RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

ANNE-MARIE SIMON

A corollary to the Evans-Griffith Syzygy theorem

Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 71-77

<http://www.numdam.org/item?id=RSMUP_1995__94__71_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1995, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

A Corollary to the Evans-Griffith Syzygy Theorem.

ANNE-MARIE SIMON (*)

ABSTRACT - Height two ideals of finite projective dimension in a Cohen-Macaulay or Gorenstein local ring are investigated, providing slight extensions of results of Serre and Evans-Griffith concerning the problem to know when they are two-generated, when the quotient ring is Cohen-Macaulay if they are three-generated.

Introduction.

This note is concerned with the height two ideals of a Cohen-Macaulay noetherian local ring: when are they two-generated, what can we say about them when they are three-generated?

In the second direction we have an important theorem of Evans-Griffith.

THEOREM 1. ([E.G.81] *Theorem* 2.1, *or* [E.G.85] *Theorem* 4.4). *Let A be a regular local ring containing a field. If I is an unmixed three generated ideal of height two, then the ring A/I is Cohen-Macaulay.*

In the first direction we have first a Serre's theorem, one formulation of it is the following.

(*) Indirizzo dell'A.: Service d'Algèbre, Université Libre de Bruxelles, Boulevard du Triomphe CP.211, B-1050 Brussel, Belgium.

THEOREM 2. ([Se, *Proposition* 5] [B-E]). *Let I be a height two ideal of a regular local ring A. If the quotient ring A/I is Gorenstein, then the ideal I is two-generated.*

However, another formulation of Serre was a little bit different.

THEOREM 2'. [Se, *corollaire à la Proposition* 2]. *Let A be a noetherian domain such that all projective A-modules of rank one or two are free, and let I be a non-zero ideal of projective dimension less or equal to one. Then the ideal I is two-generated if and only if the A-module $\mathrm{Ext}^1_A(I, A)$ is principal.*

We note that in Theorem 2' only the case where I is an ideal of projective dimension one and of height two is of some interest, at least to us.

In Theorem 2, the hypothesis on I imply that the projective dimension of I is one, they imply also that the A-module $\mathrm{Ext}^1_A(I, A)$ is principal, since $\mathrm{Ext}^1_A(I, A) \simeq \mathrm{Ext}^2_A(A/I, A)$. Indeed, the last module $\mathrm{Ext}^2_A(A/I, A)$ is the canonical module of the ring A/I, hence it is principal since A/I is Gorenstein (a local ring is Gorenstein if and only if it is Cohen-Macaulay and if its canonical module is principal).

Then we have another Evans-Griffith's theorem.

THEOREM 3. ([E.G.81], *Theorem* 2.2, *or* [E.G.85], *Theorem* 4.7). *Let A be a regular local ring containing a field and let I be a prime ideal of height two such that the A-module $\mathrm{Ext}^2_A(A/I, A)$ is principal. Then I is two generated.*

There is an evident analogy between Theorem 3 and Theorem 2', in the hypothesis, the conclusions and even the proofs. Both proofs use an extension $0 \to A \to M \to I \to 0$ whose class generates the principal A-module $\mathrm{Ext}^1_A(I, A)$, and one observes that $\mathrm{Ext}^1_A(M, A) = 0$. From this observation the conclusion of Theorem 2' follows rather quickly, while for theorem 3 one has to use the syzygy theorem; and the hypothesis that I is a prime ideal of a regular ring is strongly used. Concerning that last Theorem 3 we have also to mention [Br.E.G.].

The aim of this note is to provide slight generalizations of the abovementioned theorems. The generalizations of Theorem 1 and 2 are straightforward, indeed the proofs are essentially the same. The generalization we give of Theorem 3 requires not only the preceding extensions but also a rather different proof, using the linkage theory as developped in [P.S] or [Ul] as well as the syzygy theorem.

As a general reference for homological background we quote [E.G.85], [St], [Ul].

1. Preliminaries.

To state the syzygy theorem we must recall the Serre k-condition.

DEFINITION. *An A-module M is said to be S_k if, for all prime ideals p of A one has* $\operatorname{depth} M_p \geq \min\{k, htp\}$.

So, if an A-module is S_k for some $k > 0$, then all the associated prime ideals of M are minimal in $\operatorname{Spec} A$.

A key result is the theorem of Auslander-Bridger which shows that a finitely generated A-module of finite projective dimension is S_k if and only if it is a k^{th} syzygy (see [E.G.85] Theorem 3.8) when the ring itself is S_k.

THE SYZYGY THEOREM. ([E.G.81], *Theorem* 1.1, *or* [E.G.85], *Theorem* 3.15, *see also* [Br] *or* [Og]). *Let A be a noetherian local ring containing a field and let M be a finitely generated S_k-module over A of finite projective dimension. Then if M is not free, it has rank at least k.*

We note that the rank of a finitely generated A-module of finite projective dimension is a well-defined natural number: if $0 \to A^{n_s} \to \to A^{n_{s-1}} \to \ldots \to A^{n_0} \to M \to 0$ is a free resolution of M, then $\operatorname{rank} M = r = \sum_{i=0}^{s} (-1)^i n_i$; and, for all minimal prime ideals p of $\operatorname{Spec} A$ one has $M_p \simeq A_p^r$.

In the syzygy theorem, the hypothesis that the local ring contains a field is still essential. Indeed, the proof uses a Big Cohen-Macaulay module, only available by now in equal characteristic.

Here is the straightforward generalization of theorem 1.

PROPOSITION 1. *Let A be a Cohen-Macaulay noetherian local ring containing a field, and let I be an unmixed three generated ideal of height two of finite projective dimension. Then the ideal I is perfect, i.e. the quotient ring A/I is Cohen-Macaulay. Moreover, the module $\operatorname{Ext}_A^2(A/I, A)$ is also a perfect module, i.e. it is a Cohen-Macaulay module of projective dimension two.*

PROOF We resolve A/I and have an exact sequence $0 \to M \to A^3 \to \to A \to A/I \to 0$, where M is a second syzygy of A/I.

We need to show that M is free. (If M is free, using the Auslander-Buchsbaum equality we obtain $\operatorname{depth} A/I = \dim A - pd\, A/I = \dim A - 2 = \dim A/I$).

We observe that M is a finitely generated A-module of finite projective dimension and of rank two.

On the other hand, the A-module M is S_3. Indeed, let p be a prime ideal of A at which we localize.

If $p \not\supset I$, then $M_p \simeq A_p^2$ and depth $M_p = ht\, p \geq \min\{3, ht\, p\}$.

If $p \supset I$ and $ht\, p = 2$, the above exact sequence localized at p shows that depth $M_p = 2 \geq \min\{3, 2\}$.

If $p \supset I$ and $ht\, p > 2$, the above exact sequence localized at p shows that depth $M_p \geq 3$, because depth $(A/I)_p \geq 1$, I being unmixed.

The freeness of the module M follows now from the syzygy theorem and $M \simeq A^2$.

For the second assertion, we apply the functor $\mathrm{Hom}_A(\cdot, A) = (\cdot)^*$ to the exact sequence $0 \to A^2 \to A^3 \to A \to A/I \to 0$. We obtain a complex $0 \to A^* \to A^{3^*} \to A^{2^*} \to 0$ whose homology is concentrated in degree 2, where it is $\mathrm{Ext}_A^2(A/I, A)$.

So $pd\ \mathrm{Ext}_A^2(A/I, A) = 2$, and again the conclusion follows from the Auslander-Buchsbaum equality: $\dim A - 2 = \mathrm{depth}\ \mathrm{Ext}_A^2(A/I, A) \leq \leq \dim \mathrm{Ext}_A^2(A/I, A) \leq \dim A - 2$.

We give now the straightforward generalization of Theorem 2, though this has nothing to do with the syzygy theorem.

PROPOSITION. 2. *Let I be a height two ideal of a Gorenstein local ring. If the projective dimension of I is finite and if the quotient ring A/I s Gorenstein, then the ideal I is two-generated*

PROOF. The hypotheses imply that the ideal I is perfect, i.e. the projective dimension of A/I is two, the height of I.

A minimal resolution of A/I has the form

$$0 \to A^{m-1} \xrightarrow{\alpha} A^m \to A \to A/I \to 0\,,$$

and $\mathrm{Ext}_A^2(A/I, A) = \mathrm{coker}\ \mathrm{Hom}_A(\alpha, A)$: the sequence $A^m \xrightarrow{\alpha^t} A^{m-1} \to \to \mathrm{Ext}_A^2(A/I, A) \to 0$ is exact.

As the resolution of A/I is minimal the entries of the matrix associated to α and to its transposed α^t are in the maximal ideal of A. This shows that the minimal number of generators of $\mathrm{Ext}_A^2(A/I, A)$ is $m - 1$.

On the other hand, this number is one since A/I is assumed to be Gorenstein, so $1 = m - 1$, $m = 2$ and the ideal I is two generated.

2. An extension of Theorem 3.

PROPOSITION 3. *Let A be a Gorenstein noetherian local ring containing a field and let I be an unmixed ideal of finite projective dimen-*

sion and of height two. If the A-module $\mathrm{Ext}^2_A(A/I, A)$ *is principal, then the ideal* I *can be generated by* 2 *elements.*

PROOF. We choose in I a regular sequence x_1, x_2 and use it to make an algebraic link: if $J = (x_1, x_2)\colon I$, then $I = (x_1, x_2)\colon J$ since I is unmixed and since the ring A is Gorenstein [P.S.]; moreover, the ideal J is also unmixed. The isomorphisms $\mathrm{Ext}^2_A(A/I, A) \simeq$ $\simeq \mathrm{Hom}_A(A/I, A/(x_1, x_2)) \simeq J/(x_1, x_2)$ and the hypothesis on I imply that the linked ideal J is a height two ideal three generated: $J = (x_1, x_2, y)$ for some y in A.

As $I = (x_1, x_2)\colon J$, we have an exact sequence

$$0 \to I/(x_1, x_2) \to A/(x_1, x_2) \xrightarrow{\cdot y} J/(x_1, x_2) \to 0$$

which shows that $A/I \simeq J/(x_1, x_2)$. So we have also an exact sequence

$$0 \to A/I \to A/(x_1, x_2) \to A/J \to 0\,,$$

this shows that the A-module A/J is of finite projective dimension.

By proposition 1, we conclude that the ring A/J is Cohen-Macaulay; but then A/I is also Cohen-Macaulay by the linkage theory in a Gorenstein ring A ([P.S], or [Ul]). Consequently the ring A/I is Gorenstein since it is Cohen-Macaulay and since its canonical module $\mathrm{Ext}^2_A(A/I, A)$ is principal, and the conclusion follows from proposition 2.

NOTE. the above proposition is to be compared with a geometric result of Fiorentini and Lascu ([Fi.La.], Theorem 2 (iii)).

The hypothesis in Proposition 3 are slightly weaker then in Proposition 2; in Proposition 3, the ring A/I is not assumed to be Cohen-Macaulay in advance.

3. Some Examples.

EXAMPLE 1. To the twisted cubic curve (s^3, s^2t, st^2, t^3) of the projective space $\mathbb{P}^3_K$ is associated an ideal I of the regular local ring $A =$ $= K[\![X_0, X_1, X_2, X_3]\!]$. This ideal is a height two prime ideal which is three-generated: $I = (X_0X_3 - X_1X_2, X_1^2 - X_0X_2, X_2^2 - X_1X_3)$. Hence the ring A/I is Cohen-Macaulay, not Gorenstein. In fact the ideal I is the ideal of the 2×2 minors of the matrix

$$\phi = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_3 \\ X_0 & X_1 \end{bmatrix}.$$

A minimal projective resolution of the A-module A/I is given by

$$0 \to A^2 \xrightarrow{\phi} A^3 \to A \to A/I \to 0,$$

this shows that the canonical module of A/I, $\mathrm{Ext}^2_A(A/I, A) = \mathrm{coker}\ \phi^t$ (where ϕ^t is the transposed of ϕ) is minimally generated by 2 elements.

EXAMPLE 2. To the quartic curve (s^4, s^3t, st^3, t^4) of the projective space $\mathbb{P}^3_K$ is associated a height two prime ideal I of the ring $A = K[\![X_0, X_1, X_2, X_3]\!]$, this ideal I is four-generated: $I = (X_0X_3 - X_1X_2, X_1^3 - X_0^2X_2, X_2^3 - X_1X_3^2, X_0X_2^2 - X_1^2X_3)$. The quotient ring A/I is not Cohen-Macaulay, however it is a Buchsbaum local ring.

EXAMPLE 3. Bertini constructed an example of a non Cohen-Macaulay factorial ring B which is an image of a regular local ring A: $B = A/I$, the height g of I is greater than 3. Since B is factorial, the module $\mathrm{Ext}^g_A(B, A)$ is principal. This illustrates the fact that the hypothesis in Proposition 3 are weaker than those in Proposition 2. On the other hand, Theorem 1 is concerned with unmixed ideals I of height g generated by $g + 1$ element. When $g = 2$, when the ring A is regular, the quotient A/I is Cohen-Macaulay. When $g > 2$, this conclusion is not valid anymore. Indeed, in Bertin's example we can choose in the ideal I a regular sequence $x_1, \ldots, x_g$ such that $IA_I = (x_1, \ldots, x_g)A_I$ (I is a prime ideal of the regular ring A). This gives us a link (even a geometric link): $J = (x_1, \ldots, x_g)\colon I$. The ideal J is an unmixed ideal of height g of the ring A, the quotient ring A/J is not Cohen-Macaulay (since A/I is not), but J can be generated by $g + 1$ elements: the module $J/(x_1, \ldots, x_g) = \mathrm{Hom}_A(A/I, A/x_1, \ldots, x_g) = \mathrm{Ext}^g_A(A/I, A)$ is principal.

Schenzel gave other examples of prime ideal of height g in a regular local ring, minimally generated by $g + 1$ elements, and such that the quotient ring is not Cohen-Macaulay.

This work was done while the author was visiting the University of Ferrara.

REFERENCES

[Be] M. J. BERTIN, *Anneaux d'invariants d'anneaux de polynômes, en caractéristique p*, C. R. Acad. Sci. Paris 264 (1967), 653-656.

[Br] W. BRUNS, *The Evans-Griffith syzygy theorem and Bass numbers*, Proc. of the Amer. Math. Soc. 115 n° 4 (1992), pp. 939-946.

[Br.E.G] W. BRUNS - E. EVANS - P. GRIFFITH, *Syzygies, ideals of height two and vector bundles*, Jour. Alg. 67 (1980), pp. 143-162.

[B.E.] A. BUCHSBAUM - D. EISENBUD, *Gorenstein ideals of height 3*, in Seminar Eisenbud Singh Vogel, Teubner texte zur Mathematik, band 48 (1982), 30-47.

[Fi.La.] M. FIORENTINI - A. LASCU, *Projective embeddings and linkage*, in Rendiconti del Seminario Matematico e Fisico di Milano, vol. LXII (1987).

[E.G.81] E. G. EVANS - P. GRIFFITH, *The syzygy problem*, Annals of Math., **114** (1981), pp. 323-333.

[E.G.85] E. G. EVANS - P. GRIFFITH, *Syzygies*, London Math. Soc. Lect. Notes Ser. 106, Cambridge Univ. Press, Cambridge (1985).

[Og] T. OGOMA, *A note on the syzygy problem*, Comm. Algebra, **17** (1989), pp. 2061-2066.

[P.S.] C. PESKINE - L. SZPIRO, *Liaison des variétés algébriques I*, Invent. Math., **26** (1974), pp. 271-302.

[Sch] P. SCHENZEL, *A note on almost complete intersections*, in Seminar D. Eisenbud, B. Singh, W. Vogel, vol. 2, Teubner-Texte zur Mathematik, Band 48, Leipzig (1982), pp. 49-54.

[Se] J. P. SERRE, *Sur les modules projectifs*, dans Séminaire Dubreil-Pisot (Algèbre et théorie des nombres) 1960/1961, n° 2.

[St] J. STROOKER, *Homological questions in local Algebra*, London Math. Soc. Lect. Notes Ser., **145**, Cambridge Univ. Press, Cambridge, 1990.

[Ul] B. ULRICH, *Lectures on Linkage and deformation*, preprint, workshop on commutative algebra, Trieste 1992.

Manoscritto pervenuto in redazione il 15 dicembre 1993
e, in forma revisionata, il 31 gennaio 1994.