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On a Class of Second Order Integrodifferential Equations.

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ABSTRACT - We consider an abstract second order integrodifferential equation in a general Banach space:

$$(*)\quad \begin{cases} u''(t) = \eta A u'(t) + \beta A u(t) + \\ \qquad + \int_0^t k(t-s) A u'(s)\, ds + \int_0^t h(t-s) A u(s)\, ds, \qquad t \geqslant 0, \\ u(0) = x, \\ u'(0) = y, \end{cases}$$

where $A: D(A) \subset X \to X$ generates an analytic semigroup, $\eta > 0$ and $\beta \in \mathbf{R}$, and $k, h: [0, +\infty[\to \mathbf{R}$ are both Laplace transformable functions. Under suitable assumptions on the kernels k and h, we construct the resolvent operator for problem $(*)$, in order to represent its solutions by the variation of constants formula. Then, we state some results about the existence and uniqueness of the strict and strong solutions. Finally, we give a concrete example of $(*)$.

0. Introduction.

This paper is concerned with a class of parabolic integrodifferential equations of the second order in the Banach space X

$$(0.1)\quad \begin{cases} u''(t) = \eta A u'(t) + \beta A u(t) + \int_0^t k(t-s) A u'(s)\, ds + \\ \qquad + \int_0^t h(t-s) A u(s)\, ds, \qquad t \geqslant 0, \\ u(0) = x, \\ u'(0) = y, \end{cases}$$

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where $A\colon D(A)\subset X\to X$ is a linear operator, which generates an analytic semigroup, $\eta\geq 0$ and β are real constants and the kernels k and h are two real functions defined on the interval $[0,+\infty[$.

Problem (0.1) arises, for example, in the study of the motion of viscoelastic materials (see [11] and references therein).

In the applications one is often concerned with the corresponding problem with infinite delay (that is, with $\int_0^t$ replaced by $\int_{-\infty}^t$). This is the same as problem (0.1) with the addition of a non-homogeneous term, provided the history of u up to time $t=0$ is known; otherwise, it needs a separate treatment.

Several papers have been devoted to the problem (0.1) with $\eta=0$ and $k\equiv 0$ (see [1, 2]) and to the nonlinear case (see [3, 4, 10, 11, 13]).

We first observe that the «type» of the problem (parabolic or hyperbolic) may change if η is greater or equal to 0. In fact, if $\eta>0$ we will show in this paper that problem (0.1) is parabolic. In the case that $\eta=0$ the problem is, in general, hyperbolic, except in some special cases as, for example, when $k(t)=t^{-\alpha}$, $\alpha\in\,]0,1[$. From now on we assume $\eta>0$; the case $\eta=0$ will be treated in a forthcoming paper.

In this paper we use Laplace transform methods for studying (0.1), extending what was done in [8, 12] for first order equations. We construct the resolvent operator for (0.1) in order to represent the solution of (0.1) by the variation of constants formula.

To this aim, we assume that the kernels k and h are both Laplace transformable functions and, in addition, that the Laplace transforms $\hat{k}(\lambda)$ and $\hat{h}(\lambda)$ can be analytically extended to a suitable sector S in the complex plane, containing the positive real semiaxis, in such a way that the extensions $\hat{k}(\lambda)$ and $\hat{h}(\lambda)$ verify the estimate

$$|\lambda^{\mu}\hat{k}(\lambda)|+|\lambda^{\delta}\hat{h}(\lambda)|\leq M\,,\qquad \lambda\in S\,, \tag{0.2}$$

with $\mu,\delta\in\,]0,1]$ and $M>0$. Under these assumptions on A, η, k, h, we can define the resolvent operator for (0.1) by the formula

$$R(t)=\frac{1}{2\pi i}\int_{\gamma}e^{\lambda t}F(\lambda)\,d\lambda\,,\qquad t>0\,, \tag{0.3}$$

where γ is a suitable path contained in the sector S, and

$$F(\lambda)=(\lambda^2-\eta\lambda A-\beta A-\lambda\hat{k}(\lambda)A-\hat{h}(\lambda)A)^{-1}\,, \tag{0.4}$$

is (formally) the Laplace transform of the resolvent operator.

To study the regularity properties of the operator $R(t)$, we make a careful preliminary study of the operator $F(\lambda)$. In addition, we prove some estimates about $R(t)$, which are useful in studying the asymptotic behaviour of the solutions of linear and nonlinear problems.

Then, thanks to the properties of $R(t)$, it is possible to write the solution of (0.1) by the formula

$$u(t) = R(t)\,y + R'(t)\,x - \eta A R(t)\,x - \int_0^t k(t-s)\,AR(s)\,x\,ds\,. \tag{0.5}$$

It is well known that problem (0.1) can be recast as a first order equation, and then known results can be applied to get the existence of the resolvent operator $R(t)$ and formula (0.5) (see [8, 12]). However, we have preferred to derive them directly to make the paper as self-contained as possible, and because we are interested on the regularity of first and second derivatives of the solutions.

In fact, in the case that the initial conditions x and y are sufficiently regular, we prove that (0.5) gives the strict solution of (0.1), while if x and y are less regular, the function u, defined by (0.5), can be considered as the strong solution of (0.1).

Finally, to clarify this problem, we give a concrete example. We consider the problem

$$\begin{cases} u_{tt}(t,x) = u_{txx}(t,x) + u_{xx}(t,x) + \displaystyle\int_0^t (t-s)^{-1/2}\,u_{txx}(s,x)\,ds\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0\,, \quad x\in[0,\pi]\,, \\ u(0,x) = u_0(x)\,, \qquad x\in[0,\pi]\,, \\ u_t(0,x) = v_0(x)\,, \qquad x\in[0,\pi]\,, \\ u(t,0) = u(t,\pi) = 0\,, \qquad t>0\,. \end{cases} \tag{0.6}$$

Under suitable assumptions on the initial conditions u_0 and v_0, we prove that (0.6) has a unique strict (or strong) solution, which moreover goes to 0 exponentially as $t \to +\infty$.

Our work is organized as follows. In Section 1 we list some assumptions, which will remain valid throughout the paper. Moreover, we define the resolvent operator and we give some of its regularity properties. Section 2 is devoted to the study of the existence and uniqueness of the strict and strong solutions of (0.1). Finally, in Section 3 we present a concrete example of (0.1).

We now give some notations, which we will use in the following. Let X be a complex Banach space. If Y is another Banach space, we denote by $\mathcal{L}(X;Y)$ the Banach space of all linear bounded operators $T: X \to Y$, endowed with the norm $\|T\| = \sup\{\|T(x)\|;\ \|x\| \leq 1\}$. We set $\mathcal{L}(X) = \mathcal{L}(X;X)$.

If $0 \leq t_0 < t_1 \leq +\infty$, we denote by $C([t_0,t_1];X)$ (resp. $C^1([t_0,t_1];X)$, $C^2([t_0,t_1];X)$) the space of all continuous $u: [t_0,t_1] \to X$ (resp. continuously differentiable, twice continuously differentiable). If $t_1 < +\infty$, $C([t_0,t_1];X)$ is endowed with the norm $\|u\|_\infty = \{\sup \|u(x)\|;\ x \in [t_0,t_1]\}$. Given $\alpha \in]0,1[$ and $t_1 < +\infty$, $C^\alpha([t_0,t_1];X)$ is the subspace of $C([t_0,t_1];X)$ consisting of the α-Hölder continuous functions u, that is

$$[u]_\alpha \doteq \sup\{|t-s|^{-\alpha}\|u(t)-u(s)\|;\ t,s \in [t_0,t_1],\ t \neq s\} < +\infty .$$

It is endowed with the norm $\|u\|_{C^\alpha([t_0,t_1];X)} \doteq \|u\|_\infty + [u]_\alpha$. $C^{1,\alpha}([t_0,t_1];X)$ (resp. $C^{2,\alpha}([t_0,t_1];X)$) is the space of all (resp. twice) continuously differentiable functions u such that u' (resp. u'') belongs to $C^\alpha([t_0,t_1];X)$. Finally, we denote by $\mathcal{A}([\varepsilon,T];X)$, $0 < \varepsilon < T \leq +\infty$, the space of all analytical functions on $[\varepsilon,T]$ with values in X.

1. Existence and regularity properties of the resolvent operator.

Throughout this paper X is a complex Banach space with norm $\|\cdot\|$. In this section we shall construct the resolvent operator for the integrodifferential equation

$$(1.1) \qquad u''(t) = \eta A u'(t) + \beta A u(t) + \int_0^t k(t-s) A u'(s)\, ds + \int_0^t h(t-s) A u(s)\, ds, \qquad t \geq 0,$$

where $A: D(A) \subset X \to X$ is a linear operator satisfying:

$$(1.2) \qquad \begin{cases} \text{there exist } M_1 > 0,\ \omega \in \mathbb{R} \text{ and } \theta_0 \in]\pi/2, \pi[\text{ such that:} \\ \text{(i) the resolvent set } \rho(A) \text{ of } A \text{ contains the sector} \\ \qquad S_{\theta_0,\omega} = \{\lambda \in \mathbb{C};\ \lambda \neq \omega,\ |\arg(\lambda-\omega)| < \theta_0\}; \\ \text{(ii) for any } \lambda \in S_{\theta_0,\omega}\text{: } \|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M_1 |\lambda-\omega|^{-1} . \end{cases}$$

Assumption (1.2) means that A generates an analytic semigroup in X. Since A is a closed operator, $D(A)$ is a Banach space, endowed with the graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$, $x \in D(A)$.

Let $\eta > 0$, $\beta \in \mathbb{R}$ and k, h: $[0, +\infty[\to \mathbb{R}$ be two locally integrable and absolutely Laplace transformable functions, whose Laplace transforms $\hat{k}(\lambda)$ and $\hat{h}(\lambda)$ can be analytically extended to the sector $S_{\theta_0, 0}$ in such a way that the extensions (denoted again by $\hat{k}(\lambda)$ and $\hat{h}(\lambda)$) satisfy

$$|\lambda^{\mu} \hat{k}(\lambda)| + |\lambda^{\delta} \hat{h}(\lambda)| \leq M_2 , \qquad \lambda \in S_{\theta_0, 0} , \tag{1.3}$$

with $\mu, \delta \in]0, 1]$ and $M_2 > 0$. To define the resolvent operator for problem (1.1), we need a careful preliminary study of the operator

$$F(\lambda) = (\lambda^2 - \eta\lambda A - \beta A - \lambda \hat{k}(\lambda) A - \hat{h}(\lambda) A)^{-1} , \tag{1.4}$$

which is formally the Laplace transform of the resolvent. To this aim, we fix once and for all a maximal analytic extension of $\hat{k}(\cdot)$ (resp. $\hat{h}(\cdot)$), which we still call $\hat{k}(\cdot)$ (resp. $\hat{h}(\cdot)$), and denote by Λ_1 (resp. Λ_2) its domain of definition. We set

$$\rho_0(F) = \{\lambda \in \Lambda_1 \cap \Lambda_2 : \lambda^2 - a(\lambda) A \text{ is invertible and } (\lambda^2 - a(\lambda) A)^{-1} \in \mathcal{L}(X; D(A))\} , \tag{1.5}$$

where

$$a(\lambda) = \eta\lambda + \beta + \lambda \hat{k}(\lambda) + \hat{h}(\lambda) . \tag{1.6}$$

Using a simple perturbation argument, it is not difficult to see that $\rho_0(F)$ is an open set. The function $\lambda \to F(\lambda)$, defined by (1.4), is analytic in $\rho_0(F)$ with values in $\mathcal{L}(X; D(A))$. Moreover, the following lemma can be proved as in [9], Lemma 1.3.

LEMMA 1.1. *If $\lambda_0 \in \mathbb{C}$ is an isolated removable singularity of $F(\cdot)$ (as a function from $\rho_0(F)$ to $\mathcal{L}(X; D(A))$), then λ_0 does not belong to $\Lambda_1 \cap \Lambda_2$ and $\lim_{\lambda \to \lambda_0} F(\lambda)$ is not invertible.*

We now define an analytic extension of $F(\cdot)$ on the set

$$\rho(F) = \rho_0(F) \cup \{\lambda_0 \in \mathbb{C};\ \lambda_0 \text{ is an isolated removable singularity of } F(\cdot)\} , \tag{1.7}$$

setting, for any $\lambda_0 \in \rho(F) | \rho_0(F)$, $F(\lambda_0) = \lim_{\lambda \to \lambda_0} F(\lambda)$.

PROPOSITION 1.2. *Assume* (1.2), (1.3) *and* $\eta > 0$. *Then, there exists* $r_0 > 0$ *such that*

$$\rho(F) \supset S_{\theta_0,0} \cap \{\lambda \in C; |\lambda| \geq r_0\}, \tag{1.8}$$

and the following inequalities hold for any $\lambda \in S_{\theta_0,0}$, $|\lambda| \geq r_0$:

$$\|F(\lambda)\|_{\mathcal{L}(X)} \leq M_3 |\lambda|^{-2}, \tag{1.9}$$

$$|\lambda / a(\lambda)| \leq M_4, \tag{1.10}$$

$$\|AF(\lambda)\|_{\mathcal{L}(X)} \leq M_5 |\lambda|^{-1}, \tag{1.11}$$

with $M_3, M_4, M_5 > 0$ *constants not depending on* λ. *Moreover*

$$AF(\lambda)x = F(\lambda)Ax, \qquad x \in D(A). \tag{1.12}$$

PROOF. Consider the operator $B = A - \omega$. Since B satisfies (1.2) with the same angle θ_0 and $\omega = 0$, we have that for any $\lambda \in S_{\theta_0,0}$ the operator $\lambda^2 - \eta\lambda B = \eta\lambda(\lambda/\eta - B)$ is invertible and

$$\|(\lambda^2 - \eta\lambda B)^{-1}\| \leq M_1 |\lambda|^{-2}. \tag{1.13}$$

Therefore, by (1.6) and $A = B + \omega$, we may write

$$\lambda^2 - a(\lambda)A = \big(I - ((\beta + \lambda\hat{k}(\lambda) + \hat{h}(\lambda))B + a(\lambda)\,\omega)(\lambda^2 - \eta\lambda B)^{-1}\big)\cdot \cdot(\lambda^2 - \eta\lambda B), \qquad \lambda \in S_{\theta_0,0}. \tag{1.14}$$

Taking into account (1.3) and (1.13), it is easy to show that there exist $r_0 > 0$ such that for any $\lambda \in S_{\theta_0,0}$, $|\lambda| \geq r_0$, $\|((\beta + \lambda\hat{k}(\lambda) + \hat{h}(\lambda))B + a(\lambda)\,\omega)(\lambda^2 - \eta\lambda B)^{-1}\| < 1/2$. From this and (1.14) it follows that the operator $\lambda^2 - a(\lambda)A$ is invertible and (1.9) holds. Moreover, thanks also to the equality

$$AF(\lambda) = \frac{1}{a(\lambda)}(\lambda^2 F(\lambda) - I), \tag{1.15}$$

(1.8) and (1.12) hold.

Since

$$\frac{\lambda}{a(\lambda)} = \left(\eta + \frac{\beta}{\lambda} + \hat{k}(\lambda) + \frac{\hat{h}(\lambda)}{\lambda}\right)^{-1},$$

(1.10) easily follows, provided one chooses r_0 large enough. Finally (1.15), (1.9) and (1.10) yield (1.11).

Now, it is possible to define the resolvent operator for the problem (1.1), as the following proposition shows.

PROPOSITION 1.3. *Let* γ *be the path* $\gamma = \gamma^+ \cup \gamma^0 \cup \gamma^-$, *where*

$$\gamma^{\pm} = \{\lambda \in \mathbb{C};\ \lambda = \rho e^{\pm i\theta},\ \rho \geq r\}, \qquad \gamma^0 = \{\lambda \in \mathbb{C};\ \lambda = re^{i\varphi},\ |\varphi| < \theta\},$$

$r \geq r_0$, $\pi/2 < \theta \leq \theta_0$, *are oriented counterclockwise. Then the operator*

$$R(t) = \frac{1}{2\pi i}\int_{\gamma} e^{\lambda t} F(\lambda)\, d\lambda, \qquad t > 0, \tag{1.16}$$

is well defined and satisfies the following properties:

(1.17) $R(\cdot)$ *is analytic in* $]0, +\infty[$ *with values in* $\mathcal{L}(X)$;

(1.18) *for any* $t > 0$, $x \in X$, $R(t)x \in D(A)$ *and* $AR(\cdot)$ *is continuous in* $]0, +\infty[$ *with values in* $\mathcal{L}(X)$;

$$\|t^{-1}R(t)\|_{\mathcal{L}(X)} + \|AR(t)\|_{\mathcal{L}(X)} \leq M_6 e^{r_0 t}, \qquad t > 0, \tag{1.19}$$

with $M_6 > 0$ *a constant not depending on* t;

$$\|(k * AR)(t)\|_{\mathcal{L}(X)} \leq M_6 \int_0^t |k(s)|\, ds\, e^{r_0 t}, \qquad t > 0, \tag{1.20}$$

where $*$ *denotes the convolution*: $(f * g)(t) = \int_0^t f(t-s)g(s)\, ds$;

$$\|(h * AR)(t)\|_{\mathcal{L}(X)} \leq M_6 \int_0^t |h(s)|\, ds\, e^{r_0 t}, \qquad t > 0; \tag{1.21}$$

$$\lim_{t \to 0^+} R(t) = 0 \quad \text{in } \mathcal{L}(X); \tag{1.22}$$

$$\lim_{t \to 0^+} AR(t)x = 0 \qquad x \in \overline{D(A)}; \tag{1.23}$$

$R(\cdot)$ *is Laplace transformable and*

$$\hat{R}(\lambda) = F(\lambda), \qquad \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > r_0. \tag{1.24}$$

PROOF. (1.16)-(1.17) easily follow by the estimate (1.9). Thanks to (1.11) for any $t > 0$, $x \in X$, $\int_{\gamma} e^{\lambda t} AF(\lambda)x\, d\lambda$ is well defined. Since A is a closed operator, $\int_\gamma e^{\lambda t} F(\lambda) x \, d\lambda$ belongs to $D(A)$, and

$$AR(t)x = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} AF(\lambda) x \, d\lambda . \tag{1.25}$$

Then (1.18) follows again by (1.11). Concerning (1.19), for $t \geq 1$, due to (1.9), we have

$$\|t^{-1} R(t)\|_{\mathcal{L}(X)} \leq \frac{M_3}{\pi} \int_{r_0}^{+\infty} e^{\rho \cos \theta t} \frac{1}{\rho^2} d\rho +$$

$$+ \frac{M_3}{2\pi r_0} \int_{-\theta}^{\theta} e^{r_0 \cos \varphi t} d\varphi \leq \frac{M_3}{\pi r_0} \left(- \frac{1}{\cos \theta r_0} + \theta \right) e^{r_0 t} .$$

For $t \in]0, 1]$ we set $\lambda t = \mu$, and we have

$$t^{-1} R(t) = \frac{t^{-1}}{2\pi i} \int_{\gamma/t} e^{\lambda t} F(\lambda) \, d\lambda = \frac{1}{2\pi i} \int_\gamma e^{\mu} F\left(\frac{\mu}{t} \right) \frac{d\mu}{t^2} ,$$

from which, again by (1.9), it follows

$$\|t^{-1} R(t)\|_{\mathcal{L}(X)} \leq \frac{M_3}{\pi} \left(\int_{r_0}^{+\infty} e^{\rho \cos \theta} \frac{1}{\rho^2} d\rho + \frac{1}{2 r_0} \int_{-\theta}^{\theta} e^{r_0 \cos \varphi} d\varphi \right),$$

which proves the first inequality of (1.19). The proof of the other inequality is analogous. (1.20)-(1.22) follows from (1.19). To prove (1.23), it is sufficient to show that (1.23) holds for $x \in D(A)$, and then to use (1.19). If $x \in D(A)$, thanks to (1.12), we have

$$AR(t)x = R(t)Ax , \tag{1.26}$$

from which by (1.22) it follows $\lim_{t \to 0^+} AR(t)x = 0$. Finally, by (1.19) $R(\cdot)$ is Laplace transformable, and (1.24) is an easy consequence of the Cauchy integral formula (see also [8], Proposition 4.2).

Now, we want to study the properties of the operators

$$R'(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda F(\lambda)\,d\lambda, \qquad t>0, \tag{1.27}$$

$$R''(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^2 F(\lambda)\,d\lambda, \qquad t>0. \tag{1.28}$$

PROPOSITION 1.4. *For any $x \in X$, and $t>0$ we have that*

(1.29) *$R'(t)x \in D(A)$ and $AR'(\cdot)$ is continuous in $]0,+\infty[$ with values in $\mathcal{L}(X)$;*

(1.30) *$R(\cdot)$ is analytic in $]0,+\infty[$ with values in $\mathcal{L}(X;D(A))$;*

(1.31) $\|R'(t)\|_{\mathcal{L}(X)} + \|tAR'(t)\|_{\mathcal{L}(X)} + \|tR''(t)\|_{\mathcal{L}(X)} \leqslant M_7 e^{r_0 t}, \qquad t>0,$

with $M_7>0$ a constant not depending on t; for any $x \in X$ and $t>0$

$$R'(t)x - x = \eta AR(t)x + \beta\int_0^t AR(s)x\,ds + k * AR(t)x + \int_0^t h * AR(s)x\,ds; \tag{1.32}$$

(1.33) *for any $x \in \overline{D(A)}$,* $\quad \lim_{t\to 0^+} R'(t)x = x;$

(1.34) *for any $x \in D(A)$,* $\quad$ *with $Ax \in \overline{D(A)}$,* $\lim_{t\to 0^+} AR'(t)x = Ax;$

for any $x \in X$ and $t>0$

(1.35) $R''(t)x = \eta AR'(t)x + \beta AR(t)x + k * AR'(t)x + h * AR(t)x;$

(1.36) *for any $x \in D(A)$,* $\quad$ *with $Ax \in \overline{D(A)}$,* $\lim_{t\to 0^+} R''(t)x = \eta Ax.$

PROOF. Thanks to (1.11) for any $t>0$, $x \in X$ $\int_\gamma e^{\lambda t}\lambda AF(\lambda)x\,d\lambda$ is well defined. Since A is a closed operator, $\int_\gamma e^{\lambda t}\lambda F(\lambda)x\,d\lambda$ belongs to $D(A)$, and

$$AR'(t)x = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\,\lambda AF(\lambda)x\,d\lambda. \tag{1.37}$$

Then (1.29) follows again from (1.11). (1.30) easily follows from the estimate (1.11) and (1.17). The proof of (1.31) is analogous to that of (1.19). As both members of (1.32) are continuous in $]0, +\infty[$, to prove (1.32) it is sufficient to show that their Laplace transforms coincide, that is by (1.24), that

$$(1.38) \qquad \lambda F(\lambda)x - \lambda^{-1}x = \eta AF(\lambda)x + \beta\lambda^{-1}AF(\lambda)x + \\ + \hat{k}(\lambda)AF(\lambda)x + \lambda^{-1}\hat{h}(\lambda)AF(\lambda)x, \qquad \lambda \in \mathbb{C}, \ \operatorname{Re}\lambda > r_0 .$$

Thanks to (1.15) and (1.6), (1.38) holds, and so we get (1.32). Concerning (1.33), taking the norm into (1.32), by (1.19)-(1.21) we have

$$\|R'(t)x - x\| \leq \eta\|AR(t)x\| + M_6 e^{r_0 t}\|x\|\left(|\beta|t + \int_0^t |k(s)|ds + \int_0^t |h(s)|ds\right),$$

from which, due to (1.23), it follows (1.33). By (1.37) and (1.12) it follows for $x \in D(A)$

$$(1.39) \qquad AR'(t)x = R'(t)Ax,$$

and so, because of $Ax \in \overline{D(A)}$, (1.33) yields (1.34). Finally, by differentiating (1.32) we have (1.35), and from this, taking into account respectively (1.34), (1.23), (1.31) and (1.21), it follows (1.36).

Now, we want to emphasize the regularity properties of the function $t \mapsto R(t)x$, depending on the regularity of x.

PROPOSITION 1.5. i) *For any $x \in X$, the function $t \mapsto R(t)x$ belongs to $C([0, +\infty[; X)$;*

ii) *for any $x \in \overline{D(A)}$, the function $t \mapsto R(t)x$ belongs to $C([0, +\infty[; D(A)) \cap C^1([0, +\infty[; X)$;*

iii) *for any $x \in D(A)$, with $Ax \in \overline{D(A)}$ the function $t \mapsto R(t)x$ belongs to $C^1([0, +\infty[; D(A)) \cap C^2([0, +\infty[; X)$.*

PROOF. i) follows from (1.17) and (1.22). i), (1.23), (1.17), (1.18) and (1.33) yield ii). Finally, iii) follows from ii), (1.29), (1.34) and (1.36).

We conclude this section with some estimates, which are useful in studying the asymptotic behaviour of the solutions of linear and nonlinear problems. If we denote by $\sigma(F)$ the complementary set $\mathbb{C} \setminus \rho(F)$, and we define

$$\omega_F = \sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(F)\}, \tag{1.40}$$

then the following proposition holds:

PROPOSITION 1.6. *For any* $\varepsilon > 0$, *there exists a constant* $k(\varepsilon) > 0$ *such that the following estimate holds for* $t > 0$:

$$\|t^{-1}R(t)\| + \|R'(t)\| + \|AR(t)\| + \|k * AR(t)\| + \\ + \|h * AR(t)\| \leqslant k(\varepsilon)\, e^{(\omega_F + \varepsilon)t}, \tag{1.41}$$

where we mean $\|\cdot\| = \|\cdot\|_{\mathcal{L}(X)}$.

PROOF. We shall prove only

$$\|R(t)\| \leqslant k(\varepsilon)\, te^{(\omega_F + \varepsilon)t)}, \tag{1.42}$$

because the other inequalities may be obtained in a similar way.

We assume that $\omega_F + \varepsilon < -r_0$ (the other case can be handled in a standard way). Then it is possible to deform the path γ, in formula (1.16), into a new path $\tilde{\gamma}$ so defined

$$\tilde{\gamma} = \tilde{\gamma}_+ \cup \tilde{\gamma}_0 \cup \tilde{\gamma}_-,$$

$$\tilde{\gamma}_\pm = \left\{\lambda = \rho e^{\pm i\theta};\ \rho \geqslant \rho_0 = \frac{\omega_F + \varepsilon/2}{\cos\theta}\right\},$$

$$\tilde{\gamma}_0 = \left\{\lambda = \omega_F + \frac{\varepsilon}{2} + iy;\ |y| \leqslant h_0 = \left(\omega_F + \frac{\varepsilon}{2}\right)\tan\theta\right\},$$

with $\theta \in]\pi/2, \theta_0]$. Since $\rho_0 > r_0$, we may apply (1.9), and we have

$$\|R(t)\| \leqslant \frac{M_3}{\pi}\int_{\rho_0}^{+\infty} e^{\rho\cos\theta t}\frac{d\rho}{\rho^2} + \frac{1}{2\pi}e^{(\omega_F + \varepsilon/2)t}\int_{-h_0}^{h_0}\left\|F\left(\omega_F + \frac{\varepsilon}{2} + iy\right)\right\|dy \leqslant$$

$$\leqslant e^{\rho_0\cos\theta t}\frac{M_3}{\pi}\int_{\rho_0}^{+\infty}\frac{d\rho}{\rho^2} + \frac{1}{2\pi}e^{(\omega_F + \varepsilon/2)t}\int_{-h_0}^{h_0}\left\|F\left(\omega_F + \frac{\varepsilon}{2} + iy\right)\right\|dy =$$

$$= e^{(\omega_F + \varepsilon/2)t}\left(\frac{M_3}{\pi\rho_0} + \frac{1}{2\pi}\int_{-h_0}^{h_0}\left\|F\left(\omega_F + \frac{\varepsilon}{2} + iy\right)\right\|dy\right),$$

that is, (1.42) is true for $t \geq 1$. If $t \in]0, 1[$, by (1.19) we get

$$\|R(t)\| \leq M_6 e^{r_0 - (\omega_F + \varepsilon)} t e^{(\omega_F + \varepsilon) t} ,$$

from which the conclusion follows.

2. Existence and uniqueness of the strict and strong solutions.

In this section we want to solve the Cauchy problem

$$(2.1) \qquad \begin{cases} u''(t) = \eta A u'(t) + \beta A u(t) + \int_0^t k(t-s) A u'(s)\, ds + \\ \qquad\qquad + \int_0^t h(t-s) A u(s)\, ds , \qquad t > 0 , \\ u(0) = x , \\ u'(0) = y , \end{cases}$$

in a strict and a strong sense, where η, β, A, k and h satisfy the assumptions of Section 1. A function u: $[0, +\infty[\to X$ is said to be a strict solution of (2.1) if $u \in C^1([0, +\infty[; D(A)) \cap C^2([0, +\infty[; X)$ and satisfies (2.1). We will call a strong solution of (2.1) a function $u \in C^1([0, T]; X)$, $T > 0$, such that there exists a sequence

$$\{u_n\} \subset C^1([0, T]; D(A)) \cap C^2([0, T]; X) ,$$

with the following properties:

$$(2.2) \qquad \begin{aligned} & u_n \to u \quad \text{in } C([0, T]; X) , \quad u(0) = x ; \\ & u_n' \to u' \quad \text{in } C([0, T]; X) , \quad u'(0) = y ; \\ & u_n'' - \eta A u_n' - \beta A u_n - k * A u_u' - h * A u_n = 0 , \qquad n \in \mathbf{N} . \end{aligned}$$

We begin with a uniqueness result and a representation formula.

THEOREM 2.1. *If x, $y \in \overline{D(A)}$ and u is a strong solution of* (2.1), *then*

$$(2.3) \qquad u(t) = R(t) y + R'(t) x - \eta A R(t) x - \int_0^t k(t-s) A R(s) x\, ds , \qquad 0 \leq t \leq T .$$

PROOF. We first assume that

$$u \in C^1([0,+\infty[;D(A)) \cap C^2([0,+\infty[;X)$$

is a strict solution of (2.1). For every $t \geq 0$ the function $s \mapsto R(t-s)\,u(s)$ is twice continuously differentiable in $[0,t]$, and we have, thanks to (1.35), (1.32) and (2.1) respectively:

$$\begin{aligned}\frac{d^2}{ds^2}(R(t-s)\,u(s)) &= R''(t-s)\,u(s) - 2R'(t-s)\,u'(s) + R(t-s)\,u''(s) = \\ &= \eta AR'(t-s)\,u(s) + \beta AR(t-s)\,u(s) + (k*AR')(t-s)\,u(s) + \\ &+ (h*AR)(t-s)\,u(s) - 2u'(s) - 2\eta AR(t-s)\,u'(s) - \\ &- 2\beta(1*AR)(t-s)\,u'(s) - 2(k*AR)(t-s)\,u'(s) - \\ &- 2(1*h*AR)(t-s)\,u'(s) + \eta R(t-s)\,Au'(s) + \\ &+ \beta R(t-s)\,Au(s) + R(t-s)(k*Au')(s) + R(t-s)(h*Au)(s).\end{aligned}$$

Integrating both members of the previous identity between 0 and t, we get

$$\begin{aligned}(2.4)\qquad -u(t) + R'(t)\,x - R(t)\,y &= \eta(AR'*u)(t) + \beta(AR*u)(t) + \\ +((k*AR')*u)(t) + ((h*AR)*u)(t) &- 2u(t) + 2x - 2\eta(AR*u')(t) - \\ -2\beta((1*AR)*u')(t) - 2((k*AR)*u')(t) &- 2((1*h*AR)*u')(t) + \\ +\eta(R*Au')(t) + \beta(R*Au)(t) &+ (R*(k*Au'))(t) + (R*(h*Au))(t).\end{aligned}$$

We now observe that

$$(2.5)\qquad (AR*u')(t) = -AR(t)\,x + (AR'*u)(t),$$

$$(2.6)\qquad ((1*AR)*u')(t) = (AR*u)(t) - (1*AR)(t)\,x,$$

$$(2.7)\qquad ((k*AR)*u')(t) = -(k*AR)(t)\,x + ((k*AR')*u)(t),$$

$$(2.8)\qquad ((1*h*AR)*u')(t) = h*AR*u(t) - 1*h*AR(t)\,x.$$

Substituting (2.5)-(2.8) into (2.4), we have

$$\begin{aligned}u(t) = R(t)\,y - R'(t)\,x + \eta AR(t)\,x + 2\beta(1*AR)(t)\,x &+ \\ +(k*AR)(t)\,x + 2(1*h*AR)(t)\,x + 2x&,\end{aligned}$$

from which, taking into account (1.32), we find (2.3).

Assume now that $u \in C^1([0,T];X)$ is a strong solution of (2.1), and let $\{u_n\} \subset C^1([0,T];D(A)) \cap C^2([0,T];X)$ be a sequence satisfying (2.2). Set for any $n \in \mathbb{N}$ $x_n = u_n(0)$ and $y_n = u_n'(0)$, then $u_n(t) = R(t)y_n + R'(t)x_n - \eta AR(t)x_n - k * AR(t)x_n$, $t \in [0,T]$. Letting $n \to +\infty$ and using estimates (1.19), (1.20) and (1.31), we get (2.3), and the proof is complete.

Now, we show that, if the initial conditions x and y are sufficiently regular, then formula (2.3) gives the strict solution of (2.1).

THEOREM 2.2. *Let $x, y \in D(A)$ be such that*

$$\eta Ay + \beta Ax \in \overline{D(A)}. \tag{2.9}$$

Then the function u given by (2.3) *is the unique strict solution of* (2.1), *and belongs to $\mathcal{A}([\varepsilon, +\infty[; D(A))$ for each $\varepsilon > 0$.*

PROOF. The uniqueness obviously follows from Theorem 2.1. Concerning the existence, by (1.17)-(1.20), (1.22). (1.23), (1.29) and Proposition 1.5, i) u belongs to

$$C([0,+\infty[;X) \cap C(]0,+\infty[;D(A)) \cap C^\infty(]0,+\infty[;X)$$

and $u(t) \to x$, as $t \to 0^+$. Moreover, by (2.3) we have

$$Au(t) - Ax = \tag{2.10}$$
$$= AR(t)y + AR'(t)x - \eta AR(t)Ax - k * AR(t)Ax - Ax.$$

By adding to this equation (1.32) calculated in $(\eta/\beta)Ay$, we get

$$Au(t) - Ax = AR(t)y + \frac{1}{\beta}R'(t)(\eta Ay + \beta Ax) - \frac{\eta}{\beta}AR(t)(\eta Ay + \beta Ax) -$$
$$- \frac{1}{\beta}k * AR(t)(\eta Ay + \beta Ax) - \frac{\eta}{\beta}\int_0^t h * AR(s)Ay\,ds -$$
$$-\eta\int_0^t AR(s)Ay\,ds - \frac{1}{\beta}(\eta Ay + \beta Ax),$$

from which, by (1.23), (1.33), (2.9) and (1.19)-(1.21) we have $\lim_{t\to 0^+} Au(t) - Ax = 0$, so that $u \in C([0, +\infty[; D(A))$. By differentiating (2.3) and using (1.35), one gets

$$u'(t) = R'(t)\,y + \beta AR(t)\,x + h * AR(t)\,x\,. \tag{2.11}$$

Letting $t \to 0^+$ and using (1.33), (1.23) and (1.21), we have $u'(t) \to y$, and so that $u \in C^1([0, +\infty[; X)$. Moreover, thanks to (1.29) and (1.18) $u' \in C(]0, +\infty[; D(A))$, and we have also by (1.32) calculated in Ay

$$Au'(t) = Ay + AR(t)(\eta Ay + \beta Ax) + \beta \int_0^t AR(s)\,Ay\,ds +$$

$$+ k * AR(t)\,Ay + \int_0^t h * AR(s)\,Ay\,ds + h * AR(t)\,Ax\,.$$

Then, by (1.23), (2.9) and (1.19)-(1.21) $\lim_{t\to 0^+} Au'(t) = Ay$, so that $u \in C^1([0, +\infty[; D(A))$. To show that u satisfies (2.1), we differentiate (2.11) and we have

$$u''(t) = R''(t)\,y + \beta AR'(t)\,x + h * AR'(t)\,x\,, \tag{2.12}$$

from which, using (1.35) calculated in y, we get

$$u''(t) = \eta AR'(t)\,y + \beta AR(t)\,y + k * AR'(t)\,y +$$

$$+ h * AR(t)\,y + \beta AR'(t)\,x + h * AR'(t)\,x\,.$$

By adding and subtracting

$$\eta\beta AR(t)\,Ax + \eta h * AR(t)\,Ax + \beta k * AR(t)\,Ax + k * h * AR(t)\,Ax$$

in the previous equality and taking into account (2.11) and (2.10), we find that (2.1) holds and $u \in C^2([0, +\infty[; X)$.

Finally, the last statement follows from (1.30), and so the proof is complete.

In a forthcoming paper we will introduce a class of subspaces of X, which will enable us to give conditions on x and y, which guarantee the Hölder regularity of the strict solution up to $t = 0$.

In the case that x and y belong only to $\overline{D(A)}$ the function u defined by (2.3) is the strong solution of problem (2.1), as the following theorem shows.

THEOREM 2.3. *If* $x, y \in \overline{D(A)}$ *then the function* u *given by* (2.3) *is the unique strong solution of* (2.1) *in* $[0, T]$, $T > 0$, *and belongs to* $\mathcal{C}([\varepsilon, T]; X)$ *for each* $\varepsilon \in]0, T[$.

PROOF. The uniqueness follows from Theorem 2.1. Concerning the existence, since x and y belong to $\overline{D(A)}$ there are two sequences $\{x_n\}, \{y_n\} \subset D(A^2)$ such that $x_n \to x$, $y_n \to y$, as $n \to +\infty$ (thanks to assumption (1.2), it is not difficult to show that $D(A^2)$ is dense in $\overline{D(A)}$). Since $\eta A y_n + \beta A x_n \in D(A)$, by Theorem 2.2 for each n there exists a unique strict solution u_n of the problem

$$(2.13) \qquad \begin{cases} u_n''(t) = \eta A u_n'(t) + \beta A u_n(t) + k * A u_n'(t) + h * A u_n(t), \\ u_n(0) = x_n , \\ u_n'(0) = y_n , \end{cases}$$

given by $u_n(t) = R(t) y_n + R'(t) x_n - \eta A R(t) x_n - k * AR(t) x_n$. By (1.19), (1.31) and (1.20), u_n converges to u in $C([0, T]; X)$, $T > 0$. Moreover, by (2.11) we get $u_n'(t) = R'(t) y_n + \beta AR(t) x_n + h * AR(t) x_n$, and so, by (1.31), (1.19) and (1.21) u_n' converges in $C([0, T]; X)$, as $x \to +\infty$. Therefore, u is differentiable in $[0, T]$, $u_n' \to u'$ in $C([0, T]; X)$, as $n \to +\infty$, and we have $u'(t) = R'(t) y + \beta AR(t) x + h * AR(t) x$. From (2.13) it follows that the last condition of (2.2) holds. Finally, in virtue of (1.17) u belongs to $\mathcal{C}([\varepsilon, T]; X)$ for each $\varepsilon \in]0, T[$, and so the theorem is proved.

3. An example.

Let us consider the integrodifferential Cauchy-Dirichlet problem:

$$(3.1) \qquad \begin{cases} u_{tt}(t, x) = u_{txx}(t, x) + u_{xx}(t, x) + \displaystyle\int_0^t (t-s)^{-1/2} u_{txx}(s, x)\, ds , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0, \quad x \in [0, \pi] \\ u(0, x) = u_0(x), \qquad x \in [0, \pi], \\ u_t(0, x) = v_0(x), \qquad x \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, \qquad t > 0 . \end{cases}$$

This problem can be written in the abstract form (2.1), if we take

$X = C([0, \pi])$ and A is the operator so defined

$$\begin{cases} D(A) = \{u \in C^2([0, \pi]);\ u(0) = u(\pi) = 0\}, \\ Au = u''. \end{cases}$$

It is well known that A satisfies (1.2), while assumption (1.3) holds for $k(t) = t^{-1/2}$ and $h \equiv 0$. Therefore, all the results of Sections 1 and 2 hold for problem (3.1). If $u_0, v_0 \in C^2([0, \pi])$, with

$$u_0(0) = u_0(\pi) = v_0(0) = v_0(\pi) = 0 \tag{3.2}$$

and $u_{0xx}(0) + v_{0xx}(0) = u_{0xx}(\pi) + v_{0xx}(\pi) = 0$, then, thanks to Theorem 2.2, problem (3.1) has a unique strict solution given by

$$u(t,\cdot) = R(t)v_0 + R'(t)u_0 - \frac{d^2}{dx^2}(R(t)u_0) - \int_0^t (t-s)^{-1/2} \frac{d^2}{dx^2}(R(s)u_0)\,ds, \qquad t > 0, \tag{3.3}$$

where $R(t)$ is the resolvent operator for (3.1). In the case that u_0, v_0 belong to $C([0, \pi])$ and verify (3.2), then, by Theorem 2.3 formula (3.3) gives the unique strong solution of problem (3.1).

In addition, taking into account that the spectrum of the operator A is given by the sequence $\{-n^2, n \in \mathrm{N}\}$, we have that

$$\sigma(F) = \bigcup_{n=1}^{\infty} \{\lambda \in \mathrm{C};\ \lambda^2 + n^2\lambda + n^2 + \sqrt{\lambda}\, n^2 = 0\}.$$

Thanks to this formula, it is easy to see that $\omega_F < 0$; by numerical calculations we have $\omega_F \simeq -0.9567$. Therefore, by estimate (1.41) the strict and strong solutions, given by (3.3), go to 0 exponentially as $t \to +\infty$.

REFERENCES

[1] C. M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Diff. Eq., 7 (1970), pp. 554-569.

[2] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rat. Mech. Anal., 37 (1970), pp. 297-308.

[3] C. M. Dafermos - J. A. Nohel, *Energy methods for nonlinear hyperbolic Volterra integrodifferential equations*, Comm. Part. Diff. Eq., 4 (3) (1979), pp. 219-278.

[4] C. M. DAFERMOS - J. A. NOHEL, *A nonlinear hyperbolic Volterra equation in viscoelasticity*, Amer. J. Math. Supplement (1981), pp. 87-116.

[5] G. DA PRATO - M- IANNELLI, *Linear integro-differential equations in Banach spaces*, Rend. Sem. Mat. Univ. Padova, **62** (1980), pp. 207-219.

[6] G. DA PRATO - M. IANNELLI - E. SINESTRARI, *Temporal regularity for a class of integrodifferential equations in Banach spaces*, Boll. U.M.I. Anal. Funz. Appl. Serie VI, Vol. II-C, N. 1 (1983), pp. 171-185.

[7] G. DA PRATO - M. IANNELLI - E. SINESTRARI, *Regularity of solutions of a class of linear integrodifferential equations in Banach spaces*, J. Int. Eq., 8 (1985), pp. 27-40.

[8] G. DA PRATO - M. IANNELLI, *Existence and regularity for a class of integro-differential equations of parabolic type*, J. Math. Anal. Appl., **112** (1985), pp. 36-55.

[9] G. DA PRATO - A. LUNARDI, *Solvability on the real line of a class of linear Volterra integrodifferential equations of parabolic type*, Ann. Mat. Pura Appl. (IV), CL (1988), pp. 67-118.

[10] W. J. HRUSA - J. A. NOHEL, *The Cauchy problem in one-dimensional nonlinear viscoelasticity*, J. Diff. Eq., **59** (1985), pp. 388-412.

[11] W. J. HRUSA - M. RENARDY - J. A. NOHEL, *Mathematical Problems in Viscoelasticity*, Longman Group Limited.

[12] A. LUNARDI, *Laplace transform methods in integrodifferential equations*, J. Int. Eq., **10** (1985), pp. 185-211.

[13] R. C. MACCAMY, *A model for one-dimensional, nonlinear viscoelasticity*, Q. Appl. Math., **35** (1977), pp. 21-33.

[14] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton (1941).

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