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Fourier Integral Operators of Infinite Order on $\mathcal{D}_{L^2}^{\{\sigma\}}$ ($\mathcal{D}_{L^2}^{\{\sigma\}'}$) with an Application to a Certain Cauchy Problem.

ROSSELLA AGLIARDI (*)

Introduction.

The aim of this paper is to develop a calculus of Fourier integral operators of infinite order in the spaces $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$ and to apply it to prove some sufficient conditions for a certain Cauchy problem to be well-posed in the above-mentioned spaces.

The calculus we develop here is analogous to the one in [4] in Gevrey classes and in their dual spaces of ultradistributions. As for the spaces $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$ we consider here, we recall that they have been employed many a time in dealing with the Cauchy problem and the propagation of Gevrey singularities. For instance in [15] it is shown that some pseudo-differential and Fourier integral operators of finite order continuously map $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}})$ to themselves and the same thing is true for the fundamental solution of a hyperbolic equation with constant multiplicities constructed in [14]. Specifically in [14] the hyperbolic equation is reduced to an equivalent system. Therefore at first a fundamental solution is determined for an operator of the form

$$P = \partial_t - i\lambda(t, x, D_x) + a(t, x. D_x) \tag{I}$$

where the symbol of λ is real, λ and a are continuous in t with values in some spaces of symbols of Gevrey type σ and of order 1 and p respectively, for some $p \in [0, 1]$. A fundamental solution is found which maps $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$ to itself whenever $\sigma < 1/p$. The well-posedness of the Cauchy problem for an operator of the form (I) is well-known when $\sigma < 1/p$ (see also [11]). A necessary condition for the well-

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This work is dedicated to the memory of Prof. L. Cattabriga.

posedness in the case $\sigma > 1/p$ is proved in [10]. There it is assumed that whenever

$$a(t, x, \xi) = \mathring{a}(t, x, \xi) + \tilde{a}(t, x, \xi)$$

with $\mathring{a}$ homogeneous in ξ of degree p and order $\tilde{a} < p$, the following condition is required for the well-posedness of the Cauchy problem (with initial datum at $t = 0$) when $\sigma > 1/p$:

$$(C_{\mathrm{I}}) \qquad \operatorname{Re} \mathring{a}(0, x, \xi) \geqslant 0 \qquad \forall (x, \xi) \in R^n \times R^n .$$

In what follows we shall prove some sufficient conditions for the Cauchy problem to (I) to be well-posed when $\sigma \geqslant 1/p$ (see [2] for the case where λ and a do not depend on x). Moreover we shall confine our discussion to analytical symbols; we refer to [1] for the Gevrey case in the simplified case where $\lambda = 0$. Specifically we shall prove the following

THEOREM. Let P be an operator of the form (I) where λ and a are pseudo-differential operators whose symbols satisfy the following properties:

1) $\lambda(t, x, \xi)$ is real valued and belongs to $\mathcal{C}([0, T]; \tilde{S}^{1,1,1}(R^{2n}))$

2) $a(t, x, \xi) \in \mathcal{C}([0, T]; \tilde{S}^{p,1,1}(R^{2n}))$ with $p \in [0, 1[$. Moreover if $s \in [0, T[$ and φ solves the eiconal equation:

$$\begin{cases} \partial_t \varphi(t, s; x, \xi) = \lambda(t, x, \nabla_x \varphi(t, s; x, \xi)) \\ \qquad\qquad\qquad \text{for } t \in [s, T'], \text{ for a suitable } T' \in \,]s, T] , \\ \varphi(s, s; x, \xi) = x \cdot \xi \end{cases}$$

then we claim that

$$(C_{\mathrm{II}}) \qquad \varliminf_{\varrho \to +\infty} \varrho^{-1/\sigma} \int_s^t \operatorname{Re} a(t', x, \nabla_x \varphi(t', s; x, \varrho\eta))\, dt' \geqslant 0$$

$\forall t \in [s, T']$, uniformly with respect to $x, \eta \in R^n \times S_{n-1}$, where we assume $\sigma(2p - 1) < 1$, $\sigma \geqslant 1$. Then the Cauchy problem for P with datum at $t = s$ is well-posed in $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$.

This paper is organized as follows. In § 0 we give some preliminary definitions. § 1 is devoted to the development of a calculus for Fourier integral operators of infinite order in $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$ which allows us to prove the above mentioned result concerning the well-posedness of the Cauchy problem in these spaces. Indeed in § 2, by applying some results of § 1, we construct a parametrix for the Cauchy problem by solving the transport equations, as in [3] and [4]. Under our assumptions it turns out to be a Fourier integral operator of infinite order of the kind examined previously. Then a fundamental solution is determined. Finally we give some results concerning the propagation of Gevrey singularities. I wish to thank Prof. D. Mari and Prof. L. Zanghirati for some suggestions.

0. Main notation and definitions.

For $\xi \in R^n$ we set $\langle\xi\rangle = \sqrt{1 + \sum_{j=1}^{n} \xi_j^2}$. For $x = (x_1, \ldots, x_n) \in R^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ we write $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, where $D_{x_j} = -i\partial_{x_j}$. By $\langle D_x\rangle^N$ we mean a pseudo-differential operator of order N whose symbol is $\langle\xi\rangle^N$.

We recall here the notation concerning symbols of infinite order of Gevrey type that can be found in [3].

We shall say that $p(x, \xi) \in S_b^{\infty,\sigma,\mu}(R^n \times R^n_{B_0,B;A})$, where $A, B, B_0 > 0$, $\sigma \geqslant 1$, $\mu \geqslant 1$, if $\forall \varepsilon > 0$ there exists $C_\varepsilon \geqslant 0$ such that:

$$\sup_{x\in R^n} |\partial_\xi^\alpha D_x^\beta p(x, \xi)| \leqslant C_\varepsilon A^{|\alpha+\beta|} \alpha!^\mu \beta!^\sigma \langle\xi\rangle^{-|\alpha|} \exp\left[\varepsilon|\xi|^{1/\sigma}\right]$$

$\forall \alpha, \beta \in N^n$, $\forall \xi \in R^n$, $|\xi| \geqslant B_0 + B|\alpha|^\sigma$. We shall write $\tilde{S}_b^{\infty,\sigma,\mu}(R^n \times R^n_{B_0;A})$ for $S_b^{\infty,\sigma,\mu}(R^n \times R^n_{B_0,0;A})$. We shall denote by $S_b^{\infty,\sigma,\mu}$ the space $\varinjlim_{A, B_0, B \to +\infty} S^{\infty,\sigma,\mu}(R^n \times R)$.

As for formal series of symbols we shall say that $\sum_{j\geqslant 0} p_j(x, \xi)$ is in $FS_b^{\infty,\sigma,\mu}(R^n \times R^n_{B_0,B;A})$ if $p_j(x, \xi) \in S_b^{\infty,\sigma,\mu}(R^n \times R^n_{B_0,B;A})$ and $\forall \varepsilon > 0$ $\exists C^\circ \geqslant 0$ such that:

$$\sup_{x\in R^n} |\partial_\xi^\alpha D_x^\beta p_j(x, \xi)| \leqslant C_\varepsilon A^{|\alpha+\beta|+} \alpha!^\mu (\beta! j!)^\sigma \langle\xi\rangle^{-|\alpha|-j} \exp\left[\varepsilon|\xi|^{1/\sigma}\right]$$

$\forall \xi \in R^n$, $|\xi| \geqslant B_0 + B(\alpha| + j)^\sigma$.

We shall give the following definition of equivalence of formal series of symbols. We shall write that $\sum_{j\geqslant 0} p_j(x, \xi) \sim 0$, if $\forall \varepsilon > 0$ $\exists C_\varepsilon \geqslant 0$ and $A, B, B_0 \geqslant 0$ such that:

$$\sup_{x\in R^n} \left|\partial_\xi^\alpha D_x^\beta \sum_{j<s} p_j(x, x)\right| \leqslant C_\varepsilon A^{\alpha+\beta+s} \alpha!^\mu (\beta! s!)^\sigma \langle\xi\rangle^{-|\alpha|-s} \exp\left[\varepsilon|\xi|^{1/\sigma}\right]$$

$\xi \in R^n$, $|\xi| > B_0 + B(|\alpha| + s)^\sigma$.

In [15] it is shown how to construct a « true » symbol from a series of formal symbols denoted by $\sum_{j\geqslant 0} p_j$, i.e. $p(x,\xi)\in S_b^{\infty,\sigma,\mu}$ is found in order to have $p\sim\sum_{j\geqslant 0} p_j$.

We recall also that whenever $p\sim 0$ in $FS_b^{\infty,\sigma,\mu}$ then $\exists A, B_0, C\geqslant 0$ and $h>0$ such that:

$$\sup_{x\in R^n} |D_x^\beta p(x,\xi)|\leqslant CA^{|\beta|}\beta!^\sigma \exp[-h|\xi|^{1/\sigma}] \qquad \forall\beta\in\mathbb{N}^n, \quad \forall|\xi|\geqslant B_0 .$$

Now we recall here a few definitions concerning the spaces which are of interest in this work. We refer to [8] and [14] for a more detailed outline. For notational convenience we shall often omit to point out the domain when R^n is meant.

Let $\mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}} = \{f\in L^2; \exp[\varepsilon\langle\xi\rangle^{1/\sigma}]\cdot(\mathcal{F}f)(\xi)\in L^2\}$ where $\varepsilon>0$, $\sigma\geqslant 1$. (By $\mathcal{F}f$ we denote the L^2-Fourier transform of f).

Let $\mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}'}$ be the dual space of the Hilbert space $\mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}}$ and denote by $\mathcal{F}$ the transpose of the operator

$$\mathcal{F}\colon \{v\in L^2; \exp[\varepsilon\langle\xi\rangle^{1/\sigma}]v(\xi)\in L^2\}\to \mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}} .$$

By using the notation $\mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}}$ for any real number ε and by denoting both $\mathcal{F}f$ and ${}^t\mathcal{F}f$ by $\tilde{f}$, we get

$$\mathfrak{D}_{L^2,-\varepsilon}^{\{\sigma\}} = \mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}}$$

and we denote the norm in these spaces by

$$\|f\|_{\mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}}} = \|\exp[\varepsilon\langle\xi\rangle^{1/\sigma}]\tilde{f}(\xi)\|_{L^2(R_\xi^n)} .$$

Afterwards we define:

$$\mathfrak{D}_{L^2}^{\{\sigma\}} = \varinjlim_{\varepsilon\to 0^+} \mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}} , \qquad \mathfrak{D}_{L^2}^{(\sigma)} = \varprojlim_{\varepsilon\to +\infty} \mathfrak{D}_{L^2,\varepsilon}^{(\sigma)} ,$$

$$\mathfrak{D}_{L^2}^{\{\sigma\}'} = \varprojlim_{\varepsilon\to 0^+} \mathfrak{D}_{L^2,\varepsilon}^{\{\sigma\}'} , \qquad \mathfrak{D}_{L^2}^{(\sigma)'} = \varinjlim_{\varepsilon\to +\infty} \mathfrak{D}_{L^2,\varepsilon}^{(\sigma)'} .$$

The following spaces will also be used:

$$\mathcal{S}_{\sigma,\varepsilon} = \{f\in\mathcal{S}; \exp[\varepsilon\langle\xi\rangle^{1/\sigma}]\tilde{f}(\xi)\in\mathcal{S}\} , \qquad \varepsilon>0 , \ \sigma\geqslant 1 .$$

which are Fréchet spaces with the semi-norms:

$$|f|_{S_{\sigma,\varepsilon}} = \sup_{\xi\in R^n} \sup_{|\alpha|+k=l} |\langle\xi\rangle^k \partial_\xi^\alpha(\exp[\varepsilon\langle\xi\rangle^{1/\sigma}]\tilde{f}(\xi))| , \qquad l = 0, 1, \ldots .$$

Finally we shall denote by $\mathcal{E}_b^{\{\sigma\}}$ the space of all $f \in C^\infty(R^n)$ satisfying:

$$\sup_{x\in R^n} |\partial_x^\alpha f(x)| \leqslant C_0 C_1{}^{|\alpha|} \alpha!^\sigma \quad \text{for some } C_0, C_1 \geqslant 0 , \quad \sigma \geqslant 1 .$$

1. Fourier integral operators of infinite order on $\mathcal{D}_{L^2}^{\{\sigma\}}$ ($\mathcal{D}_{L^2}^{\{\sigma\}'}$).

In the following pages we shall study operators with amplitude in $S_b^{\infty,\sigma,\mu}$ and phase $\varphi \in S^{1,\sigma,\mu}(R^n \times R^n_{B_0(\varphi),0;A(\varphi)})$ satisfying:

$$\text{(i)} \quad \sum_{|\alpha+\beta|\leqslant 2} \sup_{\substack{x,\xi\in R^n\\ |\xi|\geqslant B_0(\varphi)}} |\partial_\xi^\alpha D_x^\beta(\varphi(x, \xi) - x\cdot\xi)|/\langle\xi\rangle^{1-|\alpha|} \leqslant \tau \text{ for some } \tau \in [0, 1[$$

(see [9], Def. 1.2 in Cap. 10).

In this paragraph the action of the above-mentioned operators on the spaces $S_{\sigma,\varepsilon}$ and $\mathcal{D}_{L^2}^{\{\sigma\}}$ is investigated. Notice that the operator defined by:

$$(1.1) \qquad P_\varphi u(x) = \int \exp[i\varphi(x, \xi)]p(x, \xi)\tilde{u}(\xi)\,đ\xi , \qquad \forall u \in S_{\sigma,\varepsilon} \text{ or } \forall u \in \mathcal{D}_{L^2}^{\{\sigma\}}$$

is well-defined. Moreover we have the following

THEOREM 1.1. If $p(x, \xi)$, $\varphi(x, \xi)$ are as above, then $\forall\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\forall\delta \in]0, \delta_\varepsilon]$ the operator P_φ defined by (1.1) is a continuous map from $S_{\sigma,\varepsilon}$ to $S_{\sigma,\delta}$ and from $\mathcal{D}_{L^2,\varepsilon}^{\{\sigma\}}$ to $\mathcal{D}_{L^2,\delta}^{\{\sigma\}}$.

Therefore P_φ: $\mathcal{D}_{L^2}^{\{\sigma\}} \to \mathcal{D}_{L^2}^{\{\sigma\}}$ is a continuous map.

Moreover P_φ extends to a continuous operator from $\mathcal{D}_{L^2}^{\{\sigma\}'}$ to $\mathcal{D}_{L^2}^{\{\sigma\}'}$.

Now let us consider *σ-regularizing operators.*

If $r(x, \xi)$ has the property:

$$\text{(i)} \sup_{x\in R^n} |D_x^\beta r(x, \xi) \leqslant C A^{|\beta|} \beta!^\sigma \exp[-h\langle\xi\rangle^{1/\sigma}]$$

with $C, A, h > 0$, $\forall\beta \in \mathbb{N}^n$, $\forall\xi \in R^n$, $|\xi|$ sufficiently large, then the operator defined by

$$R_\varphi u(x) = \int \exp[i\varphi(x, \xi)]r(x, \xi)\tilde{u}(\xi)\,đ\xi , \qquad \forall u \in \mathcal{D}_{L^2}^{\{\sigma\}}$$

extends to an operator defined on $\mathcal{D}_{L^2}^{\{\sigma\}'}$.

Furthermore it follows that

$$\sup_{x\in R^n} |D_x^\beta R_\varphi u(x)| \leqslant \tilde{C}\tilde{A}\beta!^\sigma \quad \text{for every } u \in \mathfrak{D}_{L^2}^{\{\sigma\}'},$$

that is, R_φ maps $\mathfrak{D}_{L^2}^{\{\sigma\}'}$ to $\mathcal{E}_b^{\{\sigma\}}$.

More precisely we can prove:

THEOREM 1.2. If $r(x, \xi) \in \mathcal{C}^\infty(R^n \times R^n)$ has the property

$$\text{(ii)} \sup_{x\in R^n} |\partial_\xi^\alpha D_x^\beta r(x, \xi)| \leqslant C_\alpha A^{|\beta|}\beta!^\sigma \exp[-h\langle\xi\rangle^{1/\sigma}]$$

for $|\xi| \geqslant B_0 + B|\alpha|^\sigma$ and with $h > 0$, then the operator R_φ defined on $\mathfrak{D}_{L^2}^{\{\sigma\}'}$ by

$$R_\varphi u(x) = \int \exp[i\varphi(x, \xi)]\, r(x, \xi)\tilde{u}(\xi)\, d\xi$$

is a continuous map from $\mathfrak{D}_{L^2}^{\{\sigma\}'}$ to $\mathfrak{D}_{L^2}^{\{\sigma\}}$.

THEOREM 1.3 (composition of a pseudo-differential and a Fourier integral operator). Let

$$p_1(x, \xi) \in S_b^{\infty,\sigma,1}(R^n \times R^n_{B_0^1, 0; A^1}), \qquad p_2(x, \xi) \in S_b^{\infty,\sigma,\mu}(R^n \times R^n_{B_0^2, B^2; A^2})$$

and the phase $\varphi(x, \xi) \in S_b^{1,\sigma,\mu}(R^n \times R^n_{B_0(\varphi), 0; A(\varphi)})$ with property (i).

Let

$$P_1 u(x) = \int \exp[ix\cdot\xi]\, p_1(x, \xi)\tilde{u}(\xi)\, d\xi;$$

$$P_{2\varphi} i u(x) = \int \exp[i\varphi(x, \xi)]\, p_2(x, \xi)\tilde{u}(\xi)\, d\xi$$

for every $u \in \mathfrak{D}_{L^2}^{\{\sigma\}}$.

Then there exists an operator Q_φ defined on $\mathfrak{D}_{L^2}^{\{\sigma\}}$ by

$$Q_\varphi u(x) = \int \exp[ib(x, \xi)]\, q(x, \xi)\tilde{u}(\xi)\, d\xi$$

and such that $q(x, \xi) \sim \sum_{j\geqslant 0} q_j(x, \xi)$, where

$$q_j(x, \xi) = \sum_{|\gamma|=j} 1/\gamma D_y^\gamma\Big(\big(\partial_\xi^\gamma p_1(x, \tilde{\nabla}_x\varphi(x, y, \xi)\big)p_2(y, \xi)\Big)\Big|_{y=x}$$

with

$$\tilde{\nabla}_x\varphi(x, y, \xi) = \int_0^1 \nabla_x\varphi(y + \theta(x - y), \xi)\, d\theta,$$

and there exists an operator R_φ continuously mapping $\mathcal{D}_{L^2}^{\{\sigma\}'}$ to $\mathcal{D}_{L^2}^{\{\sigma\}}$, such that:

$$P_1(x, D_x)\big(P_{2,\varphi}(x, D_x)u(x)\big) = Q_\varphi(x, D_x)u(x) + R_\varphi(x, D_x)u(x) \qquad \forall u \in \mathcal{D}_{L^2}^{\{\sigma\}}.$$

Wave front sets in Gevrey classes. When $u \in \mathcal{D}_{L^2}^{\{\sigma\}'}$ we give the following definition of wave front set.

DEFINITION (see [15]). Let $u \in \mathcal{D}_{L^2}^{\{\sigma\}'}$ with $\sigma > 1$; let $\sigma_1 \geqslant \sigma$. We say that a point $(x_0, \xi_0) \in T^*(R_x^n)$ does not belong to $WF_{\{\sigma_1\}}(u)$ when there exists a symbol $a(x, \xi)$ in $S^{0,\sigma,\sigma}$ with $a(x_0, \theta\xi_0) \neq 0$ $(\theta \geqslant 1)$ such that $a(x, D_x)u(x) \in \mathcal{E}_b^{\{\sigma\}}(R_x^n)$.

REMARK. In [14] Taniguchi has shown that this definition is equivalent to the one given by Hörmander (1971) in the case where $u \in \mathcal{E}'$.

For Fourier integral operators of the kind analyzed above we have the following result:

THEOREM 1.4. If p and φ satisfy the properties listed previously and moreover $\varphi(x, \xi)$ is homogeneous in ξ of degree 1 for $|\xi|$ sufficiently large, then for every $\sigma_1 \geqslant \sigma$ we have:

$$WF_{\{\sigma_1\}}(P_\varphi u) \subset \{(x, \theta\nabla_x\varphi(x, \xi));$$
$$\theta \geqslant 1,\ (\nabla_\xi\varphi(x, \xi), \xi) \in WF_{\{\sigma_1\}}(u),\ |\xi| \text{ sufficiently large}\}.$$

2. An application of Fourier integral operators of infinite order to the investigation of sufficient conditions for a certain Cauchy problem to be well-posed in $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$.

In this section we consider operators of the form:

$$P = \partial_t - i\lambda(t, x, D_x) + a(t, x, D_x)$$

where λ and a are pseudo-differential operators whose symbols are in $\mathcal{C}([0, T]; \mathcal{C}^\infty(R^{2n}))$ and satisfy the following properties:

$\lambda(t, x, \xi)$ is real valued and belongs to $\mathcal{B}([0, T]; \tilde{S}^{1,1,1}(R^n \times R^n))$;

$a(t, x, \xi)$ is in $\mathcal{B}([0, T]; \tilde{S}^{p,1,1}(R^n \times R^n))$ with $p \in [0, 1[$ and verifies:

(2.i) $\varliminf_{\varrho \to +\infty} \varrho^{-1/\sigma} \operatorname{Re} a(t, x, \varrho\eta) \geqslant 0$ for every $t \in [0, T]$ and uniformly with respect to $(x, \eta) \in R^n \times S_{n-1}$, where we assume $\sigma(2p - 1) < 1$, $\sigma \geqslant 1$.

Remark 2.1. Actually in dealing with the Cauchy problem with initial datum at $t = s$, $s \in [0, T[$, it is sufficient to claim:

(2.i') $\underline{\lim}_{\varrho \to +\infty} \varrho^{-1/\sigma} \int_s^t \mathrm{Re}\, a(t', x, \nabla_x \varphi(t', s; x, \varrho\eta))\, dt' \geqslant 0$, $\forall t \in [s, T']$, uniformly with respect to $(x, \eta) \in R^n \times S_{n-1}$, where φ and T' are determined in the following Prop. 2.1.

Remark 2.2. When $\sigma < 1/p$ the assumption (2.i) is trivially true and it is well-known that the Cauchy problem for P is well-posed in $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$.

We henceforth confine our discussion to the case where $\sigma \geqslant 1/p$.

We recall now the following

Proposition 2.1 (see [4], [14]). If λ is as above, then there exists $T' > 0$ and there exists a solution $\varphi(t, s)$ of the eiconal equation

$$\begin{cases} \partial_t \varphi(t, s; x, \xi) = \lambda(t, x, \nabla_x \varphi(t, s; x, \xi)) \\ \varphi(s, s; x, \xi) = x \cdot \xi \end{cases} \tag{2.1}$$

where $\varphi(x, \xi) - x \cdot \xi \in \mathcal{C}^1([0, T']^2; \mathcal{C}^\infty(R^{2n})) \cap \mathcal{B}^1([0, T']^2; \tilde{S}^{1,1,1}(R^{2n})) \cap \mathcal{T}(c_0|t - s|)$ for some $c_0 > 0$. (For the definition of $\mathcal{T}(\tau)$ we refer to [9].

Remark 2.3. In the proof of Prop. 2.1 it is shown that if T' is sufficiently small, a solution (p, q) of

$$\begin{cases} \dot{q} = -\nabla_\xi \lambda(t, q, p)\,, \quad \dot{p} = \nabla_x \lambda(t, q, p) \\ (q, p)(t = s) = (y, \eta) \end{cases}$$

can be found with the following properties:

$$q(t, s; y, \eta) - y \in \mathcal{B}^1([0, T']^2; \tilde{S}^{0,1,1}) \cap \mathcal{C}^1([0, T']^2; \mathcal{C}^\infty(R^{2n}))$$

$$p(t, s; y, \eta) \in \mathcal{B}^1([0, T']^2; \tilde{S}^{1,1,1}) \cap \mathcal{C}^1([0, T']^2; \mathcal{C}^\infty(R^{2n}))$$

$$|q(t, s; y, \eta) - y| \leqslant C_1|t - s|; \qquad |p(t, s; y, \eta) - \eta| \leqslant C_1|t - s|$$

for some $C_1 \geqslant 0$.

By choosing a suitable T' it can also be proved that there exists the inverse function $y = Y(t, s; x, \eta)$ of $x = q(t, s; y, \eta)$ and

$$|Y(t, s; x, \eta) - x| \leqslant C_2|t - s|$$

for some $C_2 \geqslant 0$,

$$Y(t, s; x, \eta) - x \in \mathcal{B}^1([0, T']^2; \tilde{S}^{0,1,1}) \cap \mathcal{C}^1([0, T']^2; \mathcal{C}^\infty(R^{2n}))\ .$$

THEOREM 2.1. If P is as above, then for every $s \in [0, T[$ there exists $T' \in]s, T]$ and there exists $e(t, s; x, \xi)$ continuously differentiable up to the first order with respect to s, t with values in $\mathcal{C}^\infty(R^n \times R^n)$ and belonging to $\tilde{S}^{\infty,\sigma,\mu}(R^n \times R^n)$ uniformly with respect to t and s, and with some $\mu \geqslant 1$, satisfying the following properties. If φ is the solution of (2.1), then the Fourier integral operator defined by:

$$(E_\varphi u)(x) = \int \exp\,[i\varphi(t, s; x, \xi)]\, e(t, s; x, \xi)\, \tilde{u}(\xi)\, d\!\!\!{}^-\xi \qquad \forall u \in \mathcal{D}_{L^2}^{\{\sigma\}}$$

satisfies

$$\begin{cases} PE_\varphi(t, s) = R_\varphi(t, s) & \forall t \in [s, T'] \\ E_\varphi(s, s) = I & \text{(identity operator)} \end{cases}$$

where $R_\varphi(t, s)$ is a continuous map from $\mathcal{D}_{L^2}^{\{\sigma\}'}$ to $\mathcal{D}_{L^2}^{\{\sigma\}}$, for every t and s.

PROOF. In view of Th. 1.3 it is sufficient to determine $e(t, s; x, \xi) \sim \sum_{h \geqslant 0} e_h(t, s; x, \xi)$ such that $e(s, s; x, \xi) = 1$ and that the following term vanishes:

$$\begin{aligned} &\partial_t e(t, s; x, \xi) - \tfrac{1}{2} \sum_{j,k=1,\dots,n} \partial^2_{\xi_k \xi_j} \lambda(t, x, \nabla_x \varphi(t, s; x, \xi)) \cdot \\ &\cdot \partial^2_{x_k x_j} \varphi(t, s; x, \xi)\, e(t, s; x, \xi) - i \sum_{j=1,\dots,n} (\partial_{\xi_j} \lambda)(t, x, \nabla_x \varphi(t, s; x, \xi)) \cdot \\ &\cdot D_{x_j} e(t, s; x, \xi) - i \sum_{|\gamma| \geqslant 2} 1/\gamma!\, D_z^\gamma\Big((\partial_\xi^\gamma \lambda)(t, x, \tilde{\nabla}_x \varphi(t, x, z, \xi)) \cdot e(t, s; z, \xi)\Big)_{z=x} + \\ &\qquad + \sum_{|\gamma| \geqslant 0} 1/\gamma!\, D_z^\gamma\Big((\partial_\xi^\gamma a)(t, x, \tilde{\nabla}_x \varphi(t, x, z, \xi)) \cdot e(t, s; z, \xi)\Big)_{z=x} \end{aligned}$$

where $\tilde{\nabla}_x \varphi(t, x, z, \xi) = \int_0^1 \nabla_x \varphi(t, s; z + \theta(x - z), \xi)\, d\theta$.

Therefore each e_h can be determined inductively as a solution of the following Cauchy problem:

$$(T_0) \qquad \begin{cases} \partial_t e_0(t, s) = \sum_{j=1}^n (\partial_{\xi_j} \lambda)(t, x, \nabla_x \varphi(t, s; x, \xi))\, \partial_{x_j} e_0(t, s) + \\ \qquad\qquad + g(t, s; x, \xi)\, e_0(t, s) = 0\ , \\ e_0(s, s) = 1\ , \end{cases}$$

where

$$g(t, s; x, \xi) = -\sum_{j,k=1,\dots,n} \tfrac{1}{2}\partial^2_{\xi_k\xi_j}\lambda(t, x, \nabla_x\varphi(t, s; x, \xi))\,\partial^2_{x_kx_j}\varphi(t, s; x, \xi) + \\ + a(t, x, \nabla_x\varphi(t, s; x, \xi))$$

and in the case $h = 1, 2, \dots$

$$(T_h)\quad \begin{cases} \partial_t e_h(t, s) - \sum_{j=1}^{n}(\partial_{\xi_j}\lambda)(t, x, \nabla_x\varphi(t, s; x, \xi))\,\partial_{x_j}e_h(t, s) + \\ + g(t, s; x, \xi)\,e_h(t, s) + \\ + \sum_{r=0}^{h-1}\Big(-i/\gamma!\,D_z^\gamma\{(\partial_\xi^\gamma\lambda)(t, x, \tilde\nabla_x\varphi(t, x, z, \xi))\,e_r(t, s; z, \xi)\}_{z=x} + \\ + \sum_{|\gamma|=h-r} 1/\gamma!\,D_z^\gamma\{(\partial_\xi^\gamma a)(t, x, \tilde\nabla_x\varphi(t, x, z, \xi))\,e_r(t, s; z, \xi)\}_{z=x}\Big) = 0\,, \\ e_h(s, s) = 0\,. \end{cases}$$

Putting $\hat e_h(t, s; y, \xi) = e_h(t, s; q(t, s; y, \xi), \xi)$, $h = 0, 1, \dots$ and solving the corresponding transport equations, we can prove inductively the following estimate (by applying for instance Lemma 4.2, p. 56 in [5]):

$$(I_h)\quad |\partial_\xi^\alpha\partial_y^\beta\hat e_h(t, s; y, \xi) \leqslant C_\varepsilon \exp[\varepsilon\langle\xi\rangle^{1/\sigma}(t-s)]A^{*|\alpha+\beta|+2h}\cdot \\ \cdot(|\alpha+\beta|+2h)!/h!\langle\xi\rangle^{-|\alpha|-h}\langle\xi\rangle^{(p-1/\sigma)(|\alpha+\beta|+2h)}\sum_{l=\min(|\alpha+\beta|,1)}^{\alpha+\beta|+3h}\frac{\{C_0(t-s)\langle\xi\rangle^{1/\sigma}\}^l}{l!}$$

which is true for every $\varepsilon' > 0$, with suitable positive constants A^*, B_0^*, B_1^*, C_0 and for $|\xi| \geqslant B_0^* + B_1^*(|\alpha| + h)^\sigma$.

From (I_h) it follows that $\forall\varepsilon > 0$, $\forall\mu' > 0$ we have:

$$|\partial_\xi^\alpha\partial_y^\beta\hat e_h(t, s; y, \xi)| \leqslant C'_\varepsilon \exp[\varepsilon\langle\xi\rangle^{1/\sigma}]A^{|\alpha+\beta|+h}\langle\xi\rangle^{-|\alpha|-h}(\alpha!\beta!)^{\sigma p+\mu'}h!^{2\sigma p-1+\mu'}$$

Since $2\sigma p - 1 < \sigma$ (by assumption $(2.i)$), we take $\mu' = \sigma + 1 - 2\sigma p$ and conclude with $\sum_{h\geqslant 0}\hat e_h(t, s) \in FS^{\infty,\sigma,\sigma+1-\sigma p}$ with respect to t and s.

In view of Th. 1.1.21 in [4] it is also $\sum_{h\geqslant 0}\hat e_h(t, s) \in FS^{\infty,\sigma,\sigma+1-\sigma p}$ uniformly with respect to t, s. Thus the theorem is proved.

Remark 2.4. The operator R_φ in Th. 2.1 can be regarded as a pseudo-differential operator, say $\tilde R$, whose amplitude is $\tilde r(x, \xi) = \exp[i\varphi(x, \xi) - ix\cdot\xi]\,r(x, \xi)$, if $r(x, \xi)$ is the amplitude of R. Moreover $\tilde r(x, \xi)$ has the same properties as $r(x, \xi)$.

Therefore, as we proved in [1], we have the following

LEMMA 2.1. If $\tilde{R}$ is as in Remark 2.4 then there exists a solution of

$$\tilde{R}(t, s) = - F(t, s) - \int_s^t \tilde{R}(t, \tau) F(\tau, s)\, d\tau$$

and F continuously maps $\mathcal{C}([s, T']; \mathcal{D}_{L^2}^{\{\sigma\}'})$ to $\mathcal{C}([s, T']; \mathcal{D}_{L^2}^{\{\sigma\}})$.

THEOREM 2.2. If $E_\varphi(t, s)$ and $R_\varphi(t, s)$ are as in Th. 2.1 and $F(t, s)$ is as in Lemma 2.1, then

$$\tilde{E}_\varphi(t, s) = E_\varphi(t, s) + \int_s^t \tilde{E}_\varphi(t, \tau) F(\tau, s)\, d\tau$$

is a fundamental solution for the Cauchy problem for P.

If $g \in \mathcal{D}_{L^2}^{\{\sigma\}}$, $f \in \mathcal{C}([0\ T]; \mathcal{D}_{L^2}^{\{\sigma\}})$ (respectively $g \in \mathcal{D}_{L^2}^{\{\sigma\}'}$, $f \in \mathcal{C}([0, T]; \mathcal{D}_{L^2}^{\{\sigma\}'})$), then for every $s \in [0, T[$ there exists $T' \in]s, T]$ such that $\forall t \in [s, T']$

$$u(t, x) = \tilde{E}_\varphi(t, s) g + \int_s^t \tilde{E}_\varphi(t, \tau) f(\tau, \cdot)\, d\tau \tag{2.2}$$

is the solution of the Cauchy problem

$$(C) \qquad \begin{cases} Pu(t, \cdot) = f(t, \cdot) & \text{in } [s, T'] \times R^n \\ u(s, \cdot) = g \end{cases}$$

and u belongs to $\mathcal{C}^1$ as a function of t with values in the space $\mathcal{D}_{L^2}^{\{\sigma\}}$ (respectively $\mathcal{D}_{L^2}^{\{\sigma\}'}$).

Moreover, whenever $\lambda(t, x, \xi)$ is homogeneous in ξ for $|\xi|$ large, then for every f in $\mathcal{C}([0, T]; \mathcal{D}_{L^2}^{\{\sigma\}})$ and for every g in $\mathcal{D}_{L^2}^{\{\sigma\}'}$, the solution $u(t, x)$ of the problem (C) (with $s \in [0, T[$, T' sufficiently small and $t \in [s, T']$) satisfies

$$(2.2)' \qquad WF_{\{\sigma\}}(u(t, \cdot)) \subset \{(q(t, s; y, \eta), \theta p(t, s; y, \eta)); (y, \eta) \in WF_{\{\sigma\}}(g); \\ \theta \geq 1 \text{ and } |\eta| \text{ sufficiently large}\}, \quad \forall t \in [s, T'],$$

where q, p are solutions to

$$\begin{cases} \dot{q} = - \nabla_\xi \lambda(t, q, p); \quad \dot{p} = \nabla_x \lambda(t, q, p), \\ (q, p)(t = s) = (y, \eta). \end{cases}$$

PROOF. In view of Th. 2.1 and Lemma 2.1 we have $P\tilde{E}_\varphi(t, s) = 0$ and $u(t, x)$ defined in (2.2) satisfies all the claims above. Uniqueness of the solution follows by a standard argument where the transpose of P is considered (see [4]).

Finally (2.2)′ follows from Th. 1.4 and Th. 2.2.

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