)$ such that

$$\frac{1}{|Q|} \int_Q (m - (\tilde{m} - \eta)) \geq \frac{\theta}{2}$$

holds for every open cube Q with edges parallel to the coordinate axis with diameter less than $1/r$ containing (x, t) . $E^{(r)}(\eta, \theta)$ is a measurable set. Note that $E^{(r)}(\eta, \theta) \subseteq E^{(s)}(\eta, \theta)$ if $r < s$. Also $E^d(\eta, \theta) \subseteq \bigcup_{r \in \mathbb{N}} E^{(r)}(\eta, \theta)$. Moreover, from $|E^{(r)}(\eta, \theta)_t| \neq 0$ a.e. $t \in [a, b]$ it follows that $|\pi(E^d(\eta, \theta))| = b - a$. Then

$$\lim_{r \rightarrow \infty} |\pi(E^{(r)}(\eta, \theta))| \geq |\pi(E^d(\eta, \theta))| = b - a.$$

Given $\varepsilon > 0$ we fix $r > 2k$ such that $|\pi(E^{(r)}(\eta, \theta))| \geq b - a - \varepsilon$, also we choose $0 < l < 1/r(n+1)$ such that $Nl = b - a$ for some natural number N . Let $\{t_i\}_{0 \leq i \leq N}$ be the partition of $[a, b]$ given by $t_i = a + il$, $i = 0, \dots, N$. Let I be the set of the indices i , $0 \leq i \leq N$ such that the strip $\mathbb{R}^n \times (t_{i-1}, t_i)$ intersects $E^{(r)}(\eta, \theta)$ and let I^c be its com-

plement. For $i \in I$ we choose a cube Q_i such that $Q_i \cap E^{(r)}(\eta, \theta) \neq \emptyset$ and $\pi(Q_i) = (t_{i-1}, t_i)$. Since $E^{(r)}(\eta, \theta) \subseteq \Omega_{1/k}$ and $\text{diam}(Q_i) < 1/2k\sqrt{n+1}$ we have that $Q_i \subseteq \Omega_{1/(2k)} \times (t_{i-1}, t_i)$. Since $|\pi E^{(r)}(\eta, \theta)| \geq b - a - \varepsilon$, I^c satisfies $\sum_{i \in I^c} (t_i - t_{i-1}) < \varepsilon$. Let $F = \bigcup_{i \in I^c} (t_{i-1}, t_i)$, then $|\Omega \times F| \leq \varepsilon |\Omega|$. Now

$$\int_{\Omega \times F} |m| < \|m\|_r |\Omega \times F|^{1/r'} \leq \varepsilon^{1/r'} \|m\|_r |\Omega|^{1/r'}$$

with r' defined by $1/r + 1/r' = 1$. To cover $\mathfrak{R}^n$ we use cubes with vertices on the points of the lattice $l\mathbb{Z}^n$. Let $Q_1^*, \dots, Q_M^*$ be the cubes in the mesh meeting Ω_k , so $Q_j^* \subseteq \Omega$, $1 \leq j \leq M$. Since $|\Omega| \leq 2|\Omega_k| \leq 2Ml^n$, then $M \geq |\Omega|/(2l^n)$. Since

$$\|m\|_r \varepsilon^{1/r'} |\Omega|^{1/r'} \geq \int_{\Omega \times F} |m| \geq \sum_{s=1}^M \int_{Q_s^* \times F} |m|$$

we have, for some s , $1 \leq s \leq M$ that

$$\int_{Q_s^* \times F} |m| \leq \|m\|_r \varepsilon^{1/r'} |\Omega|^{1/r'} M^{-1} \leq 2l^n \|m\|_r \varepsilon^{1/r'} |\Omega|^{-1/r'}.$$

We define, for $i \in I^c$, $Q_i = Q_s^*$. Then, for $i \in I$

$$\int_{Q_i} m \geq \int_{Q_i} \tilde{m} - \eta |Q_i| + \theta |Q_i|/2 = l^n \left(\int_{t_{i-1}}^{t_i} \tilde{m}_k(t) dt - \eta l + \theta l/2 \right).$$

Then

$$\begin{aligned} \sum_{i \in I} \int_{Q_i} m &\geq l^n \left(\sum_{i \in I} \int_{t_{i-1}}^{t_i} \tilde{m}_k(t) dt - \eta l \text{Card}(I) - \frac{\theta l \text{Card}(I)}{2} \right) \geq \\ &\geq l^n \sum_{i \in I} \int_{t_{i-1}}^{t_i} \tilde{m}_k(t) dt - l^n \left(\eta(b-a) + \theta \frac{b-a}{2} \right). \end{aligned}$$

Since $\sum_{i \in I^c} \left| \int_{Q_i} m \right| \leq 2l^n \|m\|_r \varepsilon^{1/r'} |\Omega|^{-1/r'}$ we get

$$\sum_{i=1}^N \int_{Q_i} m \geq l^n \left(\sum_{i \in I} \int_{t_{i-1}}^{t_i} \tilde{m}_k(t) dt - \eta(b-a) - \theta \frac{b-a}{2} - 2\|m\|_r \varepsilon^{1/r'} |\Omega|^{-1/r'} \right)$$

and so $\sum_{i=1}^N \int_{Q_i} m \geqslant cl^n$ for η , θ , and ε small enough. ■

Now, reasoning as in remarks 4.2, 4.3 and lemma 4.4 of [G-L-P], we obtain

REMARK 3.7. Suppose that $m \in L_T^r(\Omega \times \mathfrak{R})$ is bounded from above and that $\int_0^T \tilde{m}(t) dt > 0$. Then there exists a T -periodic curve $\gamma \in C^2(\mathfrak{R}, \Omega)$ and a domain Ω_0 with smooth boundary such that

(i) $\gamma(t) + \Omega_0 \subset \Omega$ for every $t \in \mathfrak{R}$.

(ii) $P_{\Gamma, \Omega_0}(m) > 0$.

THEOREM 3.8. Let m , $a_{i,j}$ $1 \leqslant i, j \leqslant n$ be as in Theorem 3.2. Suppose that there exists $\lambda > 0$, $u \in D(\tilde{L})$ $u > 0$ solution of the periodic eigenvalue problem $\tilde{L}u = \lambda mu$, $B(u) = 0$, where either $B(u) = u|_{\partial\Omega \times \mathfrak{R}}$ or $B(u) = \partial u / \partial \nu|_{\partial\Omega \times \mathfrak{R}}$. If $B(u) = \partial u / \partial \nu|_{\partial\Omega \times \mathfrak{R}}$ we also assume that $m \neq \tilde{m}$, if $B(u) = \partial u / \partial \nu|_{\partial\Omega \times \mathfrak{R}}$. Then there exists a T -periodic curve $\gamma \in C^2(\mathfrak{R}, \Omega)$ and a domain Ω_0 with smooth boundary satisfying $\gamma(t) + \Omega_0 \subset \Omega$, $t \in \mathfrak{R}$ and such that $P_{\Gamma, \Omega_0}(m) > 0$. Moreover, if $B(u) = \partial u / \partial \nu|_{\partial\Omega \times \mathfrak{R}}$, we also have $\langle \Psi^N, m \rangle < 0$.

PROOF. Taking into account lemma 3.1 and remark 2.17 and reasoning as in the regular case (see [H,1], Lemma 15.6) we obtain, in both cases, $P(m) > 0$. Then lemma 3.6 gives us the first assertion of the theorem. To see that $\langle \Psi^N, m \rangle < 0$ we choose $m_j \in C_{c,T}^\infty(\Omega \times \mathfrak{R})$, $j \in N$, such that m_j converges to m in $L_T^r(\Omega \times \mathfrak{R})$. We pick $\lambda_1 < \lambda$. Then $\mu_m(\lambda_1) > 0$, therefore, by lemma 2.16, $\mu_{m_j}(\lambda_1) > \mu_m(\lambda_1)/2$ for all large enough j . Therefore

$$-\langle \Psi^N, m_j \rangle = \mu'_{m_j}(0) \geqslant \frac{\mu_{m_j}(\lambda_1)}{\lambda_1} > \frac{\mu_m(\lambda_1)}{2\lambda_1}$$

and then $\langle \Psi^N, m \rangle \leqslant -(\mu_m(\lambda_1)/2\lambda_1) < 0$. ■

REMARK 3.9. Let m , $a_{i,j}$ $1 \leqslant i, j \leqslant n$ be as in Theorem 3.2 and let M be the operator multiplication by m . Suppose either the Dirichlet condition or the Neumann condition. Taking into account corollary 2.13 and lemma 2.18 we have, with the same proof as in the regular case, (see

[H,1], Lemma 16.9) that the positive principal eigenvalue is an M simple eigenvalue of $\tilde{L}$.

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Manoscritto pervenuto in redazione il 13 marzo 1996.