RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

IRWEN VALLE GUADALUPE

Minimal immersions of surfaces into n-dimensional space forms

Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 155-163

<http://www.numdam.org/item?id=RSMUP_1995__94__155_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1995, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

Minimal Immersions of Surfaces into n-Dimensional Space Forms.

IRWEN VALLE GUADALUPE (*)

ABSTRACT - Using the motion of the ellipse of curvature we study minimal immersions of surfaces into n-dimensional space forms. In this paper we obtain an extension of Theorem 2 of [9]. Also, we obtain some inequalities relating the integral of the normal curvature with topological invariants.

1. Introduction.

Let M be an oriented surface which is isometrically immersed into an orientable n-dimensional space form $Q^n(c)$, $n \geq 4$, where $Q^n(c)$ stands for the sphere $S^n(c)$ of radius $1/c$, the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $H^n(c)$, according to c is positive, zero or negative. If the normal curvature tensor $R^\perp$ of the immersion is nowhere zero, then exists an orthogonal bundle splitting $NM = (NM)^* \oplus (NM)^0$ of the normal bundle NM of the immersion, where $(NM)^0$ consists of the normal directions that annihilate $R^\perp$ and $(NM)^*$ is a 2-plane subbundle of NM.

Let K and K_N be the Gaussian and the normal curvature of M. Let K^* be the intrinsic curvature of $(NM)^*$.

We shall make use of the *curvature ellipse* of x: $M \to Q^n(c)$, which is, for each p in M the subset of N_pM given by

$$\varepsilon_p = \{B(X, X) \in N_pM;\ X \in T_pM \text{ and } \|X\| = 1\}$$

where B is the second fundamental form of the immersion. The first result of this paper is an extension of Theorem 2 of Rodriguez-Guadalupe [9] to the case when M is not homeomorphic to the sphere S^2.

(*) Indirizzo dell'A.: UNICAMP-IMECC, Universidade Estadual de Campinas, Caixa Postal 6065, 13081-970 Campinas, SP, Brazil.

THEOREM 1. Let $x: M \to S^n(1)$ be a minimal immersion of a complete oriented surface M into the unit sphere $S^n(1)$ with $R^\perp \neq 0$ and $K \geq 0$. If $2K \geq K^*$ at every point, then K^*, the normal curvature K_N and the Gaussian curvature K of M are constant.

REMARKS. (1) If $K > 0$, then we obtain a minimal S^2 of constant curvature in $S^n(1)$. These were classified by Do Carmo-Wallach [4]. Itoh [6] and Asperti-Ferus-Rodriguez [1] have a similar theorem.

(2) For $K = 0$ we obtain a «flat» minimal torus. These were studied by Kenmotsu [7], [8].

The second result of this paper is the following.

THEOREM 2. Let $x: M \to S^n(1)$ be a minimal immersion of a complete oriented surface M into the unit sphere $S^n(1)$. If $K \geq 0$ at every point, then either $K \equiv 0$ or the ellipse is a circle.

The following theorem relates an inequality betwen the integral of the normal curvature with topological invariants.

THEOREM 3. Let $x: M \to Q^n(c)$ be a minimal immersion of a compact oriented surface M into an oriented n-dimensional space form $Q^n(c)$ of constant curvature c with $R^\perp \neq 0$. Then we have

$$\int_M K_N \, dM \geq 4\pi\mathcal{X}(M) \tag{1.1}$$

the equality holds if and only if $(M \sim S^2)\, n = 4$.

COROLLARY 1. Let $x: M \to S^n(1)$ be a minimal immersion of a compact oriented surface M into the unit sphere $S^n(1)$ with $R^\perp \neq 0$. Then we have

$$\text{Area } (M) \geq 6\pi\mathcal{X}(M) \tag{1.2}$$

the equality holds if and only if $(M \sim S^2)\, n = 4$.

REMARK. Of course (1.2) has interest only when $M \sim S^2$, otherwise $\mathcal{X}(M) \leq 0$ and (1.2) becomes trivial.

The proofs of the above results are presented in section 4.

I want to thank Professor Asperti for bringing [2] and [3] for my attention.

2. Preliminaries.

Let M be a surface immersed in a Riemannian manifold Q^n. For each p in M, we use T_pM, TM, N_pM and NM to denote the tangent space of M at p, the tangent bundle of M, the normal space of M at p and the normal bundle of M, respectively. We choose a local field of orthonormal frames $e_1, e_2, \ldots, e_n$ in Q^n such that restricted to M, the vectores e_1, e_2 are in T_pM and $e_3, \ldots, e_n$ are in N_pM. We shall make use the following convention on the ranges of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n, \qquad 1 \leqslant i, j, k \leqslant 2,$$

$$3 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n$$

and we shall agree that reapeated indices are summed over the respective ranges. With respect to the frame field of Q^n chosen above, let $\omega^1, \omega^2, \ldots, \omega^n$ be the field of dual frames. Then the structure equations of Q^n are given by.

$$(2.1) \qquad d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$(2.2) \qquad d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \phi_{AB}, \qquad \phi_{AB} = \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

$$K_{ABCD} + K_{ABDC} = 0.$$

If we restrict these forms to M. Then

$$(2.3) \qquad \omega_\alpha = 0$$

since $0 = d\omega_\alpha = -\sum \omega_{\alpha i} \wedge \omega_i$, by Cartan' s lemma we may write

$$(2.4) \qquad \omega_{i\alpha} = \sum h^\alpha_{ij} \omega_j, \qquad h^\alpha_{ij} = h^\alpha_{ji},$$

From these formulas, we obtain

$$(2.5) \qquad d\omega_i = -\sum \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$(2.6) \qquad d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \qquad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl}\, \omega_k \wedge \omega_l,$$

$$(2.7) \qquad R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}),$$

$$(2.8) \qquad d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \qquad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta kl}\, \omega_k \wedge \omega_l,$$

$$(2.9) \qquad R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}).$$

The Riemannian connection of M is defined by (ω_{ij}). The form $(\omega_{\alpha\beta})$ defines a connection $\nabla^{\perp}$ in the normal bundle of M. We call

$$B = \sum_{\alpha, i, j} h_{ij}^{\alpha} \omega_i \omega_j e_{\alpha} \tag{2.10}$$

the *second fundamental form* of M. The *mean curvature vector* is given by

$$H = \sum_{\alpha} \left(\sum_{i} h_{ii}^{\alpha} \right) e_{\alpha} \tag{2.11}$$

An immersion is said to be minimal if $H = 0$.

Let $R^{\perp}$ be the curvature tensor associated with $\nabla^{\perp}$. Let $\{e_1, e_2\}$ be a tangent frame, if we denote $B_{ij} = B(e_i, e_j)$; $i, j = 1, 2$ then it is easy to see that

$$R^{\perp}(e_1, e_2) = (B_{11} - B_{22}) \wedge B_{12} \,. \tag{2.12}$$

An interesting notion in the study of surfaces in higher codimension is that of the *ellipse of curvature* defined as $\{B(X, X) \in N_p M : \langle X, X \rangle = 1\}$. To see that it is an ellipse, we just have to look at the following formula, for

$$\begin{cases} X = \cos\theta e_1 + \sin\theta e_2 \,, \\ B(X, X) = H + \cos 2\theta u + \sin 2\theta v \,, \end{cases} \tag{2.13}$$

where $u = (B_{11} - B_{22})/2$, $v = B_{12}$ and $\{e_1, e_2\}$ is a tangent frame. So we see that, as X goes once around the unit tangent circle, $B(X, X)$ goes twice around the ellipse. Of course this ellipse could degenerate into a line segment or a point. Everywhere the ellipse is not a circle we can choose $\{e_1, e_2\}$ orthonormal such that u and v are perpendicular. When this happens they will coincide with the semi-axes of the ellipse.

From (2.12) it follows that if $R^{\perp} \neq 0$ then u and v are linearly independent and we can define a 2-plane subbundle $(NM)^*$ of the normal bundle NM. This plane inherits a Riemannian connection from that of NM. Let R^* be its curvature tensor and define its curvature K^* by

$$d\omega_{34} = -K^* \omega_1 \wedge \omega_2 \tag{2.14}$$

if $\{e_3, e_4\}$ locally generates $(NM)^*$.

Now, if ξ is perpendicular to $(NM)^*$, then from (2.12), $R^{\perp}(e_1, e_2)\xi = 0$. Hence, it makes sense to define the normal curvature as

$$K_N = \langle R^{\perp}(e_1, e_2) e_4, e_3 \rangle \tag{2.15}$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are orthonormal oriented bases of T_pM and N_pM, respectively. If TM and $(NM)^*$ are oriented, then K_N is globally defined. In codimension 2, $NM = (NM)^*$ and K_N has a sign. In higher codimension, if $R^\perp \neq 0$, $(NM)^*$ is globally defined and oriented if TM is. In this case, it is shown in [1] that $\mathcal{X}(NM)^* = 2\mathcal{X}(M)$, where $\mathcal{X}(NM)^*$ denote the Euler characteristic of the plane bundle $(NM)^*$ and $\mathcal{X}(M)$ denote the Euler characteristc of the tangle bundle TM.

3. Minimal immersions with $R^\perp \neq 0$.

In this section we assume that M has non-zero normal curvature tensor $R^\perp$. Also if M is orientable, then we will always choose orientations in TM and in $(NM)^*$ such that K_N is positive. We have

PROPOSITION 1.1. Let $x: M \to Q^n(c)$ be a minimal immersion of an oriented surface M into an orientable n-dimensional space form $Q^n(c)$ of constant curvature c. Then we have

$$\Delta(\log |K_N - K + c|) = 2(2K - K^*) \tag{3.1}$$

if $(K - c)^2 - K_N^2 > 0$, and consequently

$$\Delta(\log |K_N + K - c|) = 2(2K + K^*). \tag{3.2}$$

PROOF. By Itoh [6] there exists isothermal coordinates $\{x_1, x_2\}$ such that putting $X_i = \partial/\partial x_i$, $i = 1, 2$ then $u = B(X_1, X_1) = -B(X_2, X_2)$ and $v = B(X_1, X_2)$ are the semi-axes of the ellipse at every point where $(K - c)^2 - K_N^2 \neq 0$. Moreover we observe that $|X_i|^2 = E = ((K - c)^2 - K_N^2)^{-1/4}$, $i = 1, 2$. If we denote $\lambda = < u, u >^{1/2}$ and $\mu = \langle v, v \rangle^{1/2}$ and following the same arguments that [10] we have

$$\lambda^2 - \mu^2 = 1, \tag{3.3}$$

$$\lambda^2 + \mu^2 = -(K - c)E^2, \tag{3.4}$$

$$2\lambda\mu = K_N E^2. \tag{3.5}$$

If $(K - c)^2 - K_N^2 > 0$ from (3.4) and (3.5) we obtain

$$\lambda + \mu = (K_N - K + c/K_N + K - c)^{1/4}. \tag{3.6}$$

Let $e_3 = \lambda^{-1}u$ and $e_4 = \mu^{-1}v$ an oriented frame in $(NM)^*$. Now, following the same computations that [10] we get

$$\omega_{34}(X_1) = -X_2(f), \tag{3.7}$$

$$\omega_{34}(X_2) = X_1(f), \tag{3.8}$$

where $f = \log|\lambda + u|$.

Hence, we have

$$\omega_{34} = -X_2(f)\,dX_1 + X_1(f)\,dX_2. \tag{3.9}$$

Deriving (3.9) and using (2.14) we get

$$\begin{aligned} -K^*\omega_1 \wedge \omega_2 &= d\omega_{34}E^{-1} \\ &= (-X_2X_2(f)\,dX_2 \wedge dX_1 + X_1X_1(f)\,dX_1 \wedge dX_2)E^{-1} = \\ &= (X_1X_1(f) + X_2X_2(f))E^{-1}\,dX_1 \wedge dX_2) = \tilde{\Delta}(f)E^{-1}\omega_1 \wedge \omega_2 \end{aligned} \tag{3.10}$$

where $\tilde{\Delta}$ denotes de Laplacian of the «flat» metric. We know $\tilde{\Delta}(f) = E\Delta(f)$, where Δ is the Laplacian of the surface. Hence, from (2.18) and (2.22) we get

$$\Delta(\log|K_N - K + c/K_N + K - c|) = -4K^*. \tag{3.11}$$

Using $E = (K-c)^2 - K_N^2)^{-1/4}$ and the Gaussian curvature K given by the equation

$$K = -\frac{1}{2}E^{-1}\tilde{\Delta}\log E. \tag{3.12}$$

we obtain

$$\Delta(\log|K_N - K + c|) + \Delta(\log|K_N + K - c|) = 8K \tag{3.13}$$

From (3.11) and (3.13) we get the equations (3.1) and (3.2).

COROLLARY 1. Let $x: M \to Q^n(c)$ be a minimal immersion with $K^* > 0$. Then the ellipse is a circle.

PROOF. Suppose that the ellipse is not a circle then from (3.11) $\Delta(\log|K_N - K + c/K_N + K - c|) < 0$. So $K_N > 0$ implies that $\log\dfrac{|K_N - K + c|}{|K_N + K - c|}$ is subharmonic and bounded from below. Therefore $\log\dfrac{|K_N - K + c|}{|K_N + K - c|}$ is constant and this implies that $K^* \equiv 0$. This is a contradiction.

COROLLARY 2. Let $x: M \to Q^n(c)$ be a minimal immersion of a compact surface M with $2K > K^*$. Then the ellipse is a circle.

PROOF. Suppose that the ellipse is not a circle then from (3.1) $\Delta(\log |K_N - K + c|) > 0$. So we have that $\log |K_N - K + c|$ is subharmonic and bounded from above and therefore is constant. This implies that $2K = K^*$. This is a contradiction.

4. Proof of Theorems.

PROOF OF THEOREM 1. First we consider the case when the ellipse is not a circle, i.e., $(K-1)^2 - K_N^2 > 0$. Now if $2K \geqslant K^*$ then from (3.1) follows that $\Delta(\log |K_N - K + 1|) \geqslant 0$. So we have that $\log |K_N - K + 1|$ is subharmonic and bounded from above. Then

$$K_N - K + 1 = \text{constant} \tag{4.1}$$

and $2K = K^*$. On the other hand from (3.2) we get $\Delta(\log |K_N + K - 1|) = 2(2K + K^*) = 8K \geqslant 0$. Similarly from above we have

$$K_N + K - 1 = \text{constant}. \tag{4.2}$$

From (4.1) and (4.2) follows that K^*, K_N and K are constant.

In the case that ellipse is a circle the theorem follows by Rodriguez-Guadalupe [9]. This complets the proof of theorem.

PROOF OF THEOREM 2. Suppose that the ellipse is not a circle. From (3.13) we obtain

$$\Delta(\log |K_N - K + 1/K_N + K - 1|) = 8K \geqslant 0\,. \tag{4.3}$$

From (3.5), Rodriguez-Guadalupe ([9], p. 9) and $K \geqslant o$ implies $2\lambda\mu E^{-2} = K_N \leqslant 1$. So from (3.4) we have $(\lambda^2 + \mu^2)E^{-2} = 1$. Therefore we get

$$0 \leqslant (\lambda + \mu)^2 E^{-2} = K_N - K + 1\,,$$

$$= (\lambda^2 + \mu^2)E^{-2} + 2\lambda\mu E^{-2} \leqslant 2\,.$$

This implies that $|K_N - K + 1|$ is bounded from above. Similarly $|K_N + K - 1|$ is bounded from above, too. Then $\log(|K_N - K + 1/K_N + K - 1|)$ is subharmonic and bounded from above and therefore is constant. From (4.3) follows that $K \equiv 0$.

PROOF OF THEOREM 3. From Asperti ([2], Prop. 3.6) we have

$$K^* = K_N - \frac{\|B^2\|^2}{2K_N} \tag{4.4}$$

where B^2 is the 3th fundamental form of M. From (4.4) we obtain

$$K_N \geq K^* \tag{4.5}$$

Integrating (4.5) over M and applying Ferus-Rodriguez-Asperti ([1], Th. 1) we get

$$\int_M K_N dM \geq \int_M K^* dM = 2\pi\chi(NM)^* = 4\pi\chi(M). \tag{4.6}$$

If $\int_M K_N dM = 4\pi\chi(M)$ then $K_N = K^*$ and from (4.4) $B^2 \equiv 0$. By Erbacher [5] the codimension is two and $n = 4$.

PROOF OF COROLLARY 1. If $\text{Area}(M) = 6\pi\chi(M)$ then $\chi(M) > 0$ and, actually $\text{Area}(M) = 12\pi$. It follows from Asperti ([3], p. 60) that

$$12\pi \geq 2\pi(s+1)(s+2) \tag{4.7}$$

where s is sucht that $n = 2 + 2s$. It is clear now from (4.7) that $s = 1$ and $n = 4$.

On the other hand, if $n = 4$ and $x: M \to S^4(1)$ is a minimal two sphere with $R^\perp \neq 0$, then $K_N = K^*$ and by theorem 3 above and Corollary 1 of Rodriguez-Guadalupe ([9]) we have

$$\text{Area}(M) = 2\pi\chi(M) + 2\pi\chi(NM) = 2\pi\chi(M) + \int_M K_N dM = 6\pi\chi(M).$$

REFERENCES

[1] A. C. ASPERTI, D. FERUS - L. RODRIGUEZ, *Surfaces with non-zero normal curvature tensor*, Atti Accad. Naz. zincei, **73** (1982), pp. 109-115.

[2] A. C. ASPERTI, *Generic minimal Surfaces*, Math. Z., **200** (1989), pp. 181-186.

[3] A. C. ASPERTI, *Superfícies mínimas genéricas e elipses de curvatura generalizadas*, Thesis de Livre-Docência (1986), Universidade de São Paulo.

[4] M. DO CARMO - N. WALLACH, *Representations of compact groups and minimal immersions of spheres into spheres*, J. Diff. Geometry, **4** (1970), pp. 91-104.
[5] J. A. ERBACHER, *Reduction of the codimension of an isometric immersion*, J. Diff. Geometry, **5** (1971), pp. 333-340.
[6] T. ITOH, *Minimal surfaces in 4-dimensional Riemannian manifolds of constant curvature*, Kodai Math. Sem. Rep., **22** (1971), pp. 451-458.
[7] K. KENMOTSU, *On a parametrization of minimal immersions of* $\mathbb{R}^2$ *into* S^5, Tôhoku Math. J., **27** (1975), pp. 83-90.
[8] K. KENMOTSU, *On minimal immersions of* $\mathbb{R}^2$ *into* S^n, Math. Soc. Japan, **28** (1976), pp. 182-191.
[9] L. RODRIGUEZ - I. V. GUADALUPE, *Normal curvature of surfaces in space forms*, Pacific J. Math., **106** (1983), pp. 95-103.
[10] R. TRIBUZY - I. V. GUADALUPE, *Minimal immersions of surfaces into 4-dimensional space forms*, Rend. Sem. Mat. Univ. Padova, **73** (1985), pp. 1-13.

Manoscritto pervenuto in redazione il 6 giugno 1994.