RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

MARIA J. FERREIRA
MARCO RIGOLI
RENATO TRIBUZY

An extension of a result of H. Hopf to Kähler submanifolds of $\mathbb{R}^n$

Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 11-15

<http://www.numdam.org/item?id=RSMUP_1995__94__11_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1995, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

An Extension of a Result of H. Hopf to Kähler Submanifolds of $\mathbb{R}^n$.

MARIA J. FERREIRA (*) - MARCO RIGOLI (**) - RENATO TRIBUZY (***)

In the early fifties H. Hopf, [H], proved that a constant mean curvature surface, homeomorphic to a sphere, immersed in Euclidean 3-space is a standard round sphere.

As Wente has recently shown, [W], the topological assumption on the Euler characteristic is an essential requirement.

Let M be a Kähler manifold of complex dimension m and let $f: M \to \mathbb{R}^n$ be an isometric immersion. Considering the complexified tangent and normal bundles of f we can split the second fundamental tensor α according to type as $\alpha = \alpha^{(2,0)} + \alpha^{(1,1)} + \alpha^{(0,2)}$. We denote with H the mean curvature vector.

It is trivial to see that, when M is a surface, the parallelism of H in the normal bundle can be equivalently expressed by

$$\nabla^{\perp} \alpha^{(1,1)} \equiv 0. \tag{1}$$

It looks thus quite natural to try to generalize the Hopf's result to higher dimensional Kähler immersed submanifolds of $\mathbb{R}^n$, under the assumption (1) (see Corollary below).

(*) Permanent address: Departamento de Matematica, Faculdade de Ciencias, Universidade' de Lisboa, rua Ernesto de Vasconcelos, B.C. 1 Lisboa, Portugal.

(**) Permanent address: Dipartimento di Matematica, Università di Milano, Via Saldini 50, Milano, Italy.

(***) Permanent address: Departamento de Matemática, ICE, Universidade do Amazonas, 69000 Manaus, AM, Brazil.

In order to state our theorem we need to recall a further ingredient: the notion of isotropy. This has been introduced (in the real case) by Calabi, [C], and (in the complex case) by Eells and Wood, [EW], in their work on minimal surfaces.

Let ∇ represent the covariant derivative on the pull-back of the trivial $\mathbb{C}^n$-bundle over $\mathbb{R}^n$ and consider its type decomposition $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)} = \nabla' + \nabla''$. Let $\langle , \rangle$ denote the complex bilinear extension of the canonical inner product of $\mathbb{R}^n$.

We say that an isometric immersion $f\colon M \to S^t \subset \mathbb{R}^n$ is second order isotropic if

$$\langle \nabla'^{\alpha} f,\ \nabla'^{\beta} f \rangle \equiv 0 \tag{2}$$

for $\alpha + \beta \geq 1$ and $\alpha, \beta \leq 2$.

THEOREM. Let M be a compact, connected, simply connected Kähler manifold with positive first Chern class $C_1(M)$. Let $f\colon M \to \mathbb{R}^n$ be an isometric immersion such that $\nabla^{\perp} \alpha^{(1,1)} \equiv 0$. Then M is isometric to a Riemannian product $M_1 \times \ldots \times M_r$ of Kähler manifolds and f splits into a product of immersions

$$f = f_1 \times \ldots \times f_r : M = M_1 \times \ldots \times M_r \to \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r} = \mathbb{R}^n \tag{3}$$

where, for each $l \in \{1, \ldots r\}$, $f_l : M_l \to \mathbb{R}^{n_l}$ is minimal is some sphere and second order isotropic.

As a consequence we have:

COROLLARY. Under the same assumptions of the theorem one has:

i) if M has codimension 1, then $n = 3$ and $f(M)$ is a round 2-sphere;

ii) if M has codimension 2, then either $f(M)$ is the product of two round 2-spheres in $\mathbb{R}^6$ or $f(M)$ is a round 2-sphere in $\mathbb{R}^4$.

REMARK. If the codimension of M is at least 3 there are other examples, beside round spheres, as we can see considering, for instance, the Veronese surface in $S^4 \subset \mathbb{R}^5$.

PROOF (of the theorem). Consider the (symmetric) 2-form

$$\omega = \langle \alpha^{(2,0)}, H \rangle.$$

We claim that ω is a holomorphic section of $\otimes^2 T^* M^{(1,0)}$. Indeed, let $1 \leq i, j, k \leq m$ and let $\{z_i\}$ be local holomorphic coordinates on M. We

then compute

$$\frac{\partial}{\partial \bar{z}_i}\left\langle \alpha\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right), H\right\rangle =$$

$$= \left\langle \nabla^{\perp}_{\partial/\partial \bar{z}_i} H, \alpha\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)\right\rangle + \left\langle H, \left(\nabla_{\partial/\partial \bar{z}_i}\alpha\right)\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)\right\rangle +$$

$$+ \left\langle H, \alpha\left(\nabla_{\partial/\partial \bar{z}_i}\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)\right\rangle + \left\langle H, \alpha\left(\frac{\partial}{\partial z_j}, \nabla_{\partial/\partial \bar{z}_i}\frac{\partial}{\partial z_k}\right)\right\rangle.$$

We now use (1) and the Codazzi equations

$$\left(\nabla_{\partial/\partial \bar{z}_i}\alpha\right)\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \left(\nabla_{\partial/\partial z_j}\alpha\right)\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_k}\right)$$

to see that the first two terms in the above sum are zero. Furthermore, since ∇ preserves type and $\left[\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_j}\right] \equiv 0$ we have

$$\nabla_{\partial/\partial \bar{z}_i}\frac{\partial}{\partial z_j} \equiv 0.$$

It follows that $\frac{\partial}{\partial \bar{z}_i}\left\langle \alpha\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right), H\right\rangle \equiv 0$ proving our claim.

On the other hand, since $C_1(M) > 0$, from Yau, [Y], we know the existence of a Kähler metric on M with positive Ricci curvature. Using then a Bochner type technique one proves the non existence of non zero holomorphic sections of $\otimes^2 T^* M^{(1,0)}$ (for instance, as in Kobayashi and WU, [KW]). Hence

$$\omega \equiv 0 \ . \tag{4}$$

We now follow [FT]. First of all observe that $f\colon M \to \mathbb{R}^n$ with M compact and the parallelism of H imply that H is never zero. Secondly, (4) and (1) imply that the Weingarten operator A_H defined on TM by

$$\langle A_H X, Y\rangle + \langle \alpha(X, Y), H\rangle$$

is parallel too. Therefore, the pointwise eigenspaces of A_H define parallel distributions $T^1, \ldots, T^r$, orthogonal to each other, such that

$$TM = T^1 \oplus \ldots \oplus T^r.$$

Using de Rham's decomposition theorem we deduce that M can be written as a Riemannian product

$$M = M_1 \times \ldots \times M_r.$$

Furthermore, indicating with J the complex structure of M, (4) implies $J \circ A_H = A_H \circ J$ so that the subbundles T^l are invariant with respect to J. Therefore each factor M_l is Kählerian.

To show that f can be written as a product of immersions as in (3) we adapt a technique of Moore, [M]. For details see [FT].

Minimality of f_l in some sphere follows from the observation that the mean curvature vector H_l is parallel and the immersion is umbilical in the direction of H_l.

In order to check the second order isotropy of f_l the only non trivial point to verify is that

$$\langle \nabla'^2 f_l, \nabla'^2 f_l \rangle \equiv 0.$$

But, if α_l is the second fundamental tensor of f_l, this is equivalent to prove that the form

$$\psi = \langle \alpha_l^{(2,0)}, \alpha_l^{(2,0)} \rangle$$

is identically null.

We proceed as we did for ω to show that ψ is a holomorphic section of $\otimes^4 T^* M^{(1,0)}$ so that the condition $C_1(M) > 0$ implies $\psi \equiv 0$. ■

PROOF (of the Corollary).

i) Since M has codimension 1 and it is compact then M cannot split into a product. It follows that $f: M \to S^{2m} \subset \mathbb{R}^{2m+1}$ is a minimal isometric immersion. Compactness of M implies that M is diffeomorphic to S^{2m}, but M is Kähler and thus $m = 1$..

ii) Suppose now that M has codimension 2. It follows from $\langle \alpha^{(2,0)}, \alpha^{(2,0)} \rangle \equiv 0$ that the image of $\alpha^{(2,0)}$, Im $\alpha^{(2,0)}$, is orthogonal to Im $\alpha^{(0,2)}$ in the Hermitian inner product. According to the theorem we have two possibilities: either $f = f_1 \times f_2 : M_1 \times M_2 \to \mathbb{R}^6$ with $f_1(M_1)$ and $f_2(M_2)$ round 2-spheres in $\mathbb{R}^3$, or $f: M^{2m} \to \mathbb{R}^{2m+2}$ is minimal in some sphere $S^{2m+1} \subset \mathbb{R}^{2m+2}$. Let us consider this latter case. Observe that $\dim_{\mathbb{R}} T^\perp M \otimes \mathbb{C} = 4$. We claim that $\alpha^{(2,0)} \equiv 0$. Indeed

$$\dim_{\mathbb{R}} \text{Im } \alpha^{(2,0)} = \dim_{\mathbb{R}} \text{Im } \alpha^{(0,2)}$$

furtherfore Im $\alpha^{(2,0)}$ and Im $\alpha^{(0,2)}$ are orthogonal and therefore if

$\alpha^{(2,0)} / \equiv 0$ we would have somewhere:

$$\dim_{\mathbb{R}} \operatorname{Im} \alpha^{(2,0)} + \dim_{\mathbb{R}} \alpha^{(0,2)} + \dim_{\mathbb{R}} T^{\perp} S^{2m+1} \otimes \mathbb{C} \geq 6$$

contradiction.

Now observe that $T^{\perp} M \cap TS^{2m+1}$ is a parallel subbundle of $T^{\perp} M$ because it is orthogonal to the parallel vector field H and M has codimension 2. Let v be a unitary smooth section of $T^{\perp} M \cap TS^{2m+1}$. Then the Weingarten operator A_v is parallel and trace $A_v = 0$ because H is orthogonal to v. Moreover, since A_v is parallel, the same argument used in the proof of the theorem shows that either $A_v \equiv 0$ or f splits into a product of factors. This latter alternative is not possible because of the codimension assumption.

We can thus use a result of Erbacher, [E], on the reduction of codimension, to have that $f(M^{2m})$ is contained in some (affine) $\mathbb{R}^{2m+1}$. Hence $f(M^{2m}) \subseteq S^{2m}$ and from i) it follows that $m = 1$. ■

REFERENCES

[C] E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Diff. Geom., **1** (1976), pp. 111-125.

[E] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Diff. Geom., **5** (1971) pp. 333-340.

[EW] J. Eells - J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Advances in Math., **49** (1983), pp. 217-263.

[FT] M. J. Ferreira - R. Tribuzy, *Kählerian submanifolds of $\mathbb{R}^n$ with pluriharmonic Gauss maps*, to appear in Bulletin de la Societé Mathématique de Belgique.

[H] H. Hopf, *Differential Geometry in the Large*, LNM, **1000** (Springer, 1983).

[KW] S. Kobayashi - H. H. Wu, *On holomorphic sections of certain Hermitian vector bundles*, Math. Ann., **189** (1970), pp. 1-4.

[M] J. D. Moore, *Isometric immersions of Riemannian products*, J. Diff. Geom., **5** (1971), pp. 159-168.

[W] H. C. Wente, *Counter example to a conjecture of H. Hopf*, Pac. J. Math., **121** (1986), pp. 193-243.

[Y] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Mauge-Ampere equation*, I Comm. Pure Appl. Math., **31** (1978), pp. 339-411.

Manoscritto pervenuto in redazione il 7 gennaio 1994.