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A Property of the Variety of 2-Engel Groups.

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Introduction.

Suppose that $\mathfrak{V}$ is a variety of groups defined by the law $w(x_1, \ldots, x_n) = 1$, and assume that n is the least number of variables required to determine $\mathfrak{V}$. Following [KRS] we denote by $\mathfrak{V}^*$ the class of groups G satisfying the following property:

«For every n infinite subsets $X_1, \ldots, X_n$ of G, there exist elements x_i in X_i, $i = 1, \ldots, n$, such that the subgroup generated by $\{x_1, \ldots, x_n\}$ is a $\mathfrak{V}$-group».

Clearly all finite groups satisfy the property for any $\mathfrak{V}$. The question we are interested in is:

«For which varieties $\mathfrak{V}$ is very infinite $\mathfrak{V}^*$-group a $\mathfrak{V}$-group?»

For example, if $\mathfrak{V}$ is the variety $\mathfrak{A}$ of the abelian groups, then the law defining $\mathfrak{A}$ is $w(x, y) = [x, y] = 1$, and, by definition of $\mathfrak{A}^*$, for every pairs X, Y of infinite subsets of $G \in \mathfrak{A}^*$, there exist $x \in X$, $y \in Y$ such that $xy = yx$. It follows, from a theorem proved by B. H. Neumann in [N], that G is centre-by-finite, so that $Z(G)$ is infinite. For any $x, y \in G$, we consider the infinite subsets $Z(G)x$, $Z(G)y$. By hypothesis we can find $z_1, z_2 \in Z(G)$ such that $1 = [z_1 x, z_2 y] = [x, y]$, so that $G \in \mathfrak{A}$.

Problems of similar nature are discussed in [KRS], where the variety $\mathfrak{V}$ considered is the class $\mathfrak{A}^2$ of metabelian groups, and in [RS], where the authors studied the classes of locally nilpotent, locally soluble and locally finite groups. Furthermore in [LMR], the authors answer the question affirmatively, using a considerably weaker hypothe-

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sis, when $\mathfrak{V}$ is the variety of nilpotent groups of nilpotency class $n-1$. In fact they assume only that $[x_1, \ldots, x_n] = 1$ instead of supposing that $\langle x_1, \ldots, x_n \rangle$ is nilpotent of class $n-1$. In the present paper we establish a positive result for the class $\mathcal{E}_2$ of 2-Engel groups, by proving the following:

THEOREM. *Let G be an infinite group. If for every pair X, Y of infinite subsets of G there exist some x in X and y in Y such that $[x, y, y] = 1$, then G is a 2-Engel group.*

Our notation and terminology are standard (see for instance [Ro]). We shall write $\mathcal{E}_k^*$ to denote the class of groups G for which, whatever X, Y are infinite subsets of G, there exist x in X and y in Y such that $[x, \underbrace{y, \ldots, y}_{k}] = 1$. Thus our Theorem states that $\mathcal{E}_2^* = \mathcal{E}_2 \cup \mathcal{F}$, where $\mathcal{F}$ is the class of all finite groups. The proof we give relies upon a lemma proved in [S] which we restate here below for the reader's convenience:

LEMMA. *Let G be an infinite group in $\mathcal{E}_k^*$. Then $C_G(x)$ is infinite for every x in G.*

Proofs.

We will need some preliminary results before proving our statement. The first of these is actually a straightforward consequence of the above Lemma.

LEMMA 1. *If G is an infinite group in the class $\mathcal{E}_k^*$, then for any $x \in G$ there exists an infinite abelian subgroup A of G containing x.*

Furthermore we point out that:

REMARK. *If G is in $\mathcal{E}_k^*$ and its centre $Z(G)$ is infinite, then for any $x, y \in G$ the subsets $xZ(G)$, $yZ(G)$ are infinite, hence there are $z_1, z_2 \in Z(G)$ such that:*

$$1 = [xz_1, \underbrace{yz_2, \ldots, yz_2}_{k}] = [x, \underbrace{y, \ldots, y}_{k}] \quad \forall x, y \in G.$$

Therefore G is a k-Engel group.

LEMMA 2. *Let $G = \langle y, A \rangle$ be an infinite group in $\mathcal{E}_2^*$, where A is an infinite abelian subgroup of G. Then there exists an infinite subset T of the set $B = \{a \in A \mid [a, y, y] = 1\}$ such that $t_1 t_2^{-1} \in B$ for any t_1, t_2 in T.*

PROOF. Consider the set $Y = \{ y^a \mid a \in A\}$. If Y is finite, then the index $|A: C_A(y)|$ is finite too, hence $C_A(y)$ is infinite and contained in the centre of G, $Z(G)$. This means that $Z(G)$ is infinite, and, by the previous remark, G is a 2-Engel group. In this case we choose $T = B = A$.

So we may assume, without loss of generality, that Y is infinite. Suppose now that the set $A \setminus B$ is infinite and consider the two infinite sets Y and $A \setminus B$. By hypothesis there are elements $a \in A \setminus B$ and $b \in A$ such that $1 = [a, y^b, y^b] = [a, y, y]$. But this is a contradiction since a is not in B. Thus $A \setminus B$ has to be finite and B is an infinite subset of A.

If A has a torsion-free element a, then it is possible to construct an infinite strictly decreasing chain of infinite subgroups of A

$$A \geqslant \langle a \rangle > \langle a^2 \rangle > \dots > \langle a^{2^n} \rangle > \dots .$$

Since $A \setminus B$ is finite, there exists $n \in \boldsymbol{N}$ such that $\langle a^{2^n} \rangle$ is completely contained in B. Then we set $T = \langle a^{2^n} \rangle$. We have now to examine what happens when A is a torsion group. In this case the subgroup H generated by $A \setminus B$ is finite, and A/H is infinite. Choose any transversal T for H in A containing 1. This is an infinite subset of A contained in B and, for any pair of distinct elements of T, t_1, t_2, we have $t_1 t_2^{-1} \notin H$. Since $1 \in T$, we have $t_1 t_2^{-1} \in B$, for every t_1, t_2 in T. This proves our claim.

We are now in a position to prove the theorem stated in the introduction.

THEOREM. *If G is an infinite group in the class $\mathcal{E}_2^*$, then G is a 2-Engel group.*

PROOF. Our purpose is to show that $[x, y, y] = 1$, for every x, y in G. By Lemma 1 we may assume, without loss of generality, that G is the group generated by y and A, where A is an infinite abelian subgroup of G containing x, i.e. $G = \langle y, A \rangle$.

If we consider the subset $B = \{a \in A \mid [a, y, y] = 1\}$ of A, Lemma 2 guarantees the existence of an infinite subset T of B such that for any $t_1, t_2 \in T$, $t_1 t_2^{-1} \in B$. Set $\bar{T} = \{ y^t \mid t \in T\}$ and consider the following two cases:

Case 1. $\bar{T}$ finite. Since T is contained in the union of finitely many cosets of $C_A(y)$, it follows that $C_A(y)$ is infinite. Thus the centre of G, containing $C_A(y)$, is infinite too and the claim follows from the Remark.

Case 2. $\bar{T}$ infinite. We will show that $[a, y, y] = 1$ for every a in A. The subsets $a\bar{T}$, $\bar{T}$ of G are infinite for every $a \in A$ and, therefore, we can find $t_1, t_2 \in T$ such that $1 = [ay^{t_1}, y^{t_2}, y^{t_2}]$. But

$$[ay^{t_1}, y^{t_2}, y^{t_2}] = [(ay)^{t_1}, y^{t_2}, y^{t_2}] = [(ay)^{t_1 t_2^{-1}}, y, y]^{t_2} =$$

$$= [ay^{t_1 t_2^{-1}}, y, y] = [[a, y]^{y^{t_1 t_2^{-1}}}, y][y^{t_1 t_2^{-1}}, y, y].$$

Now we notice that, since $t_1 t_2^{-1}$ is in B, $y^{t_1 t_2^{-1}} \in C_G(y)$ for every t_1, t_2 in T. Hence $[y^{t_1 t_2^{-1}}, y, y] = 1$, and $y^{y^{(t_1 t_2^{-1})^{-1}}}$, so that we have

$$1 = [[a, y]^{y^{t_1 t_2^{-1}}}, y] = [a, y, y] \quad \forall a \in A,$$

and the theorem is proved.

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