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Generalized Solutions of Time Dependent Impulsive Control Systems (*).

CHANG EON SHIN - RYU JI HYUN (**)

ABSTRACT - This paper is concerned with the impulsive Cauchy problem

$$\dot{x}(t) = f(t, x) + g(t, x) \dot{u}(t), \quad t \in [0, T], \quad x(0) = \bar{x}$$

where u is a possibly discontinuous control function and the vector fields $f, g: \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ are measurable in t and Lipschitz continuous in x . If g is smooth w.r.t. the variable x and satisfies $\|g(t, \cdot) - g(s, \cdot)\|_{\infty} \leq \phi(t) - \phi(s)$, for some increasing function ϕ and every $s < t$, we show that the above Cauchy problem is well posed as u ranges in the space $L^1(d\phi)$.

1. Introduction.

Consider the Cauchy problem for an impulsive control system of the form

$$(1.1) \quad \dot{x}(t) = f(t, x) + g(t, x) \dot{u}(t), \quad t \in [0, T], \quad x(0) = \bar{x} \in \mathbb{R}^n,$$

where u is a scalar control function and the dot denotes a derivative w.r.t. time. We assume that the vector fields $f, g: \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ are bounded, measurable in t and Lipschitz continuous in x , so that

$$(1.2) \quad |f(t, x)| \leq M, \quad |g(t, x)| \leq M,$$

$$(1.3) \quad |f(t, x) - f(t, y)| \leq L|x - y|, \quad |g(t, x) - g(t, y)| \leq L|x - y|,$$

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for some constants M, L . Under these conditions, for any continuously differentiable scalar function u , the right hand side of (1.1) is measurable in t and Lipschitz in x . Therefore, a well known theorem of Carathéodory [1] provides the existence and uniqueness of the corresponding solution $t \mapsto x(t, u)$. Aim of this paper is to show that, under suitable assumptions on g , the map $u \mapsto x(T, u)$ can be continuously extended to a much larger space of (possibly discontinuous) control functions. Besides (1.2)-(1.3), let g be twice continuously differentiable w.r.t. x , say

$$(1.4) \quad \|g(t, \cdot)\|_{\mathcal{C}^2} \doteq \\ \doteq \sup_x \left\{ |g(t, x)| + \sum_{i=1}^n \left| \frac{\partial g(t, x)}{\partial x_i} \right| + \sum_{i,j=1}^n \left| \frac{\partial^2 g(t, x)}{\partial x_i \partial x_j} \right| \right\} \leq M.$$

Moreover, we shall assume that the total variation of g w.r.t. time is bounded:

$$(1.5) \quad \|g(t, \cdot) - g(s, \cdot)\|_{\mathcal{C}^2} \leq \phi(t) - \phi(s), \quad 0 \leq s < t \leq T$$

for some non-decreasing function ϕ . Observe that, if u is a $\mathcal{C}^1$ function, the solution of (1.1) is not affected by changing g on a set of times of measure zero. For simplicity, we shall thus assume that both g and ϕ are right continuous functions of time. By possibly replacing ϕ with

$$\tilde{\phi}(t) \doteq \begin{cases} 0 & \text{if } t < 0, \\ 1 + t + \phi(t) & \text{if } 0 \leq t < T, \\ 2 + T + \phi(T) & \text{if } t \geq T, \end{cases}$$

it is not restrictive to assume that

$$(1.6) \quad \phi(0+) - \phi(0-) \geq 1, \quad \phi(T+) - \phi(T-) \geq 1, \quad \dot{\phi}(t) \geq 1 \text{ a.e.}$$

By (1.6), the positive Radon measure $d\phi$ contains an atom at $t = 0$ and at $t = T$, and satisfies $d\phi \geq dx$, where dx denotes the standard Lebesgue measure. We can now state the main result of this paper.

THEOREM 1.1. *Consider a set of bounded, measurable control functions of the form $\mathcal{U} \doteq \{u: [0, T] \mapsto [-M_1, M_1] \mid u \in \mathcal{C}^1\}$. For $u \in \mathcal{U}$, call $x(t, u)$ the corresponding solution of the Cauchy problem (1.1). Then,*

under the assumptions (1.2)-(1.6), the map $u \mapsto x(T, u)$ satisfies

$$(1.7) \quad |x(T, u) - x(T, v)| \leq C \int_0^T |u(t) - v(t)| d\phi(t),$$

for some constant C and all $u, v \in \mathcal{U}$.

As a consequence, the map $x(T, u)$ can be uniquely extended by continuity to the closure of $\mathcal{U}$ in the space $L^1(d\phi)$. This provides a natural definition of solution of (1.1) also for a discontinuous control u ,

$$x(T, u) \doteq \lim_{n \rightarrow \infty} x(T, v_n),$$

where $\{v_n\}_{n \geq 1}$ is any bounded sequence of $\mathcal{C}^1$ functions, tending to u in the space $L^1(d\phi)$.

REMARK 1.2. In the case where g is a piecewise smooth function of t, x , with finitely many jumps at times $0 = t_0 < t_1 < \dots < t_n = T$, one can always construct a function ϕ such that (1.5) holds. Indeed, for suitable constants C_1, C_2 , one can take

$$\phi(t) \doteq C_1 t + C_2 \cdot \sup \{k; t_k \leq t\}.$$

REMARK 1.3. Our results can be extended to systems of the form

$$\dot{x} = f(t, x, u) + g(t, x, u) \dot{u}.$$

Indeed, the dependence on u is easily removed by introducing an additional coordinate $x_0 = u$, with $\dot{x}_0 = \dot{u}$.

In the case where the vector fields f, g do not depend on time, solutions of the impulsive Cauchy problem (1.1) were studied in [2]. For a special class of Lagrangean systems with piecewise continuous dependence on a time-like variable, the impulsive control problem was recently considered in [6]. The present approach is simpler than [6], since it does not require any smoothing approximation of the vector field g .

The proof of Theorem 1.1 is given in the next two sections. We first introduce a suitable definition of solution of (1.1), valid when u lies in the set

$$\mathcal{U}' \doteq \{u: [0, T] \mapsto [-M_1, M_1] \mid u \text{ is piecewise constant and all}$$

of its jumps occur at times $t \neq 0, T$ where ϕ is continuous\}.

For $u \in \mathcal{U}'$, we show that the inequality (1.7) holds, hence the map $u \mapsto x(T, u)$ can be continuously extended to the closure of $\mathcal{U}'$ in the space $L^1(d\phi)$. When $u \in \mathcal{C}^1$, this continuous extension coincides with the usual Carathéodory definition. Since the closures of $\mathcal{U}$ and $\mathcal{U}'$ coincide, the result will be proved.

2. Definition of generalized solutions and preliminary lemmas.

Let $k: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a (time dependent) vector field, and fix a time $\tau \in [0, T]$. Denote by $t \mapsto \exp \{t k(\tau)\} \tilde{x}$ the solution of the Cauchy problem

$$(2.1) \quad \dot{x}(t) = k(\tau, x(t)), \quad x(0) = \tilde{x}.$$

We assume that for every $x \in \mathbb{R}^n$, the map $t \mapsto k(t, x)$ is measurable and for every $t \in [0, T]$, the map $x \mapsto k(t, x)$ is continuously differentiable. Moreover, denote by $t \mapsto \Phi(t, k(\tau), \tilde{x})$ the fundamental matrix solution of the linear differential equation

$$(2.2) \quad \dot{v}(t) = D_x k(\tau, \exp \{t k(\tau)\} \tilde{x}) \cdot v(t),$$

with $\Phi(0, k(\tau), \tilde{x})$ the identity matrix. Here $D_x k(\tau, \cdot)$ represents the Jacobian matrix of first order partial derivatives of $k(\tau, \cdot)$ with respect to x .

The matrix $\Phi(t, k(\tau), \tilde{x})$ has the following properties.

LEMMA 2.1. *Let M_2 be a constant such that*

$$|D_x k(\tau, \exp \{t k(\tau)\} \tilde{x}) \cdot w| \leq M_2 |w|$$

for every $\tilde{x}, w \in \mathbb{R}^n$, $\tau \in [0, T]$ and $|t| \leq M_1$. Then $|\Phi(t, k(\tau), \tilde{x}) \cdot w| \leq |w| e^{M_2 |t|}$.

PROOF. Since $d/dt |\Phi(t, k(\tau), \tilde{x}) \cdot w| \leq M_2 |\Phi(t, k(\tau), \tilde{x}) \cdot w|$, by Gronwall's inequality $|\Phi(t, k(\tau), \tilde{x}) \cdot w| \leq |w| e^{M_2 |t|}$. ■

LEMMA 2.2. *Let k be twice continuously differentiable w.r.t. x and let $\tau \in [0, T]$. Suppose that for any $x, y \in \mathbb{R}^n$*

$$|k(\tau, x) - k(\tau, y)| \leq L |x - y|$$

and $\|k(\tau, \cdot)\|_{C^2} \leq M$. Then for any $0 \leq t \leq M_1$ and $x_1, x_2, w \in \mathbb{R}^n$,

$$|\Phi(t, k(\tau), x_1) \cdot w - \Phi(t, k(\tau), x_2) \cdot w| \leq n^3 M M_1 |x_1 - x_2| |w| e^{2LM_1 + n^2 M M_1}.$$

PROOF. Let $\tau \in [0, T]$ and $x_1, x_2, w \in \mathbb{R}^n$. We put $v_1(t) = \Phi(t, k(\tau), x_1) \cdot w$ and $v_2(t) = \Phi(t, k(\tau), x_2) \cdot w$. For $i = 1$ and 2 , $v_i(t)$ is the value at time t of the solution of the linear differential equation

$$\dot{v}_i(t) = D_x k(\tau, \exp \{t k(\tau)\} x_i) \cdot v_i(t), \quad v_i(0) = w.$$

Observing that for any $v, x, y \in \mathbb{R}^n$,

$$|D_x k(\tau, x) \cdot v| \leq n^2 M |v|$$

and

$$|D_x k(\tau, x) \cdot v - D_x k(\tau, y) \cdot v| \leq n^3 M |x - y| |v|,$$

due to Lemma 2.1

$$\begin{aligned} \frac{d}{dt} |v_1(t) - v_2(t)| &\leq \\ &\leq |D_x k(\tau, \exp \{t k(\tau)\} x_1) \cdot v_1(t) - D_x k(\tau, \exp \{t k(\tau)\} x_2) \cdot v_2(t)| \leq \\ &\leq |D_x k(\tau, \exp \{t k(\tau)\} x_1) \cdot v_1(t) - D_x k(\tau, \exp \{t k(\tau)\} x_1) \cdot v_2(t)| + \\ &+ |D_x k(\tau, \exp \{t k(\tau)\} x_1) \cdot v_2(t) - D_x k(\tau, \exp \{t k(\tau)\} x_2) \cdot v_2(t)| \leq \\ &\leq n^2 M |v_1(t) - v_2(t)| + n^3 M |x_1 - x_2| |w| e^{LM_1 + n^2 M M_1}. \end{aligned}$$

Gronwall's inequality implies that

$$|v_1(t) - v_2(t)| \leq n^3 M M_1 |x_1 - x_2| |w| e^{LM_1 + 2n^2 M M_1}. \quad \blacksquare$$

When $u \in \mathcal{U}'$, the corresponding generalized solution $x(t, u)$ of (1.1) can be defined in a straightforward manner. Indeed, let u have jumps at points t_i , with $0 < t_1 < \dots < t_n < T$. In this case, $x(t, u)$ is the function which solves the differential equation

$$(2.3) \quad \dot{x}(t) = f(t, x(t))$$

on each subinterval $]t_{i-1}, t_i[$, together with the boundary conditions

$$(2.4) \quad x(0) = \bar{x}, \quad x(t_i +) = \exp \{(u(t_i +) - u(t_i -)) g(t_i)\} x(t_i -),$$

$$i = 1, \dots, n.$$

To study the continuous dependence of these solutions on the control $u \in \mathcal{U}'$, it is convenient to introduce an alternative representation, in terms of a new variable ξ , which will remove the discontinuities due to the jumps in u .

Choose points c_i with $c_1 = 0$, $c_{n+1} = T$, such that $c_i < t_i < c_{i+1}$ for each $i = 1, \dots, n$. Since u is constant outside the points t_i , on each subinterval $I_i = [c_i, c_{i+1}]$, the function $x(t, u)$ provides a solution to

$$(2.5) \quad \dot{x}(t) = f(t, x(t)) + g(t_i, x(t)) \dot{u}(t), \quad c_i \leq t \leq c_{i+1}.$$

Defining the auxiliary variable $\xi(t) \doteq \exp \{ -u(t) g(t_i) \} x(t)$, it is known [2, 3] that ξ is an absolutely continuous function which satisfies

$$(2.6) \quad \dot{\xi}(t) = F^*(t, t_i, \xi(t), u(t)), \quad \text{a.e. on } [c_i, c_{i+1}],$$

where $F^*: [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is defined by

$$(2.7) \quad F^*(t, \tau, \xi, u) = \Phi(-u, g(\tau), \exp \{ u g(\tau) \} \xi) \cdot f(t, \exp \{ u g(\tau) \} \xi).$$

For $u \in \mathcal{U}'$, the corresponding solution $t \mapsto x(t, u)$ can thus be obtained by setting

$$(2.8) \quad x(t, u) = \exp \{ u(t) g(t_i) \} \xi(t, u), \quad t \in I_i,$$

where $t \mapsto \xi(t, u)$ is the piecewise continuous function such that

$$(2.9) \quad \dot{\xi}(t) = F^*(t, t_i, \xi(t), u(t)), \quad t \in I_i,$$

$$(2.10) \quad \begin{cases} \xi(0) = \exp \{ -u(0) g(t_1) \} \bar{x}, \\ \xi(c_i +) = \exp \{ -u(c_i) g(t_i) \} (\exp \{ u(c_i) g(t_{i-1}) \} \xi(c_i -)). \end{cases}$$

The main advantage of the representation (2.9)-(2.10) is the following. The total variation of u , and hence of x , can be arbitrarily large. On the other hand, the total variation of ξ is related to the total variation of g , which by (1.5) is bounded in terms of ϕ . For this reason, it is convenient to study the solution of (1.1) in terms of the variable ξ , which is much better behaved than u or x .

From now on, we assume that f and g satisfy all the hypotheses in Theorem 1.1. The following lemma shows that the map F^* defined in (2.7) is Lipschitz continuous w.r.t. both variables ξ, u .

LEMMA 2.3. *There exists $L_1 > 0$ such that for any $t, \tau \in [0, T]$, $\xi_1, \xi_2 \in \mathbb{R}^n$ and $|u_1|, |u_2| \leq M_1$,*

$$(2.11) \quad |F^*(t, \tau, \xi_1, u_1) - F^*(t, \tau, \xi_2, u_2)| \leq L_1(|\xi_1 - \xi_2| + |u_1 - u_2|).$$

PROOF. By (1.4), we can easily see that for any $\xi, w \in \mathbb{R}^n$ and $|t| \leq M_1$, $|D_x g(\tau, \exp\{tg(\tau)\} \xi) \cdot w| \leq n^2 M |w|$. By Lemma 2.1 and Lemma 2.2,

$$\begin{aligned} & |F^*(t, \tau, \xi_1, u_1) - F^*(t, \tau, \xi_2, u_1)| \leq \\ & \leq |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_1) \cdot f(t, \exp\{u_1 g(\tau)\} \xi_1) - \\ & - \Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_1 g(\tau)\} \xi_1)| + \\ & + |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_1 g(\tau)\} \xi_1) - \\ & - \Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_1 g(\tau)\} \xi_2)| \leq \\ & \leq n^3 M^2 M_1 e^{2LM_1 + 2n^2 MM_1} |\xi_1 - \xi_2| + L e^{LM_1 + n^2 MM_1} |\xi_1 - \xi_2| = \\ & = C_1 |\xi_1 - \xi_2| \end{aligned}$$

and

$$\begin{aligned} & |F^*(t, \tau, \xi_2, u_1) - F^*(t, \tau, \xi_2, u_2)| \leq \\ & \leq |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_1 g(\tau)\} \xi_2) - \\ & - \Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_2 g(\tau)\} \xi_2)| + \\ & + |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_2 g(\tau)\} \xi_2) - \\ & - \Phi(-u_1, g(\tau), \exp\{u_2 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_2 g(\tau)\} \xi_2)| + \\ & + |\Phi(-u_1, g(\tau), \exp\{u_2 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_2 g(\tau)\} \xi_2) - \\ & - \Phi(-u_2, g(\tau), \exp\{u_2 g(\tau)\} \xi_2) \cdot f(t, \exp\{u_2 g(\tau)\} \xi_2)| \leq \\ & \leq L M e^{n^2 MM_1} |u_1 - u_2| + n^3 M^3 M_1 e^{LM_1 + 2n^2 MM_1} |u_1 - u_2| + \\ & + n^2 M^2 e^{n^2 MM_1} |u_1 - u_2| = C_2 |u_1 - u_2| \end{aligned}$$

where $C_1 = n^3 M^2 M_1 e^{2LM_1 + 2n^2 MM_1} + L e^{LM_1 + n^2 MM_1}$ and $C_2 = e^{n^2 MM_1} (n^3 M^3 M_1 e^{LM_1 + n^2 MM_1} + L M + n^2 M^2)$. Thus (2.11) holds for $L_1 = C_1 + C_2$. ■

LEMMA 2.4. *Let $\tilde{x} \in \mathbb{R}^n$ and $0 \leq \tau_1 < \tau_2 \leq T$. Then for any $t \in \mathbb{R}$,*

$$(2.12) \quad |\exp \{tg(\tau_1)\} \tilde{x} - \exp \{tg(\tau_2)\} \tilde{x}| \leq e^{L|t|} (\phi(\tau_2) - \phi(\tau_1)) |t|.$$

PROOF. Replacing t with $-t$, it is not restrictive to assume $t > 0$. Observing that

$$\begin{aligned} \frac{d}{dt} |\exp \{tg(\tau_1)\} \tilde{x} - \exp \{tg(\tau_2)\} \tilde{x}| &\leq \\ &\leq |g(\tau_1, \exp \{tg(\tau_1)\} \tilde{x}) - g(\tau_2, \exp \{tg(\tau_2)\} \tilde{x})| \leq \\ &\leq |g(\tau_1, \exp \{tg(\tau_1)\} \tilde{x}) - g(\tau_1, \exp \{tg(\tau_2)\} \tilde{x})| + \\ &\quad + |g(\tau_1, \exp \{tg(\tau_2)\} \tilde{x}) - g(\tau_2, \exp \{tg(\tau_2)\} \tilde{x})| \leq \\ &\leq L |\exp \{tg(\tau_1)\} \tilde{x} - \exp \{tg(\tau_2)\} \tilde{x}| + (\phi(\tau_2) - \phi(\tau_1)), \end{aligned}$$

Gronwall's inequality implies

$$|\exp \{tg(\tau_1)\} \tilde{x} - \exp \{tg(\tau_2)\} \tilde{x}| \leq e^{L|t|} (\phi(\tau_2) - \phi(\tau_1)) |t|. \quad \blacksquare$$

Let $t_1, t_2 \in [0, T]$ and $p, q, v \in \mathbb{R}^n$. From (1.4) and (1.5), we can easily see that

$$(2.13) \quad |(D_x g(t_1, p) - D_x g(t_2, p)) \cdot v| \leq n^2 |\phi(t_2) - \phi(t_1)| |v|,$$

$$(2.14) \quad |g(t_1, p) - g(t_1, q)| \leq n^2 M |p - q|,$$

$$(2.15) \quad |D_x g(t_1, p) \cdot v| \leq n^2 M |v|$$

and

$$(2.16) \quad |(D_x g(t_1, p) - D_x g(t_1, q)) \cdot v| \leq n^3 M |p - q| |v|.$$

We define a map

$$k(t_i, \tau) = g(t_i, (1 - \tau)q + \tau p), \quad \tau \in [0, 1], \quad i = 1, 2.$$

Then k is differentiable w.r.t. τ and we have

$$\begin{aligned} (2.17) \quad &|g(t_2, p) - g(t_2, q) - g(t_1, p) + g(t_1, q)| = \\ &= |k(t_2, 1) - k(t_2, 0) - k(t_1, 1) + k(t_1, 0)| = \left| \int_0^1 \frac{d}{d\tau} (k(t_2, \tau) - k(t_1, \tau)) d\tau \right| = \end{aligned}$$

$$= \left| \int_0^1 (D_x g(t_2, (1-\tau)q + \tau p) \cdot (p-q) - D_x g(t_1, (1-\tau)q + \tau p) \cdot (p-q)) d\tau \right| \leq \\ \leq n^2 |\phi(t_2) - \phi(t_1)| \cdot |p - q|.$$

In the similar way, we have that

$$(2.18) \quad |(D_x g(t_2, p) - D_x g(t_2, q) - D_x g(t_1, p) + D_x g(t_1, q)) \cdot v| \leq \\ \leq n^3 |\phi(t_2) - \phi(t_1)| |p - q| |v|.$$

PROPOSITION 2.5. Let $x_0, y_0 \in \mathbb{R}^n$ and let t_1 and t_2 be points on $[0, T]$. Define a map $K: [-M_1, M_1] \rightarrow \mathbb{R}^n$ by

$$(2.19) \quad K(s) = \\ = \exp \{ -s g(t_2) \} \exp \{ s g(t_1) \} x_0 - \exp \{ -s g(t_2) \} \exp \{ s g(t_1) \} y_0.$$

Then there exists $B_1 > 0$ such that for any $s \in [-M_1, M_1]$,

$$(2.20) \quad |K(s)| \leq |x_0 - y_0| e^{B_1 |\phi(t_2) - \phi(t_1)|}.$$

PROOF. Let

$$p_1 = \exp \{ s g(t_1) \} x_0, \quad p_2 = \exp \{ -s g(t_2) \} p_1, \\ q_1 = \exp \{ s g(t_1) \} y_0 \quad \text{and} \quad q_2 = \exp \{ -s g(t_2) \} q_1.$$

Then

$$K(s) = p_2 - q_2$$

and

$$K'(s) = -g(t_2, p_2) + \Phi(-s, g(t_2), p_1) \cdot g(t_1, p_1) + g(t_2, q_2) - \\ - \Phi(-s, g(t_2), q_1) \cdot g(t_1, q_1) = -g(t_2, p_2) + g(t_2, q_2) + \\ + \Phi(-s, g(t_2), \exp \{ s g(t_2) \} p_2) \cdot g(t_1, \exp \{ s g(t_2) \} p_2) - \\ - \Phi(-s, g(t_2), \exp \{ s g(t_2) \} q_2) \cdot g(t_1, \exp \{ s g(t_2) \} q_2).$$

Define a map $H: [-M_1, M_1] \rightarrow \mathbb{R}^n$ by

$$(2.21) \quad H(s) = \Phi(-s, g(t_2), \exp \{ s g(t_2) \} p_2) \cdot g(t_1, \exp \{ s g(t_2) \} p_2) - \\ - \Phi(-s, g(t_2), \exp \{ s g(t_2) \} q_2) \cdot g(t_1, \exp \{ s g(t_2) \} q_2).$$

Observing that for any $p \in \mathbb{R}^n$,

$$\begin{aligned} \frac{d}{ds} \Phi(-s, g(t_2), \exp\{s g(t_2)\} p) \cdot g(t_1, \exp\{s g(t_2)\} p) &= \\ &= -\Phi(-s, g(t_2), \exp\{s g(t_2)\} p) \cdot D_x g(t_2, \exp\{s g(t_2)\} p) \cdot \\ &\cdot g(t_1, \exp\{s g(t_2)\} p) + \Phi(-s, g(t_2), \exp\{s g(t_2)\} p) \cdot \\ &\cdot D_x g(t_1, \exp\{s g(t_2)\} p) \cdot g(t_2, \exp\{s g(t_2)\} p), \end{aligned}$$

we have

$$\begin{aligned} (2.22) \quad H'(s) &= \Phi(-s, g(t_2), \exp\{s g(t_2)\} p_2) \cdot \\ &\cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp\{s g(t_2)\} p_2) - \\ &- \Phi(-s, g(t_2), \exp\{s g(t_2)\} q_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp\{s g(t_2)\} q_2), \end{aligned}$$

where $[g(t_2, \cdot), g(t_1, \cdot)](p) = D_x g(t_1, p) \cdot g(t_2, p) - D_x g(t_2, p) \cdot g(t_1, p)$. If there exists $B_2 > 0$ such that

$$(2.23) \quad |K'(s)| \leq B_2 |\phi(t_2) - \phi(t_1)| \cdot |p_2 - q_2|,$$

then by Gronwall's inequality

$$(2.24) \quad |K(s)| \leq |x_0 - y_0| e^{B_1 |\phi(t_2) - \phi(t_1)|},$$

where $B_1 = M_1 \cdot B_2$. We thus only have to show that inequality (2.23) holds for some $B_2 > 0$. Since

$$\begin{aligned} (2.25) \quad K'(s) &= H(s) - g(t_2, p_2) + g(t_2, q_2) = \\ &= H(s) - (g(t_1, p_2) - g(t_1, q_2)) + \\ &+ (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) = \\ &= H(s) - H(0) + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) = \\ &= \int_0^s H'(\tau) d\tau + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) \end{aligned}$$

and by (2.17)

$$|g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)| \leq n^2 |\phi(t_2) - \phi(t_1)| \cdot |p_2 - q_2|,$$

to claim inequality (2.23) we shall show that there exists $B_3 > 0$ such that

for any $\tau \in [-M_1, M_1]$

$$(2.26) \quad |H'(\tau)| \leq B_3 |\phi(t_2) - \phi(t_1)| |p_2 - q_2|.$$

We fix $\tau \in [-M_1, M_1]$ and define maps $v_1, v_2: [-M_1, M_1] \rightarrow \mathbb{R}^n$ by

$$v_1(\sigma) = \Phi(\sigma, g(t_2), \exp\{\tau g(t_2)\} p_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp\{\tau g(t_2)\} p_2)$$

and

$$v_2(\sigma) = \Phi(\sigma, g(t_2), \exp\{\tau g(t_2)\} q_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp\{\tau g(t_2)\} q_2).$$

Then

$$(2.27) \quad H'(\tau) = v_1(-\tau) - v_2(-\tau)$$

and v_1, v_2 satisfy

$$(2.28) \quad \begin{cases} \frac{d}{d\sigma} v_1(\sigma) = D_x g(t_2, \exp\{\sigma g(t_2)\} p_3) \cdot v_1(\sigma), \\ v_1(0) = [g(t_2, \cdot), g(t_1, \cdot)](p_3), \end{cases}$$

$$(2.29) \quad \begin{cases} \frac{d}{d\sigma} v_2(\sigma) = D_x g(t_2, \exp\{\sigma g(t_2)\} q_3) \cdot v_2(\sigma), \\ v_2(0) = [g(t_2, \cdot), g(t_1, \cdot)](q_3), \end{cases}$$

where $p_3 = \exp\{\tau g(t_2)\} p_2$ and $q_3 = \exp\{\tau g(t_2)\} q_2$. We compute a bound for $|v_1(0) - v_2(0)|$ to get

$$\begin{aligned} (2.30) \quad & |v_1(0) - v_2(0)| = |D_x g(t_1, p_3) \cdot g(t_2, p_3) - D_x g(t_2, p_3) \cdot g(t_1, p_3) - \\ & - D_x g(t_1, q_3) \cdot g(t_2, q_3) + D_x g(t_2, q_3) \cdot g(t_1, q_3)| = \\ & = |(D_x g(t_1, p_3) - D_x g(t_2, p_3)) \cdot (g(t_2, p_3) - g(t_2, q_3)) + \\ & + (D_x g(t_2, q_3) - D_x g(t_2, p_3)) \cdot (g(t_1, p_3) - g(t_1, q_3)) + \\ & + (D_x g(t_1, p_3) - D_x g(t_2, p_3) - D_x g(t_1, q_3) + D_x g(t_2, q_3)) \cdot g(t_2, q_3) + \\ & + D_x g(t_2, q_3) (g(t_2, p_3) - g(t_1, p_3) - g(t_2, q_3) + g(t_1, q_3))| \leq \\ & \leq 4n^4 M |\phi(t_2) - \phi(t_1)| |p_3 - q_3| \leq 4n^4 M e^{LM_1} |\phi(t_2) - \phi(t_1)| |p_2 - q_2|. \end{aligned}$$

By considering $-\sigma$ instead of σ , we assume that $\sigma \geq 0$. Observing that

$$\begin{aligned}
 (2.31) \quad |v_1(\sigma)| &\leq |v_1(0)| e^{n^2 M \sigma} \leq \\
 &\leq |D_x g(t_1, p_3) \cdot g(t_2, p_3) - D_x g(t_2, p_3) \cdot g(t_1, p_3)| e^{n^2 M M_1} \leq \\
 &\leq |D_x g(t_1, p_3) \cdot g(t_2, p_3) - D_x g(t_2, p_3) \cdot g(t_2, p_3)| + \\
 &\quad + |D_x g(t_2, p_3) \cdot g(t_2, p_3) - D_x g(t_2, p_3) \cdot g(t_1, p_3)| e^{n^2 M M_1} \leq \\
 &\leq 2n^2 M e^{n^2 M M_1} |\phi(t_2) - \phi(t_1)|,
 \end{aligned}$$

we have a bound for $|v_1(\sigma) - v_2(\sigma)|$ as

$$\begin{aligned}
 |v_1(\sigma) - v_2(\sigma)| &= \\
 &= \left| \int_0^\sigma (D_x g(t_2, \exp\{\eta g(t_2)\} p_3) \cdot v_1(\eta) - D_x g(t_2, \exp\{\eta g(t_2)\} q_3) \cdot v_1(\eta) + \right. \\
 &\quad + D_x g(t_2, \exp\{\eta g(t_2)\} q_3) \cdot v_1(\eta) - \\
 &\quad \left. - D_x g(t_2, \exp\{\eta g(t_2)\} q_3) \cdot v_2(\eta)) d\eta + (v_1(0) - v_2(0)) \right| \leq \\
 &\leq \int_0^\sigma n^3 M |\exp\{\eta g(t_2)\} p_3 - \exp\{\eta g(t_2)\} q_3| |v_1(\eta)| d\eta + \\
 &\quad + \int_0^\sigma n^2 M |v_1(\eta) - v_2(\eta)| d\eta + |v_1(0) - v_2(0)| \leq \\
 &\leq 2n^5 M^2 M_1 e^{2LM_1 + n^2 M M_1} |\phi(t_2) - \phi(t_1)| |p_2 - q_2| + \\
 &\quad + 4n^4 M e^{LM_1} |\phi(t_2) - \phi(t_1)| |p_2 - q_2| + \int_0^\sigma n^2 M |v_1(\eta) - v_2(\eta)| d\eta.
 \end{aligned}$$

By Gronwall's inequality,

$$(2.32) \quad |v_1(\sigma) - v_2(\sigma)| \leq B_3 |p_2 - q_2| |\phi(t_2) - \phi(t_1)| \text{ for any } 0 \leq \sigma \leq M_1,$$

where $B_3 = (2n^5 M^2 M_1 e^{2LM_1 + n^2 M M_1} + 4n^4 M e^{LM_1}) e^{n^2 M M_1}$. By (2.27),

$$(2.33) \quad |H'(\tau)| \leq B_3 |p_2 - q_2| |\phi(t_2) - \phi(t_1)|.$$

We thus have that for any $s \in [-M_1, M_1]$

$$(2.34) \quad \left| \int_0^s H'(\tau) d\tau \right| \leq B_4 |p_2 - q_2| |\phi(t_2) - \phi(t_1)| ,$$

where $B_4 = B_3 M_1$. By (2.25)

$$|K'(s)| \leq B_2 |p_2 - q_2| |\phi(t_2) - \phi(t_1)| ,$$

where $B_2 = B_4 + n^2$. As a consequence, the proposition is proved. ■

3. Proof of the theorem.

Before proving that (1.7) holds for $u, v \in \mathcal{U}$, we show that it holds for $u, v \in \mathcal{U}'$. Let $u, v \in \mathcal{U}'$. Recall that the generalized solutions $x(t, u)$ and $x(t, v)$ can be defined in terms of (2.8)-(2.10). Assume that either u or v jumps at t_i where

$$0 < t_1 < t_2 < \dots < t_n < T ,$$

moreover, we may assume that u and v are left continuous since ϕ is continuous at each t_i and the integral $\int_0^T |u(t) - v(t)| d\phi(t)$ is not affected by changing the value $|u(t_i) - v(t_i)|$. Let $c_1 = 0$ and $d_n = T$. We choose $c_i, d_{i-1} \in (t_{i-1}, t_i]$ with $c_i < d_{i-1}$ for $i = 2, \dots, n$. Define the time intervals $I_i = [c_i, d_i]$, $i = 1, \dots, n$. Since u and v are left continuous, it is not restrictive to assume that $d_{i-1} = t_i$. Define

$$X(t) = \exp \{ -u(t) g(t) \} x(t, u) ,$$

$$Y(t) = \exp \{ -v(t) g(t) \} x(t, v) .$$

Since

$$|x(T, u) - x(T, v)| = |\exp \{ u(T) g(T) \} X(T) - \exp \{ v(T) g(T) \} Y(T)| ,$$

we need to estimate the increase of $|X(t_i) - Y(t_i)|$ as i increases. On each interval I_i , we define

$$X_i(t) = \exp \{ -u(t) g(t_i) \} x(t, u) , \quad Y_i(t) = \exp \{ -v(t) g(t_i) \} x(t, v) .$$

By (2.6), on the interval I_i , X_i and Y_i satisfy the differential equations

$$\dot{X}_i(t) = F^*(t, t_i, X_i(t), u(t)), \quad \dot{Y}_i(t) = F^*(t, t_i, Y_i(t), v(t)),$$

respectively. Due to Lemma 2.3, on the interval $[t_i, t_{i+1}]$ we have

$$(3.1) \quad \frac{d}{dt} |X_i(t) - Y_i(t)| \leq L_1 (|X_i(t) - Y_i(t)| + |u(t) - v(t)|).$$

We thus have an estimate by Gronwall's inequality

$$(3.2) \quad |X_i(t_{i+1}) - Y_i(t_{i+1})| \leq \\ \leq |X_i(t_i) - Y_i(t_i)| e^{L_1(t_{i+1}-t_i)} + L_1 e^{L_1 T} \int_{t_i}^{t_{i+1}} |u(s) - v(s)| ds.$$

Next, we estimate the difference between

$$|X(t_{i+1}) - Y(t_{i+1})| \quad \text{and} \quad |X_i(t_{i+1}) - Y_i(t_{i+1})|.$$

If we put $x_0 = X_i(t_{i+1})$ and $y_0 = Y_i(t_{i+1})$, then

$$(3.3) \quad |X(t_{i+1}) - Y(t_{i+1})| = \\ = |\exp\{-u(t_{i+1})g(t_{i+1})\} \exp\{u(t_{i+1})g(t_i)\} x_0 - \\ - \exp\{-v(t_{i+1})g(t_{i+1})\} \exp\{v(t_{i+1})g(t_i)\} y_0| \leq \\ \leq |\exp\{-u(t_{i+1})g(t_{i+1})\} \exp\{u(t_{i+1})g(t_i)\} x_0 - \\ - \exp\{-u(t_{i+1})g(t_{i+1})\} \exp\{u(t_{i+1})g(t_i)\} y_0| + \\ + |\exp\{-u(t_{i+1})g(t_{i+1})\} \exp\{u(t_{i+1})g(t_i)\} y_0 + \\ - \exp\{-v(t_{i+1})g(t_{i+1})\} \exp\{v(t_{i+1})g(t_i)\} y_0| \doteq E_1 + E_2.$$

If in (3.3) $E_2 \leq C_3(\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$ for some $C_3 > 0$, then by Proposition 2.5

$$(3.4) \quad |X(t_{i+1}) - Y(t_{i+1})| \leq \\ \leq |X_i(t_{i+1}) - Y_i(t_{i+1})| e^{B_1(\phi(t_{i+1}) - \phi(t_i))} + C_3 \int_{t_i}^{t_{i+1}} |u(s) - v(s)| d\phi(s).$$

By (3.2) and (3.4),

$$(3.5) \quad |X(t_{i+1}) - Y(t_{i+1})| \leq e^{B_1(\phi(t_{i+1}) - \phi(t_i))} \left(|X(t_i) - Y(t_i)| e^{L_1(t_{i+1} - t_i)} + L_1 e^{L_1 T} \int_{t_i}^{t_{i+1}} |u(s) - v(s)| ds \right) + C_3 \int_{t_i}^{t_{i+1}} |u(s) - v(s)| d\phi(s).$$

Observing that on the interval $[0, t_1]$ equation (1.1) is $\dot{x} = f(t, x)$, $x(t_1, u) = x(t_1, v)$ and

$$(3.6) \quad |X(t_1) - Y(t_1)| \leq M |u(0) - v(0)|.$$

Due to (3.5) and (3.6), we can use the induction to obtain

$$\begin{aligned} |X(T) - Y(T)| &\leq e^{B_1(\phi(T) - \phi(0)) + L_1 T} |X(t_1) - Y(t_1)| + \\ &\quad + L_1 e^{2L_1 T + B_1(\phi(T) - \phi(0))} \int_0^T |u(s) - v(s)| d\phi(s) + \\ &\quad + C_3 e^{B_1(\phi(T) - \phi(0)) + L_1 T} \int_0^T |u(s) - v(s)| ds \leq e^{B_1(\phi(T) - \phi(0)) + L_1 T} \cdot \\ &\quad \cdot \left(M |u(0) - v(0)| + (L_1 e^{L_1 T} + C_3) \int_0^T |u(s) - v(s)| d\phi(s) \right) \leq \\ &\quad \leq C_4 \left(\int_0^T |u(s) - v(s)| d\phi(s) \right), \end{aligned}$$

where $C_4 = e^{B_1(\phi(T) - \phi(0)) + L_1 T} (M + L_1 e^{L_1 T} + C_3)$. We can estimate $|x(T, u) - x(T, v)|$:

$$\begin{aligned} |x(T, u) - x(T, v)| &= |\exp\{u(T)g(T)\}X(T) - \exp\{v(T)g(T)\}Y(T)| \leq \\ &\leq |\exp\{u(T)g(T)\}X(T) - \exp\{u(T)g(T)\}Y(T)| + \\ &\quad + |\exp\{u(T)g(T)\}Y(T) - \exp\{v(T)g(T)\}Y(T)| \leq \\ &\leq e^{LM_1} |X(T) - Y(T)| + M |u(T) - v(T)| \leq C_5 \int_0^T |u(s) - v(s)| d\phi(s), \end{aligned}$$

where $C_5 = C_4 e^{LM_1} + M$. Hence (1.7) holds for $u, v \in \mathcal{U}'$.

Now we need to show that, in (3.3), $E_2 \leq C_3 (\phi(t_{i+1}) - \phi(t_i)) |u(t_{i+1}) - v(t_{i+1})|$ for some constant $C_3 > 0$. By Lemma 2.4,

$$\begin{aligned} & |\exp \{u(t_{i+1}) g(t_i)\} y_0 - \\ & \quad - \exp \{(u(t_{i+1}) - v(t_{i+1})) g(t_{i+1})\} \exp \{v(t_{i+1}) g(t_i)\} y_0| \leq \\ & \leq e^{2LM_1} (\phi(t_{i+1}) - \phi(t_i)) |u(t_{i+1}) - v(t_{i+1})| \end{aligned}$$

and

$$\begin{aligned} & |\exp \{-u(t_{i+1}) g(t_{i+1})\} \exp \{u(t_{i+1}) g(t_i)\} y_0 - \\ & \quad - \exp \{-v(t_{i+1}) g(t_{i+1})\} \exp \{v(t_{i+1}) g(t_i)\} y_0| = \\ & = |\exp \{-u(t_{i+1}) g(t_{i+1})\} \exp \{u(t_{i+1}) g(t_i)\} y_0 - \\ & \quad - \exp \{-u(t_{i+1}) g(t_{i+1})\} \exp \{(u(t_{i+1}) - v(t_{i+1})) g(t_{i+1})\} \cdot \\ & \quad \cdot \exp \{v(t_{i+1}) g(t_i)\} y_0| \leq e^{3LM_1} (\phi(t_{i+1}) - \phi(t_i)) |u(t_{i+1}) - v(t_{i+1})|. \end{aligned}$$

Therefore $E_2 \leq C_3 (\phi(t_{i+1}) - \phi(t_i)) |u(t_{i+1}) - v(t_{i+1})|$ for $C_3 = e^{3LM_1}$.

Next, we claim that (1.7) holds for $u, v \in \mathcal{U}$. Suppose that for any $w \in \mathcal{U}$, there exists a sequence $\{w_n\}$ in $\mathcal{U}'$ such that $w_n \rightarrow w$ in $L^1(d\phi)$ and $x(T, w_n) \rightarrow x(T, w)$ as $n \rightarrow \infty$. For u and $v \in \mathcal{U}$, we have sequences $\{u_n\}$ and $\{v_n\}$ so that $u_n \rightarrow u$, $v_n \rightarrow v$ in $L^1(d\phi)$, $x(T, u_n) \rightarrow x(T, u)$ and $x(T, v_n) \rightarrow x(T, v)$ as $n \rightarrow \infty$. We thus have that for any $n \in \mathbb{N}$

$$\begin{aligned} (3.7) \quad & |x(T, u) - x(T, v)| \leq \\ & \leq |x(T, u) - x(T, u_n)| + |x(T, v) - x(T, v_n)| + |x(T, u_n) - x(T, v_n)| \leq \\ & \leq |x(T, u) - x(T, u_n)| + |x(T, v) - x(T, v_n)| + C \int_0^T |u_n(s) - v_n(s)| d\phi(s) \end{aligned}$$

and take $n \rightarrow \infty$ in (3.7) to get

$$|x(T, u) - x(T, v)| \leq C \int_0^T |u(s) - v(s)| d\phi(s).$$

Hence we only have to show that for any $w \in \mathcal{U}$, there exists a sequence $\{w_n\}$ in $\mathcal{U}'$ such that $w_n \rightarrow w$ in $L^1(d\phi)$ and $x(T, w_n) \rightarrow x(T, w)$ as $n \rightarrow \infty$. Let $w \in \mathcal{U}$. We can construct a sequence $\{w_n\}$ in $\mathcal{U}'$ such that $w_n \rightarrow w$ in $L^1(d\phi)$ and $w_n(t) = \sum_{i=1}^{\delta(n)} \alpha_i^n \chi_{I_i^n}(t)$, where $I_1^n = [a_1^n, b_1^n]$, $I_i^n =$

$= (a_i^n, b_i^n]$ for $i = 2, \dots, \delta(n)$,

$$(3.8) \quad 0 = a_1^n < b_1^n = a_2^n < b_2^n < \dots < b_{\delta(n)-1}^n = a_{\delta(n)}^n < b_{\delta(n)}^n = T$$

and

$$(3.9) \quad b_i^n - a_i^n < \frac{1}{2n}.$$

For $n \in \mathbb{N}$ and $i = 2, \dots, \delta(n)$, we choose $c_i^n \in (a_i^n, b_i^n)$. Put $c_1^n = 0$ and $c_{\delta(n)+1}^n = T$. Define the time intervals $J_1^n = [c_1^n, c_2^n]$ and $J_i^n = (c_i^n, c_{i+1}^n]$ for $i = 2, \dots, \delta(n)$.

Before proving that $\lim_{n \rightarrow \infty} x(T, w_n) = x(T, w)$, we observe that for a control function $\tilde{w}$, if the solution or the generalized solution of the initial value problem

$$(3.10) \quad \dot{x}(t) = f(t, x) + g(b_i^n, x) \dot{\tilde{w}}(t)$$

$$\text{for } t \in J_i^n, \quad i = 1, \dots, \delta(n) \text{ and } x(0) = \bar{x}$$

exists, then we denote by $y_n(t, \tilde{w})$ the solution or the generalized solution of equation (3.10) corresponding to a control function $\tilde{w}$. If $\tilde{w} \in \mathcal{U}$, then $y_n(t, \tilde{w})$ is the usual solution of (3.10). If $\xi(t, \tilde{w})$ is the solution of the differential equation such that on each interval J_i^n ,

$$(3.11) \quad \begin{cases} \dot{\xi}(t) = F^*(t, b_i^n, \xi(t), \tilde{w}(t)), \\ \xi(c_i^n) = \exp \{ -\tilde{w}(c_i^n) g(b_i^n) \} y_n(c_i^n, \tilde{w}), \end{cases}$$

then $y_n(t, \tilde{w})$ also satisfies that

$$(3.12) \quad y_n(t, \tilde{w}) = \exp \{ \tilde{w}(t) g(b_i^n) \} \xi(t, \tilde{w}), \quad t \in J_i^n.$$

On the other hand, if $\tilde{w} \in \mathcal{U}'$, then $y_n(t, \tilde{w})$ is inductively defined by (3.11) and (3.12), in this case $y_n(t, \tilde{w}) = x(t, \tilde{w})$ is the generalized solution of (1.1) corresponding to $\tilde{w}$.

Simple computation yields that

$$(3.13) \quad \lim_{n \rightarrow \infty} y_n(T, w) = x(T, w),$$

when we take $y_n(t, w)$ as a usual Carathéodory solution of (3.10) corresponding to w . By Theorem 5 in [2],

$$(3.14) \quad \lim_{n \rightarrow \infty} |y_n(T, w_n) - y_n(T, w)| = 0,$$

when $y_n(t, w)$ is defined by (3.11) and (3.12) corresponding to w . As a consequence,

$$\lim_{n \rightarrow \infty} x(T, w_n) = x(T, w)$$

and (1.7) holds for $u, v \in \mathcal{U}$. ■

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REFERENCES

- [1] A. F. FILLIPOV, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Pulishers, Dordrecht (1988).
- [2] A. BRESSAN, *Impulsive control systems*, in *Nonsmooth Analysis and Geometric Methods in Deterministic Optimal Control*, Springer (1995), pp. 1-22.
- [3] A. BRESSAN, *On differential systems with impulsive Controls*, Rend. Sem. Mat. Univ. Padova, **77** (1987), pp. 1-9.
- [4] A. BRESSAN - F. RAMPAZZO, *Impulsive systems with commutative vector fields*, J. Optim. Theory Appl., **71** (1991), pp. 67-83.
- [5] A. BRESSAN - F. RAMPAZZO, *On differential systems with vector-valued impulsive controls*, Boll. Un. Mat. Ital., (7), **2-B** (1988), pp. 641-656.
- [6] A. BRESSAN - M. MONICA MOTTA, *Some optimization problems with monotone impulsive character. Approximation by means of structural discontinuities*, Atti Accad. Naz. Lincei, Serie IX, **2** (1994), pp. 31-52.
- [7] G. B. FOLLAND, *Real Analysis*, Wiley Interscience (1984).
- [8] S. G. PANDIT - S. G. DEO, *Differential Systems involving Impulses*, Lecture Notes in Math., vol. **954**, Springer (1982).
- [9] W. W. SCHMAEDEKE, *Optimal control theory for nonlinear vector differential equations containing measures*, SIAM J. Control, **3** (1965), pp. 231-280.
- [10] H. J. SUSSMANN, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. of Probability, **6** (1987), pp. 17-41.

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