# Rendiconti del Seminario Matematico della Università di Padova



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*Rendiconti del Seminario Matematico della Università di Padova*, tome 93 (1995), p. 103-107

<http://www.numdam.org/item?id=RSMUP_1995__93__103_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 93 (1995)

# An Algebraic Summation Over the Set of Partitions and Some Strange Evaluations.

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ABSTRACT - By means of formal power series operation, a general algebraic summation formula over the set of partitions is established. Several combinatorial identities are demonstrated as special cases.

Let $\Omega$ be a subset of non-negative integers and $x_0, x_1, \ldots, x_n$ the indeterminates. Evaluate

$$(1) \quad E_{m-n}(x|\Omega) := \sum_{k \in \tau(m, n|\Omega)} \{x_{k_1}(x_{k_1} + x_{k_2})\ldots(x_{k_1} + x_{k_2} + \ldots + x_{k_n})\}^{-1}$$

where $\tau(m, n | \Omega)$ is the set of all $n$-tuples $\kappa = (k_1, k_2, \ldots, k_n) \in \Omega^n$ with sum $m$.

For each solution $\kappa = (k_1, k_2, \ldots, k_n) \in \tau(m, n|\Omega)$ of the equation $k_1 + k_2 + \ldots + k_n = m (k_i \in \Omega)$, define its type by the partition $\rho = [0^{p_0} 1^{p_1} \ldots m^{p_m}]$, if number $k$ appears $p_k$ times in this solution $(k_1, k_2, \ldots, k_n)$ for $0 \leq k \leq m$ (it is obvious that $p_k = 0$ if $k \notin \Omega$). Then the solutions with the same type $\rho$ are generated by the different permutations $S(\rho)$ of multi-set $\mathbb{p} = \{0^{p_0}, 1^{p_1}, \ldots, m^{p_m}\}$. Thus, we can classify the solution-set $\tau(m, n|\Omega)$ of that equation according to the partitions $\sigma(m, n|\Omega) = \{\rho = [0^{p_0} 1^{p_1} \ldots m^{p_m}]: \sum kp_k = m, \sum p_k = n$ and $p_k = 0$ for $k \notin \Omega\}$, of number $m$ into $n$-parts with each part restricted in $\Omega$. Based on this observation, the summation defined by (1) can be decom-

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(**) Partially supported by NSF (Chinese) Youth Grant # 19901033.

posed as

(2) $E_{m_-\cdot n}(x|\Omega) =$

$$= \sum_{\rho\in\sigma(m,\,n|\Omega)} \sum_{\pi\in S(\rho)} \{x_{\pi(1)}(x_{\pi(1)}+x_{\pi(2)})\ldots(x_{\pi(1)}+x_{\pi(2)}+\ldots+x_{\pi(n)})\}^{-1}.$$

As a crucial lemma, it is an easy exercise (Chu [2], 1989) to show, by the induction principle on $n = \sum p_k$, that.

LEMMA.

(3)
$$\sum_{\pi\in S(\rho)} \{x_{\pi(1)}(x_{\pi(1)}+x_{\pi(2)})\ldots(x_{\pi(1)}+x_{\pi(2)}+\ldots+x_{\pi(n)})\}^{-1} = \left\{\prod_{k=0}^{m} p_k!\, x_k^{p_k}\right\}^{-1}.$$

It follows from substituting (3) into (2), that

(4)
$$E_{m_-\cdot n}(x|\Omega) = \sum_{\rho\in\sigma(m,\,n|\Omega)} \left\{\prod_{k=0}^{m} p_k!\, x_k^{p_k}\right\}^{-1}$$

which could be expressed equivalently as

(5)
$$E_{m_-\cdot n}(x|\Omega) = (n!)^{-1} \sum_{\kappa\in\tau(m,\,n|\Omega)} \left\{\prod_{i=1}^{n} x_{k_i}\right\}^{-1}$$

in view of the fact that the number of different permutations $S(\rho)$ of multi-set $\mathbb{p} = \{0^{p_0}, 1^{p_1}, \ldots, m^{p_m}\}$ is equal to $n!/\prod p_i!$. Both (4) and (5) lead to a general algebraic summation formula if we denote by $[t^m]\,P(t)$ the coefficients of $t^m$ in the power series expansion of function $P(t)$.

THEOREM.

(6)
$$E_{m_-\cdot n}(x|\Omega) := [t^m]\left\{\sum_{k\in\Omega} t^k/x_k\right\}^n/n!.$$

Let $\Gamma$ be the Gamma function. Denote the lower and upper factorials, respectively, by

$$[z]_n = z(z-1)\ldots(z-n+1) = \Gamma(1+z)/\Gamma(1+z-n)$$

and

$$(z)_n = z(z+1)\ldots(z+n-1) = \Gamma(z+n)/\Gamma(z).$$

Now we are ready to exhibit some special evaluations. For convenience, we denote by $\Phi(t, x\,|\,\Omega)$ the power series inside the bracket in equation (6).

($A$) Let $\Omega = \mathbb{N}_0$, the set of non-negative integers, and $x_k = y^{-k}/\binom{x}{k}$. Then the corresponding $\Phi$-function is $\Phi(t, x\,|\,\Omega) = (1 + yt)^x$ which results in

$$(7)\qquad E_{m_-\cdot n}\left(x_k = y^{-k}/\binom{x}{k}\,|\,\mathbb{N}_0\right) = y^m\binom{nx}{m}/n!\,.$$

Taking $x = -1$ and $y = -1/a$ in (6), we get

$$(8)\qquad E_{m_-\cdot n}(x_k = a^k\,|\,\mathbb{N}_0) = a^{-m}\binom{m+n-1}{m}/n!$$

whose special case corresponding to $a = 2$ is due to Knuth and Pittel [4].

$$\sum_{k_1+k_2+\ldots+k_n=m}\{2^{k_1}(2^{k_1}+2^{k_2})\ldots(2^{k_1}+2^{k_2}+\ldots+2^{k_n})\}^{-1} = 2^{-m}(n)_m/m!n!\,.$$

Alternatively, the limiting case of (6) for $1/x$ and $y$ tending to zero under condition $xy = 1/c$ yields another formula

$$(9)\qquad E_{m_-\cdot n}(x_k = k!c^k\,|\,\mathbb{N}_0) = n^m c^{-m}/m!n!$$

which gives, for $c = 1$, another evaluation of Knuth and Pittel [4].

$$\sum_{k_1+k_2+\ldots+k_n=m}\{k_1!(k_1!+k_2!)\ldots(k_1!+k_2!+\ldots+k_n!)\}^{-1} = n^m/m!n!\,.$$

($B$) If we take $\Omega = \mathbb{N}$, the positive integers. Then for $x_k = y/\binom{x}{k-1}$, it holds that $\Phi(t, x\,|\,\Omega) = yt(1 + yt)^x$. From this, the shifted version of (7) follows

$$(10)\qquad E_{m_-\cdot n}\left(x_k = y^{-k}/\binom{x}{k-1}\,|\,\mathbb{N}\right) = y^m\binom{nx}{m-n}/n!\,,\qquad (m \geqslant n)$$

which reduces when $x = -1$, $y = -1/a$ and $x = -2$, $y = -1/c$, respectively, to the formulae:

$$(11)\qquad E_{m_-\cdot n}(x_k = a^k\,|\,\mathbb{N}) = a^{-m}\binom{m-1}{n-1}/n!\,,\qquad (m \geqslant n),$$

$$(12)\qquad E_{m_-\cdot n}(x_k = c^k/k\,|\,\mathbb{N}) = c^{-m}\binom{m+n-1}{m-n}/n!\,,\qquad (m \geqslant n).$$

Similarly, one can compute the following summations:

$$(13)\qquad E_{m_-\cdot n}(x_k = (-1)^{k-1}\,kx^k\,|\,\mathbb{N}) = x^{-m}S_1(m, n)/m!\,,\qquad (m \geqslant n),$$

$$(14)\qquad E_{m_-\cdot n}(x_k = kx^k\,|\,\mathbb{N}) = x^{-m}S_1^*(m, n)/m!\,,\qquad (m \geqslant n),$$

$$(15)\qquad E_{m_-\cdot n}(x_k = k!x^k\,|\,\mathbb{N}) = x^{-m}S_2(m, n)/m!\,,\qquad (m \geqslant n),$$

where $S_1(m, n)$, $S_1^*(m, n)$ and $S_2(m, n)$ are the Stirling numbers defined by

$$[z]_m = \sum_{n \leqslant m} S_1(m, n)\,z^n\,,$$

$$(z)_m = \sum_{n \leqslant m} S_1^*(m, n)\,z^n\,,$$

and

$$z^m = \sum_{n \leqslant m} S_2(m, n)[z]_n\,.$$

(*C*) Recall the generating function of Hagen-Rothe coefficients (Gould [3], 1956)

$$\sum_{k \geqslant 0} \frac{a}{a+bk}\binom{a+bk}{k}u^k = v^a\,,\qquad u = (v-1)\,v^{-b}\,.$$

We can extend the results displayed in (*A*) as an identity on binomial coefficients

$$(16)\qquad E_{m_-\cdot n}\left(x_k = c^{-k}\left\{\frac{a}{a+bk}\binom{a+bk}{k}\right\}^{-1}\,|\,\mathbb{N}_0\right) = $$

$$= \frac{an}{an+bm}\binom{an+bm}{m}\frac{c^m}{n!}$$

as well as its Abel-analogue

$$(17)\qquad E_{m_-\cdot n}(x_k = ac^{-k}(a+bk)^{-1}/k!\,|\,\mathbb{N}_0) = anc^m(an+bm)^{m-1}/n!$$

where the latter is the limiting version of the former under replacements $a \to aM$, $b \to bM$ and $c \to cM$ when $M \to \infty$.

More generally, for any Sheffer-sequences generated by

$$\sum_{k \in \Omega} t^k / X_k(\lambda) = t^\delta \exp[\lambda \psi(t)], \qquad (\delta \in \Omega)$$

all the formulas exhibited above could be formally unified as

$$E_{m_- \cdot n}(x_k = y^{-k} X_k(\lambda) \,|\, \Omega) = y^m X_{m - n\delta}(n\lambda)/n!\,.$$

For particular settings of $\psi(t)$, this identity could be used to create numerous other evaluations. But the resulting relations are too messy to be stating unless when necessary.

REMARK. The evaluations (7)-(17) demonstrated in this note may also be reformulated in the summations of (4)-(5). Some of them can be found in Chu [2] (1989).

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Manoscritto pervenuto in redazione l'8 giugno 1993
e, in forma revisionata, il 12 agosto 1993.