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A Pseudocompact Space with Kelley's Property has a Strictly Positive Measure.

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ABSTRACT - A pseudocompact space with Kelley's property $(**)$ (resp. c.c.c.) has a Baire strictly positive measure (resp. (ω_1, ω)-caliber). These results were known for the class of compact spaces.

In this paper by a space we mean a completely regular Hausdorff space. Our notation and terminology follow [3].

A family $\mathcal{B}$ of nonempty open subsets of a space X is a *pseudobase* for X if every nonempty open subset of X contains an element of $\mathcal{B}$.

DEFINITIONS. Let X be a space.

(*a*) X has a *strictly positive measure* if there are a pseudobase $\mathcal{B}$ for X and a probability measure μ defined on the σ-algebra generated by $\mathcal{B}$, such that $\mu(B) > 0$, for all $B \in \mathcal{B}$.

(*b*) X has (Kelley's) property $(**)$ if its Stone-Čech compactification has a Borel, regular strictly positive measure.

(*c*) If $\tau \geqslant \lambda$ are cardinals, then X has (τ, λ)-caliber if for every family γ of nonempty open subsets of X with $|\gamma| = \tau$ there is a subfamily $\gamma_1 \subset \gamma$ such that $|\gamma_1| = \lambda$ and $\bigcap \gamma_1 \neq \emptyset$. If X has (τ, τ)-caliber we say that X has τ-caliber.

In [3], property $(**)$ is defined in terms of intersection numbers of the set of nonempty open subsets. It is easy to see that property $(**)$ is

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preserved in dense subspaces. A space with a strictly positive measure has property (**) ([3], p. 127) and also has (ω_1, ω)-caliber. If $\{U_\xi: \xi < \omega_1\}$ is a family of nonempty open subsets in a space with a strictly positive measure, we may suppose that $\mu(U_\xi) \geqslant \delta$, $\xi < \omega_1$ for a $\delta > 0$. Then $\bigcap \{U_\xi: \xi \in A\} = \emptyset$ for every infinite $A \subset \omega_1$, implies that $\chi_{U_{\xi_n}} \to 0$ (χ_U denotes the characteristic function on U) for every sequence ξ_n. The result follows from Lebesgue's dominated convergence theorem.

The space $\Sigma_\omega \{0, 1\}^{\omega_1} = \{x \in \{0, 1\}^{\omega_1}: |\{i < \omega_1: x(i) = 1\}| < \omega\}$ has property (**), since it is dense in $\{0, 1\}^{\omega_1}$, but fails to have a strictly positive measure, since it has not (ω_1, ω)-caliber.

It is also known that a compact space with c.c.c. (countable chain condition) has (ω_1, ω)-caliber ([3]). The space $\Sigma_\omega \{0, 1\}^{\omega_1}$ is again a counterexample for the non-compact case.

We prove that a space X has property (**) (resp. (ω_1, ω)-caliber) iff vX (the real compactification of X) does so. It follows that a pseudocompact space with property (**) (resp. c.c.c.) has a Baire strictly positive measure (resp. (ω_1, ω)-caliber).

If X is a space, then $C_p(X)$ is the space of all real-valued continuous functions on X with the topology of pointwise convergence.

It is known that if X is compact with c.c.c. then every pseudocompact subspace K of $C_p(X)$ is metrizable ([2]). This is no true if compactness of X is relaxed to pseudocompactness. This follows by a construction of Shakhmatov ([7]). Also A. Tulcea proved that if X has a strictly positive measure and K is either sequentially compact or convex and compact, then K is metrizable ([9]). It is an open problem whether the same result is valid for a simply compact K ([5]). Anyway this happens if X contains a continuous image of a product of separable spaces as a dense subspace ([8]) (such a space is called to be an *almost-separable* space). Every almost-separable space has a strictly positive measure and ω_1-caliber. This is a consequence of the following properties: 1) a product of separable spaces has a strictly positive measure and also has ω_1-caliber, and 2) an almost-separable space is a continuous image of a product of separable spaces.

We prove that not always a space with a strictly positive measure is almost-separable.

LEMMA 1. *A space X has a Baire strictly positive measure (resp. (ω_1, ω)-caliber) if vX does so.*

PROOF. For a continuous function f on X, put $\widehat{f}$ for its continuous extension on vX.

CLAIM. A sequence f_n, $n = 1, 2, \ldots$ of continuous functions on X converges pointwise to the constant function 0 iff the sequence $\widehat{f}_n$, $n = 1, 2, \ldots$ converges pointwise to 0 in vX. [Assume that for $p \in vX \setminus X$ the sequence $\widehat{f}_n(p)$ does not converge to 0; there is $\varepsilon > 0$ such that $S = \{n \in \mathbb{N}: |\widehat{f}_n(p)| \geqslant \varepsilon\}$ is infinite. The set $Z = \{y \in vX: |\widehat{f}_n(y)| \geqslant \varepsilon\}$ is a zero-set in vX, and it is nonempty, since it contains p. The result follows from the fact that X meets every nonempty zero-set in vX ([4], p. 118).]

If μ is a Baire strictly positive measure on vX, define a Baire strictly positive measure on X by the type

$$\nu(f) = \int \widehat{f} \, d\mu \ ([3], \text{ p. } 275).$$

Now let $\{U_\xi: \xi < \omega_1\}$ be a family of nonempty open subsets of X. For $\xi < \omega_1$ choose a continuous function $f_\xi: X \to [0, 1]$ such that $\|f_\xi\| = 1$ and support $(f_\xi) \subset U_\xi$. If vX has (ω_1, ω)-caliber, then there exists a sequence $W_{\xi_n} = \{x \in vX: \widehat{f}_{\xi_n}(x) > 1/2\}$, $n = 1, 2, \ldots$ such that $\bigcap W_{\xi_n}$ is nonempty. If the family $\{U_{\xi_n}: n = 1, 2, \ldots\}$ contains no infinite subfamily with nonempty intersection then $f_{\xi_n} \to 0$. By the claim it follows that $\widehat{f}_{\xi_n} \to 0$, contradiction.

PROPOSITION 2. *A pseudocompact space with property* (**) *(resp. c.c.c.) has a Baire strictly positive measure (resp.* (ω_1, ω)*-caliber).*

PROOF. If X is pseudocompact then $vX = \beta X$.

As a consequence of Lemma 1 we have that the sigma-product spaces $\Sigma_\omega\{0, 1\}^{\omega_1}$ and $\Sigma_{\omega_1}(\mathbb{R}^{\omega_1}) = \{x \in \mathbb{R}^{\omega_1}: |\{i < \omega_1: x(i) \neq 0\}| \leqslant \omega\}$ have a Baire strictly positive measure. Indeed, they are both C-embedded ([3], p. 224) and dense in the respective product spaces $\{0, 1\}^{\omega_1}$ and $\mathbb{R}^{\omega_1}$, which are their respective real compactifications; being separable, these spaces have a Baire strictly positive measure.

The next lemma appears in [8]. We give a different proof.

LEMMA 3. *If X is almost-separable then every pseudocompact subspace K of $C_p(X)$ is metrizable.*

PROOF. On X, define an equivalence relation $\sim$ by saying $x \sim y$ iff $f(x) = f(y)$ for every $f \in K$. Let $\widetilde{X}$ be the quotient space $X/\sim$; then $\widetilde{X}$ is also almost-separable. For $f \in K$ we have a function $\widetilde{f}: \widetilde{X} \to \mathbb{R}$ given by writting $\widetilde{f}([x]) = f(x)$; set $\widetilde{K} = \{\widetilde{f}: f \in K\}$. Then $\widetilde{K} \underset{\to}{\subseteq} C_p(\widetilde{X})$ is homeomorphic to K.

We may suppose that $\tilde{X}$ contains a continuous image of $\mathbb{N}^\tau$, for a cardinal τ, as a dense subspace. For $n = 1, 2, \ldots$ put $E_n = \{1, 2, \ldots, n\}^\tau$. Then E_n is compact with c.c.c. and $\cup E_n$ is dense in $\mathbb{N}^\tau$. It follows that $\tilde{X}$ contains a sequence of compact subspaces $\tilde{E}_n$, $n = 1, 2, \ldots$ with c.c.c. and such that $\cup \tilde{E}_n$ is dense in $\tilde{X}$. Since $\tilde{K}$ separates points in $\tilde{X}$, it follows that $\tilde{E}_n$ embeds in $C_p(\tilde{K})$. By well known facts (see [2], p. 30) it follows that $\tilde{E}_n$ is metrizable. Consequently $\tilde{X}$ is separable and the result follows.

PROPOSITION 4. *If X is pseudocompact then $C_p(X)$ is almost-separable iff it is separable.*

PROOF. The result follows for the embedding $X \underset{\rightarrow}{\subset} C_p(C_p(X))$ and Lemma 3.

The space $\Sigma_\omega\{0, 1\}^{\omega_1} = \{x \in \{0, 1\}^{\omega_1} \colon |\{i < \omega_1 \colon x(i) = 1\}| < \omega\}$ has a Baire strictly positive measure, but it is not almost separable, since it has not ω_1-caliber.

Shakhmatov in [7] constructed a non-metrizable pseudocompact space X_s with c.c.c. and $C_p(X_s, I)$ pseudocompact (I is the unit interval). The space $C_p(X_s, I)$ has property (**), since it is dense in $I^{|X_s|}$, so it has a Baire strictly positive measure, but by Lemma 3 it follows that it is not almost separable.

So, we have the following

PROPOSITION 5. *There exists a non-almost separable space with a Baire strictly positive measure.*

REMARKS. (i) Shakhmatov's example shows that almost-separability in Lemma 3 cannot be replaced by the existense of a strictly positive measure.

(ii) If X is discrete with $|X| > 2^\omega$ then $C_p(X)$ is almost-separable but not separable.

(iii) Since X embeds into $C_p(C_p(X))$, Shakhmatov's example shows that Tulcea's result for the metrizability of convex compact subspaces K of $C_p(X)$, with X having a strictly positive measure, is not valid for simply convex pseudocompact subspaces. We have the same assertion for the Lindelöf subspaces. This follows from the fact that $\Sigma_{\omega_1}(\mathbb{R}^{\omega_1})$ is homeomorphic to $C_p(L_{\omega_1})$, where L_{ω_1} is the one point Lindelöfication of ω_1 ([1]).

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