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# On the Convolution in the Space $\mathcal{D}'^{(M_p)}_{L^2}$.

STEVAN PILIPOVIĆ (*)

SUMMARY. - We investigate the convolution and the Fourier transformation in the space of Beurling ultradistributions $\mathcal{D}'^{(M_p)}_{L^2}$. We give some simple conditions on a convolutor $S$ for the solvability and the hypoelipticity of a convolution equation $S * U = V$ in $\mathcal{D}'^{(M_p)}_{L^2}$.

**1.** The space $\mathcal{D}'^{(M_p)}_{L^2}$ introduced in [3] is a subspace of the space of Beurling ultradistributions ([2]). This is a natural generalization of the Schwartz space $\mathcal{D}'_{L^2}$ and we investigated them in [3] in connection with the Hilbert transformation of ultradistributions. With suitable assumptions on the sequence $M_p$, we determined in [4] elements of $\mathcal{D}'^{(M_p)}_{L^2}$ as boundary values of certain holomorphic functions.

In [3] some questions on the convolution in $\mathcal{D}'^{(M_p)}_{L^2}$ occured. This was the motivation for our investigations of the convolution in $\mathcal{D}'^{(M_p)}_{L^2}$. The Fourier transformation maps $\mathcal{D}'^{(M_p)}_{L^2}$ into a subspace of $L^2_{\text{loc}}$. So by proving the exchange formula we obtain simple conditions for the solvability and the hypoelipticity of a convolution equation in $\mathcal{D}'^{(M_p)}_{L^2}$.

**2.** Our notation is the same as in [2]. Let $M_p$, $p \in \mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$ be a sequence of positive numbers such that

$$\text{(M.1)} \qquad M_p^2 \leqslant M_{p-1} M_{p+1}, \quad p \in \mathfrak{N};$$

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(M.2) There are constants $A$ and $H$ such that

$$M_p \leqslant AH^p \min_{0\leqslant q\leqslant p} \{M_q M_{p-q}\}, \qquad p \in \mathfrak{N}_0;$$

(M.3) There is a constant $A$ such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leqslant Ap\, M_p/M_{p+1}, \qquad p \in \mathfrak{N}.$$

We assumed in [3] instead of (M.2) and (M.3) the weaker conditions: (M.2)′ and (M.3)′. (All these conditions were analysed in [2].) The reason for that is the structural theorem for $\mathcal{D}'^{(M_p)}_{L^2}$ which we need for the full characterization of the convolution in $\mathcal{D}'^{(M_p)}_{L^2}$.

The associated function $M$ is defined by

$$M(\varrho) = \sup_{p\in\mathfrak{N}_0} \left\{\log \frac{M_0 \varrho^p}{M_p}\right\}, \qquad \varrho > 0.$$

From [2, Proposition 3.6] it follows

(1) $\quad 2M(\varrho) \leqslant M(H\varrho) + \log AM_0 \quad$ ($A$ and $H$ are from (M.2)).

The space of Beurling ultradistributions is defined ([2]) as the strong dual of the space

$$\mathcal{D}^{(M_p)} = \operatorname{inj\,lim}_{m\in\mathfrak{N}} \operatorname{proj\,lim}_{n\in\mathfrak{N}} \mathcal{D}^{(M_p)}_{m,n}$$

where

$$\mathcal{D}^{(M_p)}_{m,n} = \{\varphi \in C^\infty;\ \operatorname{supp}\varphi \subset K(m),\ \|\varphi\|_{m,n} < \infty\},$$

$C^\infty = C^\infty(\mathfrak{R}^q)$is the space of smooth functions on $\mathfrak{R}^q$, $K(m)$ is the closed ball with the center at zero and with the radius $m > 0$,

$$\|\varphi\|_{m,n} = \sup_{\substack{\alpha\in\mathfrak{N}_0^q \\ x\in K(m)}} \left\{\frac{n^{|\alpha|}|\varphi^{(\alpha)}(x)|}{M^{|\alpha|}}\right\} \qquad (|\alpha| = \alpha_1 + \ldots + \alpha_q).$$

The space $\mathcal{D}^{(M_p)}_{L^2}$ is defined ([3]) as follows:

$$\mathcal{D}^{(M_p)}_{L^2,h} = \left\{\varphi \in C^\infty \cap L^2;\ \gamma_h(\varphi) = \sum_{\alpha\in\mathfrak{N}^q_0} \frac{h^{|\alpha|}\|\varphi^{(\alpha)}\|_2}{M_{|\alpha|}} < \infty\right\}, \qquad h>0\,,$$

$$\mathcal{D}^{(M_p)}_{L^2} = \operatorname*{proj\,lim}_{\alpha\in\mathfrak{N}} \mathcal{D}^{(M_p)}_{L^2,n}\,.$$

The space $\mathcal{D}^{(M_p)}_{L^2}$ is an $FG$-space (Gelfand space, see [1]) and

$$\mathcal{D}^{(M_p)} \hookrightarrow \mathcal{D}^{(M_p)}_{L^2} \hookrightarrow \mathcal{D}_{L^2}\ ([3])$$

where « $A \hookrightarrow B$ » means that $A$ is a dense subspace of $B$ and the inclusion mapping is continuous. ($\mathcal{D}_{L^2}$ is well-known Schwartz space.)

The Fourier transformation of an $f\in L^2$ is defined by

$$(\mathcal{F}_2 f)(\xi) = \operatorname*{l.i.m.}_{A\to\infty} \int_{-A}^{A}\!\!\cdots\!\int_{-A}^{A} f(x)\exp\left(i\langle x,\xi\rangle\right) dx_1, \ldots, dx_q\,,$$

$$\xi\in\mathfrak{R}^q\left(\langle x,\xi\rangle = x_1\xi_1 + \ldots + x_q\xi_q\right),$$

where l.i.m. means the square-mean limit. Since we shall use in the paper the Fourier transformations of tempered distributions and of $\mathcal{D}'^{(M_p)}_{L^2}$-ultradistributions, we indicate these transformations by $\mathcal{F}_t$ and $\mathcal{F}_M$, respectively. It is well-known that for $f\in L^2\subset S'$, $\mathcal{F}_2 f = \mathcal{F}_t f$.

Obviously, the sequence of norms $\gamma_k$, $k\in\mathfrak{N}$, on $\mathcal{D}^{(M_p)}_{L^2}$ is equivalent to the following one:

$$\tilde{\gamma}_k(\varphi) = \sup_{\alpha\in\mathfrak{N}^q_0}\left\{\frac{k^{|\alpha|}\|\varphi^{(\alpha)}\|_2}{M_{(\alpha)}}\right\} = (2\pi)^{-q/2}\sup_{\alpha\in\mathfrak{N}^q_0}\left\{\frac{k^{|\alpha|}\|(\xi^\alpha(\mathcal{F}_2\varphi)(\xi))\|_2}{M_{|\alpha|}}\right\}.$$

$$k\in\mathfrak{N}_0\left(\xi^\alpha = \xi_1^{\alpha_1},\ldots,\xi_q^{\alpha_q}\right).$$

This implies that $\mathcal{D}^{(M_p)}_{L^2}$ is isomorphic to the space $D^{(M_p)}_{L^2} = \mathcal{F}_2(\mathcal{D}^{(M_p)}_{L^2})$ in which we transport the convergence structure from $\mathcal{D}^{(M_p)}_{L^2}$.

The inverse Fourier transformation of an $f\in L^2$ is defined by

$$(\mathcal{F}_2^{-1} f)(\xi) = (2\pi)^{-q}(\mathcal{F}_2 f)(-\xi)\,, \qquad \xi\in\mathfrak{R}^q\,.$$

Clearly, $\mathcal{F}_2^{-1}$ is an isomorphism of $D^{(M_p)}_{L^2}$ onto $\mathcal{D}^{(M_p)}_{L^2}$. This implies that

the adjoint mappings

$$\mathcal{F}_M: D'^{(M_p)}_{L^2} \to \mathcal{D}'^{(M_p)}_{L^2}, \qquad \mathcal{F}_M^{-1}\, \mathcal{D}'^{(M_p)}_{L^2} \to D'^{(M_p)}_{L^2}$$

are isomorphisms. One can easily prove that for $f \in L^2$

$$\mathcal{F}_M f = \mathcal{F}_2 f \quad \text{and} \quad \mathcal{F}_M^{-1} f = \mathcal{F}_2 f .$$

Similarly, we define $\mathcal{F}_M: \mathcal{D}'^{(M_p)}_{L^2} \to D'^{(M_p)}_{L^2}$ and $\mathcal{F}_M^{-1}: D'^{(M_p)}_{L^2} \to \mathcal{D}'^{(M_p)}_{L^2}$. Let $f \in \mathcal{D}'^{(M_p)}_{L^2}$ and $\varphi \in \mathcal{D}^{(M_p)}_{L^2}$. We have:

$$\langle \mathcal{F}_M^{-1} f, \mathcal{F}_2 \varphi \rangle = \langle f, \varphi \rangle = \langle \mathcal{F}_M f, \mathcal{F}_2^{-1} \varphi \rangle . \tag{2}$$

Let

$$D^\alpha = D_1^{\alpha_1} \dots D_q^{\alpha_q} = \left(i \frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(i \frac{\partial}{\partial x_q}\right)^{\alpha_q}.$$

An operator of the form $P(D) = \sum_{\alpha \in \mathfrak{N}_0^q} a_\alpha D^\alpha$, $a_\alpha$ are complex numbers, is called the ultradifferential operator of class $(M_p)$ if there are $L > 0$ and $C > 0$ such that

$$|a_\alpha| \leqslant C L^{|\alpha|} / M_{|\alpha|}, \qquad \alpha \in \mathfrak{N}_0^q \ ([2]) . \tag{3}$$

It was proved in [4] that the mapping $\varphi \mapsto P(D)\varphi$ from $\mathcal{D}^{(M_p)}_{L^2}$ into $\mathcal{D}^{(M_p)}_{L^2}$ is continuous. We have

$$\mathcal{F}_2(P(D)\varphi)(\xi) = P(\xi)\, \mathcal{F}_2(\varphi)(\xi), \qquad \xi \in \mathfrak{N}^q .$$

We proved in [4] the following structural theorem: $f \in \mathcal{D}^{(M_p)}_{L^2}$ iff there exist an ultradifferential operator of class $(M_p)$, $P(D)$, and an $F \in L^2$ such that

$$f = P(D) F . \tag{4}$$

By (2) and (4) we have

$$(\mathcal{F}_M f)(\xi) = P(\xi)(\mathcal{F}_2 F)(\xi), \qquad \xi \in \mathfrak{N}^q . \tag{5}$$

Thus, we see that $\mathcal{F}_M f$ is a function from $L^2_{loc}$.

For the analytic function $P(\zeta) = \sum_{\alpha \in \mathfrak{N}_0^q} a_\alpha \zeta^\alpha$, $\zeta \in \mathfrak{C}^q$, ($a_\alpha$ satisfy (3)) we have

$$(6) \qquad |P(\zeta)| \leqslant (C/M_0) \sup_{\alpha \in \mathfrak{N}_0^q} \left\{ \frac{L^{|\alpha|} M_0 |\zeta|^{|\alpha|}}{M_{|\alpha|}} \right\} \leqslant C_1 \exp\left(M(L_1|\zeta|)\right), \qquad \zeta \in \mathfrak{C}^q,$$

where $C_1 = (C/M_0) \sum_{\alpha \in \mathfrak{N}_0^q} (L/L_1)^{|\alpha|}$ and $L_1 > L$.

**3.** Let $f \in \mathcal{D}'^{(M_p)}_{L^2}$ and $\varphi \in \mathcal{D}^{(M_p)}_{L^2}$; we define

$$(f * \varphi)(x) = \langle f(t), \varphi(x - t) \rangle, \qquad x \in \mathfrak{R}^q.$$

Let us put $\psi(x) = (f * \varphi)(x)$, $x \in \mathfrak{R}^q$, and assume that $f$ is of the form (4). We have

PROPOSITION 1. *For any $\alpha \in \mathfrak{N}_0^q$, $\psi^{(\alpha)}$ is bounded and continuous. Moreover, for any $k > 0$*

$$\beta_k(\psi) = \sup_{\substack{x \in \mathfrak{R}^q \\ \alpha \in \mathfrak{N}_0^q}} \left\{ \frac{k^{|\alpha|} |\psi^{(\alpha)}(x)|}{M_{|\alpha|}} \right\} < \infty.$$

PROOF. For any $x \in \mathfrak{R}^q$ we have

$$\psi^{(\alpha)}(x) = \langle f(t), \varphi^{(\alpha)}(x - t) \rangle = \langle F(t), P(-D)\varphi^{(\alpha)}(x - t) \rangle = \int_{\mathfrak{R}^q} F(t) P(-D)\left(\varphi^{(\alpha)}(x - t)\right) dt.$$

Since $P(D)\, \varphi^{(\alpha)} \in L^2$, the first part of the assertion follows.

Using (6), for $x \in \mathfrak{R}^q$, $\alpha \in \mathfrak{N}_0^p$, we have

$$\frac{k^{|\alpha|}}{M_{|\alpha|}} |\psi^{(\alpha)}(x)| \leqslant \|F\|_2 \frac{k^{|\alpha|}}{M_{|\alpha|}} \left\| \left(P(D)\varphi^{(\alpha)}\right)(\xi) \right\|_2 \leqslant$$

$$\leqslant C_1 \frac{k^{|\alpha|}}{M_{|\alpha|}} \|F\|_2 \left\| \exp\left(M(L_1|\xi|)\right) \xi^\alpha (\mathcal{F}_2 \varphi)(\xi) \right\|_2.$$

Thus by (1) we have

$$\sup_{\substack{x \in \mathfrak{R}^q \\ \alpha \in \mathfrak{N}_0^q}} \left\{ \frac{k^{|\alpha|}}{M_{|\alpha|}} |\psi^{(\alpha)}(x)| \right\} \leqslant C \left\| \exp\left(M(L_1|\xi|) + M(k|\xi|)\right)(\mathcal{F}_2 \varphi)(\xi) \right\|_2 \leqslant$$

$$\leqslant C_0 \left\| \exp\left(M((L_1 + k)|\xi|)\right)(\mathcal{F}_2 \varphi)(\xi) \right\|_2$$

where $C$ and $C_0$ are suitable constants. Since the sequence of norms

$$\nu_k(\varphi) = \|\exp\big(M(k|\xi|)\big)(\mathcal{F}_2\varphi)(\xi)\|_2\,, \qquad k \in \mathfrak{N}\,,$$

is equivalent to the sequence $\gamma_k$, $k \in \mathfrak{N}$ ([4, Proposition 2.1]), the second assertion of the proposition follows.

If $\varphi$, $\psi \in \mathcal{D}_{L^2}^{(M_p)}$, then $\mathcal{F}_2(\varphi * \psi) = (\mathcal{F}_2\varphi)(\mathcal{F}_2\psi)$ ([5]). (In this case $*$ is the ordinary convolution.)

PROPOSITION 2. *Let $f \in \mathcal{D}_{L^2}'^{(M_p)}$ and $\varphi \in \mathcal{D}_{L^2}^{(M_p)}$. Then*

$$\mathcal{F}_t(f * \varphi)(\xi) = (\mathcal{F}_M f)(\xi)(\mathcal{F}_2\varphi)(\xi)\,, \qquad \xi \in \mathfrak{R}^q\,.$$

PROOF. We assume that $f$ is of the form (4). We have

$$(f * \varphi)(x) = \langle P(D)F(t), \varphi(x-t)\rangle = \langle F(t), P(-D)\varphi(x-t)\rangle = \\ = \int_{\mathfrak{R}^q} F(t)P(-D)\varphi(x-t)\,dt = (F * P(D)\varphi)(x)\,.$$

since $F \in L^2$ and $P(D)\varphi \in L^2$, from the remark given before Proposition 2 and (5), the assertion follows.

**4.** Let $g \in \mathcal{D}_{L^2}'^{(M_p)}$ such that for every $\varphi \in \mathcal{D}_{L^2}^{(M_p)}$, $g * \varphi \in \mathcal{D}_{L^2}^{(M_p)}$. Then we call $g$ the convolution-operator or convolutor. The space of all convolutors is denoted by $\mathcal{O}_C'(\mathcal{D}_{L^2}'^{(M_p)}, \mathcal{D}_{L^2}'^{(M_p)})$ or in short, by $\mathcal{O}_C'^{(M_p)}$. (Note that we do not assume the continuity of the mapping $\varphi \to g * \varphi$, in $\mathcal{D}_{L^2}^{(M_p)}$. This will be proved in Proposition 5.)

Proposition 2 implies.

PROPOSITION 3. *Let $g \in \mathcal{O}_C'^{(M_p)}$ and $\varphi \in \mathcal{D}_{L^2}^{(M_p)}$. Then*

$$\mathcal{F}(g * \varphi)(\xi) = (\mathcal{F}_M g)(\xi)(\mathcal{F}_2\varphi)(\xi)\,, \qquad \xi \in \mathfrak{R}^q\,.$$

PROPOSITION 4. *A $g \in \mathcal{D}_{L^2}'^{(M_p)}$ is from $\mathcal{O}_C'^{(M_p)}$ iff there exists $k > 0$ such that*

$$(\mathcal{F}_M g)\exp\big(-M(k|\cdot|)\big) \in L^\infty\,. \tag{7}$$

PROOF. From (7) it follows that for any $\varphi \in \mathcal{D}^{(M_p)}_{L^2}$ and any $\alpha \in \mathfrak{N}^q_0$

$$\mathcal{F}_M(g * \varphi^{(\alpha)}) = (\mathcal{F}_M g)\, \mathcal{F}_2(\varphi^{(\alpha)}) \in L^2$$

and

$$\mathcal{F}_M(g * \varphi^{(\alpha)}) = \mathcal{F}_2(g * \varphi^{(\alpha)}) = (\mathcal{F}_M g)(-i\xi)^\alpha (\mathcal{F}_2 \varphi)\,.$$

Let $r > 0$. From (7) and (1) we obtain

$$\sup_{\alpha \in \mathfrak{N}^q_0} \left\{ \frac{r^{|\alpha|}}{M_{|\alpha|}} \|g * \varphi^{(\alpha)}\|_2 \right\} \leqslant (2\pi)^{-q/2} \left\| \sup_\alpha \left\{ \frac{r^{|\alpha|}}{M_{|\alpha|}} |(\mathcal{F}_M g)(\xi)|\, |\xi|^\alpha |(\mathcal{F}_2 \varphi)(\xi)| \right\} \right\|_2 \leqslant$$

$$\leqslant C \| \exp\Big(M(r|\xi|)\Big) \exp\Big(M(k|\xi|)\Big)(\mathcal{F}_2 \varphi)(\xi) \|_2$$

$$\leqslant \bar{C} \| \exp\Big(M\big((r+k)|\xi|\big)\Big)(\mathcal{F}_2 \varphi)(\xi) \|_2\,.$$

Thus, we have proved

$$\tilde{\gamma}_r(g * \varphi) \leqslant \bar{C} \nu_{r+k}(\varphi)\,, \qquad \varphi \in \mathcal{D}^{(M_p)}_{L^2}. \tag{8}$$

Conversely, let $g \in \mathcal{O}'^{(M_p)}_C$ and $\varphi$ be an arbitrary element from $\mathcal{D}^{(M_p)}_{L^2}$. Since $\mathcal{F}_2(g * \varphi) = (\mathcal{F}_M g)(\mathcal{F}_2 \varphi)$ and $\nu_k(g * \varphi) < \infty$ for every $k > 0$, we have

$$\| (\mathcal{F}_M g) \exp\Big(M(k|\cdot|)\Big)(\mathcal{F}_2 \varphi) \|_2 < \infty \quad \text{for every } k > 0\,. \tag{9}$$

It can be proved by (9) that $\mathcal{F}_M g$ is from $L^\infty_{\mathrm{loc}}$. Namely, for any open ball $\overset{\circ}{K}(m)$, $L^2\big(\overset{\circ}{K}(m)\big)$ can be embedded into $D^{(M_p)}_{L^2}$ in a natural way:

$$L^2\big(\overset{\circ}{K}(m)\big) \in \psi \mapsto \tilde{\psi} = \begin{cases} \psi & \text{on } \overset{\circ}{K}(m) \\ 0 & \text{on } \mathfrak{R}^q \setminus \overset{\circ}{K}(m) \end{cases} \in D^{(M_p)}_{L^2}\,.$$

Since the condition

$$\| H(\xi) \psi(\xi) \|_{L^2(\overset{\circ}{K}(m))} < \infty \quad \text{for every } \psi \in L^2\big(\overset{\circ}{K}(m)\big)\,,$$

implies $H \in L^\infty\big(\overset{\circ}{K}(m)\big)$, we obtain that $\mathcal{F}_M g \in L^\infty_{\mathrm{loc}}$.

Let us prove (7) by proving that $g$ does not belong to $\mathcal{O}'^{(M_p)}_C$ if (7) does not hold.

If (7) does not hold there exists a sequence $a_j$ such that $a_1 > 1$,

$a_{j+1} > a_j + 1$ and

$$|(\mathcal{F}_M g)(\xi)| \exp\left(-M(j|\xi|)\right) \geqslant j \quad \text{if } |\xi| \in A_j \subset (a_j, a_{j+1})$$

with $\operatorname{mes} A_j = \varepsilon_j > 0$, $j \in \mathfrak{N}$.

Put

$$\psi_j(\xi) = \begin{cases} \varepsilon_j^{-1} \exp\left(-M(j|\xi|)\right), & |\xi| \in A_j,\ j \in \mathfrak{N}, \\ 0 & \text{elsewhere} \end{cases}$$

and $\psi = \sum_{j=1}^{\infty} \psi_j$. Since $\exp\left(-M(|\xi|)\right)$ decreases monotonically faster than any power of $1/|\xi|$ when, $|\xi| \to \infty$, one can prove that $\psi \in D_{L^2}^{(M_p)}$ and that $(\mathcal{F}_M g)\psi$ does not belong to $D_{L^2}^{(M_p)}$.

The proof is completed.

From (8) directly follows:

PROPOSITION 5. *If $g \in \mathcal{O}_C'^{(M_p)}$ then the mapping*

$$\varphi \mapsto g * \varphi \quad \textit{from } \mathcal{D}_{L^2}^{(M_p)} \textit{ into } \mathcal{D}_{L^2}^{(M_p)}$$

*is continuous.*

The last assertion enables us to define the convolution of an $f \in \mathcal{D}_{L^2}'^{(M_p)}$ and a $g \in \mathcal{O}_C'^{(M_p)}$ in a usual way:

$$\langle f \mathbin{\text{☆}} g, \varphi \rangle = \langle f, \check{g} * \varphi \rangle \quad \text{where } \check{g}(x) = g(-x),\ x \in \mathfrak{N}.$$

PROPOSITION 6. *If $f \in \mathcal{D}_{L^2}'^{(M_p)}$ and $g \in \mathcal{O}_C'^{(M_p)}$ then,*

$$\mathcal{F}_M(f \mathbin{\text{☆}} g) = (\mathcal{F}_M f)(\mathcal{F}_M g).$$

PROOF. For $\psi \in D_{L^2}^{(M_p)}$ we have

$$\begin{aligned} \langle \mathcal{F}_M(f \mathbin{\text{☆}} g), \psi \rangle &= \langle f \mathbin{\text{☆}} g, \mathcal{F}_2 \psi \rangle = \langle f, \check{g} * \mathcal{F}_2 \psi \rangle = \\ &= \langle \mathcal{F}_M f, \mathcal{F}_2^{-1}(\check{g} * \mathcal{F}_2 \psi) \rangle = \langle (\mathcal{F}_M f)(\xi), (2\pi)^{-q} \mathcal{F}_2(\check{g} * \mathcal{F}_2 \psi)(-\xi) \rangle = \\ &= \langle (\mathcal{F}_M f)(\xi), (2\pi)^{-q} (\mathcal{F}_M g)(\xi) \big(\mathcal{F}_2(\mathcal{F}_2 \psi)\big)(-\xi) \rangle = \\ &= \langle (\mathcal{F}_M f)(\xi), (\mathcal{F}_M g)(\xi) \psi(\xi) \rangle. \end{aligned}$$

This implies the assertion.

If $g$ and $f$ are from $\mathcal{O}_C'^{(M_p)}$ then $f \mathbin{\text{☆}} g \in \mathcal{O}_C'^{(M_p)}$.

This follows from the definition of $\mathcal{O}'^{(M_p)}_C$ and the fact that

$$(f \mathbin{\text{☆}} g) * \varphi = f \mathbin{\text{☆}} (g * \varphi) = f * (g * \varphi), \quad \varphi \in \mathcal{D}^{(M_p)}_{L^1}.$$

Properties of the convolution are given in the next

PROPOSITION 7. *Let* $g, h \in \mathcal{O}'^{(M_p)}_C$ *and* $f \in \mathcal{D}'^{(M_p)}_{L^1}$.

(i) $g \mathbin{\text{☆}} h = h \mathbin{\text{☆}} g$;

(ii) $(f \mathbin{\text{☆}} g) \mathbin{\text{☆}} h = f \mathbin{\text{☆}} (g \mathbin{\text{☆}} h)$.

PROOF. (i) We have

$$\langle g \mathbin{\text{☆}} h, \varphi \rangle = \langle g, \check{h} * \varphi \rangle = (g * (\check{h} \check{*} \varphi))(0);$$

$$\langle h \mathbin{\text{☆}} g, \varphi \rangle = (h * (\check{g} \check{*} \varphi))(0).$$

Proposition 3 implies that $(h * (\check{g} \check{*} \varphi))(x) = (g * (\check{h} \check{*} \varphi))(x)$, $x \in \mathfrak{R}^q$.

(ii) Since

$$\langle f \mathbin{\text{☆}} (g \mathbin{\text{☆}} h), \varphi \rangle = \langle f, (g \check{\mathbin{\text{☆}}} h) * \varphi \rangle = \langle f(x), (g \mathbin{\text{☆}} h)(t), \varphi(x+t) \rangle =$$

$$= \big\langle f(x), \langle g(t), \langle \check{h}(u), \varphi(x+t-\alpha) \rangle \rangle \big\rangle =$$

$$= \langle f(x), \langle g(t), (\check{h} * \varphi)(x+t) \rangle \rangle = \langle f(x), \check{g} * (\check{h} * \varphi)(x) \rangle,$$

and

$$\langle (f \mathbin{\text{☆}} g) \mathbin{\text{☆}} h, \varphi \rangle = \langle (f \mathbin{\text{☆}} g), (\check{h} * \varphi) \rangle = \langle f, \check{g} * (\check{h} * \varphi) \rangle,$$

the assertion is proved.

**5.** Observe the convolution equation in $\mathcal{D}'^{(M_p)}_{L^1}$:

$$S \mathbin{\text{☆}} U = V, \tag{10}$$

where $S \in \mathcal{O}'^{(M_p)}_C$ and $V \in \mathcal{D}'^{(M_p)}_{L^1}$ are known ultradistributions and $U \in \mathcal{D}'^{(M_p)}_{L^1}$ is the unknown one.

Denote by $\mathcal{O}'^{(M_p)}_{C,\mathcal{A}}$ the space of all convolutors $S \in \mathcal{O}'^{(M_p)}_C$ for which $\mathcal{F}_M S$ is a smooth function which has the analytic continuation onto the whole $\mathfrak{C}^q$.

PROPOSITION 8. *Let $S \in \mathcal{O}_C'^{(M_p)}$. The sufficient condition that the equation* (10) *is solvable in $\mathcal{D}_{L^2}'^{(M_p)}$ for any $V \in \mathcal{D}_{L^2}'^{(M_p)}$ is the following one:*

*There exist $C > 0$ and $k > 0$ such that*

$$|(\mathcal{F}_M S)(\zeta)| \geqslant \frac{C}{\exp\left(M(k|\zeta|)\right)}, \qquad \zeta \in \mathbb{C}^q. \tag{11}$$

PROOF. Assume that (11) holds. Put

$$u(\xi) = \frac{(\mathcal{F}_M V)(\xi)}{(\mathcal{F}_M S)(\xi)}, \qquad \xi \in \mathbb{R}^q.$$

From (5) it follows that for some ultradifferential operator $P$ of class $(M_p)$ and some $v \in L^2$

$$u(\xi) = \frac{P(\xi)}{(\mathcal{F}_M S)(\xi)}\, v(\xi), \qquad \xi \in \mathbb{R}^q.$$

Let $P_1(\xi) = P(\xi)/(\mathcal{F}_M S)(\xi)$, $\xi \in \mathbb{R}^q$. From [2, Proposizion 4.5] it follows that $P_1(D)$ is an ultradifferential operator of class $(M_p)$. Thus, by (5) again, we obtain that the solution of (10) is

$$U = P_1(D)\, \mathcal{F}_2^{-1} v.$$

PROPOSITION 9. *Let $S \in \mathcal{O}_C'^{(M_p)}$. Then the necessary condition for the solvability of* (10) *for any $V \in \mathcal{D}_{L^2}'^{(M_p)}$ is the following one:*

*There exist $C > 0$, $D > 0$ and $k > 0$ such that*

$$|(\mathcal{F}_M S)(\xi)| \geqslant C \exp\left(-M(k|\xi|)\right), \qquad |\xi| \geqslant D. \tag{12}$$

PROOF. Assume that (10) is solvable for any $V \in \mathcal{D}_{L^2}'^{(M_p)}$ but (12) does not hold. This implies that there exists a sequence of sets $A_j$, $j \in \mathbb{N}$, such that $\operatorname{mes} A_j = \varepsilon_j > 0$, $A_j \subset (a_j, a_{j+1})$ $a_1 > 1$, $a_{j+1} > a_j + 1$, $j \in \mathbb{N}$ and

$$|(\mathcal{F}_M S)(\xi)| \leqslant j^{-1} \exp\left(-M(j^2|\xi|)\right), \qquad |\xi| \in A_j,\ j \in \mathbb{N}.$$

As in the proof of Proposition 4, put

$$v_j = \begin{cases} \varepsilon_j^{-1} \exp\left(-M(j|\xi|)\right), & |\xi| \in A_j,\ j \in \mathbb{N}, \\ 0 & \text{elsewhere} \end{cases}$$

and

$$v = \sum_{j=1}^{\infty} v_j .$$

Clearly $v \in D'^{(M_p)}_{L^2}$ but

$$v/(\mathcal{F}_M S) \notin D'^{(M_p)}_{L^2} .$$

Thus, for such $V = \mathcal{F}_M^{-1}(v)$ and $S$ the solution of (10) does not exist in $\mathcal{D}'^{(M_p)}_{L^2}$.

**6.** We say that equation (10) satisfies the hypoelipticity condition if the existence of the solution $U \in \mathcal{D}'^{(M_p)}_{L^2}$ of (10) and $V \in \mathcal{D}^{(M_p)}_{L^2}$ imply that $U \in \mathcal{D}^{(M_p)}_{L^2}$. In this case we say that $U$ is the hypoeliptic solution of (10).

PROPOSITION 10. *Let $S \in \mathcal{O}'^{(M_p)}_C$. Equation* (10) *is hypoeliptic iff* (12) *holds.*

PROOF. Let $\tilde{\psi}(\xi) = 1$ for $|\xi| \leqslant D + 1$ and $\tilde{\psi}(\xi) = 0$ for $|\xi| > D + 1$. This is an element of $D^{(M_p)}_{L^2}$.

Put

$$\tilde{P}(\xi) = \frac{1 - \tilde{\psi}(\xi)}{(\mathcal{F}_M S)(\xi)} .$$

We have

$$|\tilde{P}(\xi)| \leqslant C \exp M(k|\xi|) , \qquad \xi \in \mathfrak{R}^q .$$

Obviously, $P = \mathcal{F}_M^{-1} \tilde{P} \in \mathcal{O}'^{(M_p)}_C$ and

$$S \mathbin{\text{☆}} P = \delta - \psi , \qquad \text{where } \psi = \mathcal{F}^{-1}(\tilde{\psi}) .$$

Since

$$U = U * \delta = V \mathbin{\text{☆}} P + U \mathbin{\text{☆}} \psi$$

one can easily prove that $U \in \mathcal{D}^{(M_p)}_L$.

By the same arguments as in the second part of the proof of Proposition 9 one can prove that if (10) is hypoeliptic then (12) holds for $S$.

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*Added in proof.* Proposition 4 implies that $(\mathcal{F}_M S)(\xi) = s(\xi) \exp[M(k|\xi|)]$, $\xi \in \mathfrak{R}^q$, for some $k > 0$ and some $s \in L^\infty$, and from (4) we have $(\mathcal{F}_M V)(\xi) = P(\xi) v(\xi)$, $\xi \in \mathfrak{R}^2$, where $P$ is an ultradifferential operator of class $(M_p)$ and $v \in L^2$. So, by using the Fourier transformation we have:

« (10) is solvable in $\mathcal{D}'^{(M_p)}_{L^2}$ if the equation $su = v$ has a solution $u \in L^2$ ».

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